Migration Equation Coefficients for a Shot-Offset Frame in Layered Media

by Raul Estevez

In the present paper I intend to complete the work begun in my last paper, "Migration Equation Coefficients for an Emergent Angle Frame ... ", carrying out similar calculations but in relation to the second frame proposed by Claerbout in his paper of July 3 ("A Shot Offset Frame ..."). I would like to point out that I still don't see very clearly the connection between these frames and a possible generalization of the slant frames equation, valid for sections and gathers. In particular, I think that a better understanding of this relationship will probably yield to certain improvements in the proposed transformations and corresponding equations.

Nevertheless, the introduction of recording variables such as s, g, z, t and interpretation variables such as h, y, r, d (especially in the case of CDP sections) seem to be necessary if we want to apply the wave equation to sections. Moreover, the integral transformations and integration techniques developed in the previous paper as well as the ones that will be considered in the present paper, may be of interest in future related studies. Therefore, at least as a matter of further reference, I thought that it may be helpful to carry on the present calculations.

H-Frame and Its Jacobian

As previously, we begin from a coordinate system with s, g, z, t as independent variables and seek a transformation that allows us to express the wave equation in terms of h, y, r and d as independent variables.

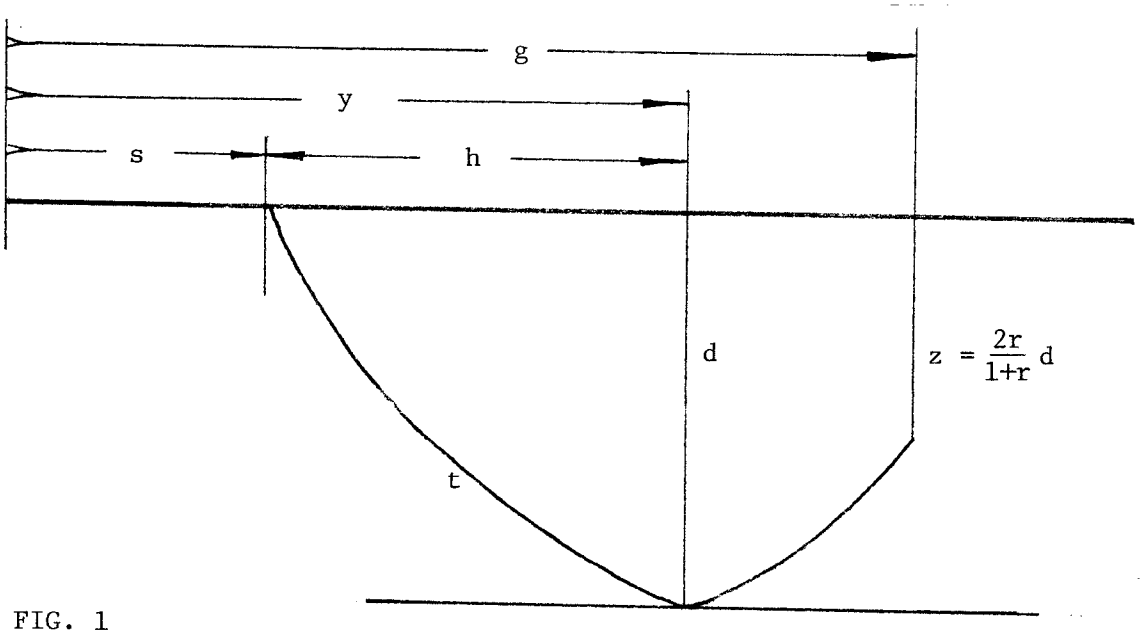

FIG. 1

According to figure 1 we could define this transformation in different ways, but when trying to write it for stratified media we may run into trouble in some cases. For example, if we take the transformation originally proposed by Claerbout (July 3), we would have to deal with integrals in relation to the travel time, which would imply knowing velocity and other parameters as functions of travel time instead of depth. This last problem can be avoided if we define the transformation in the same way Claerbout did in the first of the two frames (p-frame), but expressing instead, p as a function of h and d, $p(h,d)$. Following our last paper, this would mean:

$$s = y - h \tag{1-1a}$$

$$g = y + h - u\left(p(h,d), \frac{2r}{1+r}d\right) = y + h - \tag{1-1b}$$

$$- p(h,d) \int_0^{2rd/(1+r)} v(z)\left[1 - (p(h,d)v(z))^2\right]^{-1/2} dz$$

$$z = 2rd/(1+r) \tag{1-1c}$$

$$t = 2\tau(p(h,d),d) - \tau\left(p(h,d), \frac{2r}{1+r}d\right) = 2\int_0^d \frac{1}{v(z)}\left[1 - (p(h,d)v(z))^2\right]^{-1/2} dz$$

$$-\int_0^{2rd/(1+r)} \frac{1}{v}\left[1 - (pv)^2\right]^{-1/2} dz = \int_0^{2rd/(1+r)} \frac{1}{v}\left[1 - (pv)^2\right]^{-1/2} dz + 2\int_{2rd/(1+r)}^d \frac{1}{v}\left[1 - (pv)^2\right]^{-1/2} dz \tag{1-1d}$$

The Jacobian of this transformation looks simpler than the one corresponding to the p-frame (which opens the possibility of an analytical computation of its inverse):

$$\begin{bmatrix} s_h & s_y & s_d & s_r \\ g_h & g_y & g_d & g_r \\ z_h & z_y & z_d & z_r \\ t_h & t_y & t_d & t_r \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ g_h & 1 & g_d & g_r \\ 0 & 0 & z_d & z_r \\ t_h & 0 & t_d & t_r \end{bmatrix} \qquad (1\text{-}2)$$

If, following our previous paper, we define:

$$d' = 2rd / (1+r) \qquad (1\text{-}3a)$$

$$\tilde{I}_1(h,d) = \int_0^{d'} v(z)[\, 1 - (p(h,d)v(z))^2\,]^{-1/2}\, dz \qquad (1\text{-}3b)$$

$$\tilde{I}_2(h,d) = \int_0^{d'} v^3[\, 1 - (pv)^2\,]^{-3/2}\, dz \qquad (1\text{-}3c)$$

$$\tilde{I}_3(h,d) = \int_0^{d'} v[\, 1 - (pv)^2\,]^{-3/2}\, dz \qquad (1\text{-}3d)$$

$$\tilde{I}_4(h,d) = \int_{d'}^{d} v[\, 1 - (pv)^2\,]^{-3/2}\, dz \qquad (1\text{-}3e)$$

$$SQ\,1\,(h,d) = [\, 1 - (p(h,d)v(d))^2\,]^{-1/2} \qquad (1\text{-}3f)$$

$$SQ\,2\,(h,d) = [\, 1 - (p(h,d)v(d')^2\,]^{-1/2} \quad , \qquad (1\text{-}3g)$$

the elements of the Jacobian (1-2) can be rewritten as:

$$s_{h,y,d,r} = -1,\ 1,\ 0,\ 0 \qquad (1\text{-}4a)$$

$$g_{h,y,d,r} = 1 - p_h(h,d)(\tilde{I}_1 + p^2\tilde{I}_2),\ 1,\ -p_d(\tilde{I}_1 + p^2 I_2) - \frac{2r}{1+r}\, pv(d')SQ2,\ -\frac{2d}{(1+r)^2}\, pv(d')SQ2 \qquad (1\text{-}4b)$$

$$z_{h,y,d,r} = 0,\ 0,\ 2r/(1+r),\ 2d/(1+r)^2 \qquad (1\text{-}4c)$$

$$t_{h,y,d,r} = p\,p_h\,(\tilde{I}_3 + 2\,\tilde{I}_4),\ 0,\ p\,p_d\,(\tilde{I}_3 + 2\,\tilde{I}_4) + \frac{2}{v(d)}\,SQ1 - \frac{2r}{1+r}\cdot\frac{SQ2}{v(d')}\ ,$$

$$,\ -\frac{2d}{(1+r)^2}\ \frac{SQ2}{v(d')}\ . \qquad (1\text{-}4d)$$

As we see, in all the previous expressions p, p_h and p_d are functions of h and d. So, unless we find a practical way to compute $p(h,d)$, $p_h(h,d)$ and $p_d(h,d)$, the Jacobian (1-2) will remain undefined.

Computation of $p(h,d)$, $p_h(h,d)$ and $p_d(h,d)$

In our last paper we found an integral relation that links p, h and d. If in the expression for $u(p,w)$:

$$u(p,w) = p\int_0^w v(z)[\ 1-(pv)^2\ ]^{-1/2}\,dz\ , \qquad (2\text{-}1)$$

we make $u=h$ and $w=d$, we will have that

$$p(h,d) = \frac{h}{\int_0^d v(z)[1-(p(h,d)v(z))^2]^{-1/2}\,dz}\ . \qquad (2\text{-}2)$$

This relation represents an integral equation for $p(h,d)$, and its analytical reduction to an explicit equation does not seem to be a very easy task. Nevertheless, equation (2-2) is similar to another equation well known in theoretical seismology (Herglotz-Wiechert's inversion) and it seems to me quite possible that an explicit equation for $p(h,d)$ can be obtained following a method attributed to Rasch (see Garland or Bullen). Rather than following this course I tried instead to get a good approximate solution.

In terms of numerical approximations, the first thing we may think about is a perturbation scheme for (2-2). For small angles (i.e., $h/d \ll 1$) $p \sim 0$, and for a given set (h,d) we could start a recursion beginning with:

$$p(h,d) \approx \frac{h}{\int_0^d v(z)dz} \quad . \tag{2-3}$$

Subsequently, as the angles become larger, we can begin recursions with previously calculated values of p , contiguous to the region of interest. The implicit assumption is that p is slowly varying. As noted at the end of the previous paper, the presence of the square root with a singularity at $pv=1$ makes these schemes very unstable. If any success is possible at all, it will depend on the initial value being very close to the solution, especially in regions near the singularity. I did not investigate deeply the convergence of various recursive schemes, but in several tests I made, I got non-convergent recursions for values of $h \gtrsim d$.

Since we are interested as well in computing P_h and P_d , another possibility is to convert (2-2) into an initial-valued problem by analytically calculating p_h and p_d as functions of h , d and p . In this direction I tried with positive results the Runge-Kutta method but, in general, this kind of solution was quite expensive and the precision very poor for $h \gtrsim d$.

Noticing that the main trouble lies on the presence of the singularity in the integral of (2-2), it is not difficult to realize that a satisfactory solution could be achieved by applying the previously discussed method of product integration combined with a fast converging recursive scheme. As pointed out before, this method allows us to integrate analytically the singularity and is described in our previous paper.

Starting with an initial value for p ,

$$p(h,d) \cong \frac{h}{v(0)v(d)\ (h^2+d^2)^{1/2}} \cdot \frac{v(0)+v(d)}{2} \qquad (2\text{-}4)$$

and computing the integrals according to the referred method, I got a very fast and cheap convergence with the Newton recursion (2 or 3 recursions gave reasonable precision).

If we call

$$I_1(h,d) = \int_0^d v[\ 1 - (pv)^2\ ]^{-1/2}\ dz \qquad (2\text{-}5)$$

$$I_3(h,d) = \int_0^d v^3[\ 1 - (pv)^2\ ]^{-3/2}\ dz \quad , \qquad (2\text{-}6)$$

the algorithm becomes:

$$p_{n+1} = p_n - \frac{P_n I_{1n}(h,d) - h}{I_{1n}(h,d) + p_n{}^2 I_{3n}(h,d)} \quad , \qquad (2\text{-}7)$$

where,

$$I_{1n} = \frac{1}{p_n{}^2} \sum_{j=0}^{NZ-1} \frac{1}{b_j} \{[\ 1 - (p_n v_j)^2\ ]^{1/2} - [\ 1 - (p_n v_{j+1})^2\ ]^{1/2}\} \ , \qquad (2\text{-}8)$$

$$I_{3n} = \frac{1}{p_n^4} \sum_{j=0}^{NZ-1} \frac{1}{b_j} \{[1-(p_n v_{j+1})^2]^{1/2} + [1 - (p_n v_{j+1})^2]^{-1/2} - [1 - (p_n v_j)^2]^{1/2} - [1 - (p_n v_j)^2]^{-1/2}\ \} \qquad (2\text{-}9)$$

and

$$v_i = v_{i-1} + b_i z \ . \qquad (2\text{-}10)$$

Once we know p for a given set of (h,d) , the computation of $p_h(h,d)$ and $p_d(h,d)$ can be carried out analytically, since from (2-2) it follows that:

$$p_h(h,d) = \frac{I_1(h,d)}{I_1^2(h,d) + hp(h,d)I_3(h,d)} \qquad (2\text{-}11)$$

and

$$p_d(h,d) = - \frac{hv(d)[1 - (pv(d))^2]^{-1/2}}{I_1^2(h,d) + hp(h,d)I_3(h,d)} \quad . \qquad (2\text{-}12)$$

It may seem that too much effort has been put in simply trying to solve a very small aspect of a particular transformation. The reason this was done is because I feel that the computation of $p(h,d)$ may be of importance not only for the treated transformation but also for future applications. Notice that the ray parameter p seems to be the best, if not the only, reasonable way to characterize a given ray in a variable velocity medium. Therefore, its computation from some surface parameters with a given velocity model, may be very helpful in some cases.

With $p(h,d)$, $p_h(h,d)$ and $p_d(h,d)$, the Jacobian (1-2) is fully defined, and the computation of its inverse, and therefore the transformed wave equation's coefficients, can be done in the same way it was proposed for the previously considered p-frame. I shall point out that the simplicity of the Jacobian (1-2) indicates that the analytical computation of its inverse may be a relatively easy task. This would provide us with analytical expressions for the equation's coefficients. Further computations, including the consideration of some simple velocity models (as we did before), should wait for a deeper understanding of the two considered frames.

VARIATIONAL PRINCIPLES AND APPROXIMATE SOLUTIONS FOR THE WAVE EQUATION

by W. Scott Dunbar

Traditionally, geophysicists have described the earth in terms of a partial differential equation. We are concerned with the two dimensional scalar wave equation in the displacement P :

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 P}{\partial t^2} \tag{1}$$

with the initial and boundary conditions

$$P(0, z, t) = P(x, B, t) = P(A, z, t) = 0 \tag{2}$$

$$P(x, z, 0) = f(x, z) \tag{3}$$

$$\left. \frac{\partial P}{\partial t} \right|_{t=0} = g(x, z) \tag{4}$$

where A and B are the spatial limits of the problem and c is the velocity. For relatively simple geometries, (1) - (4) can be quite easily solved. However, for non-trivial geometries, some kind of approximation must be used.

One approximation technique is afforded through the use of a variational principle. This involves finding a functional $F(P)$ that is minimized (or maximized) by the true solution P . The most common variational principle for the wave equation is Hamilton's principle. This states that for kinetic energy T and potential energy U , the functional

$$F = \int_0^T (T - U)\, dt \tag{5}$$

is minimized by P subject to the conditions (2) and (3) and to the additional condition

$$P(x, z, T) = h (x, z) \tag{6}$$

where T is some later time (arbitrary, but fixed). This condition is necessary in order to define a minimum.

We will now restrict ourselves to one dimension. This will make the algebra less confusing. Everything that follows can be easily extended to two dimensions. In one dimension, the kinetic energy is

$$T = \frac{1}{2}\int_x \rho \left(\frac{\partial P}{\partial t} \right)^2 dx$$

where ρ is the material density. The potential energy is

$$U = \frac{1}{2}\int_x E \left(\frac{\partial P}{\partial x} \right)^2 dx$$

where E is an elastic modulus. Substitution into (5) gives

$$F(P) = \frac{1}{2}\int_0^T \int_x \left\{ \left(\frac{\partial P}{\partial t} \right)^2 - c^2 \left(\frac{\partial P}{\partial x} \right)^2 \right\} dx\, dt \tag{7}$$

where $c^2 = E/\rho$. The integrand is known as the Lagrangian, L . The Euler-Lagrange equation

$$\frac{\partial}{\partial x} \left(\frac{\partial L}{\partial P'} \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{P}} \right) = 0$$

is the one dimensional wave equation. ($\dot{P}$ is the time derivative of P ; P' is the x derivative.)

The Rayleigh-Ritz technique [1] can be used to find an approximate solution to (7). We begin by selecting n trial functions $f_1,\ldots,f_n$, defined over the domains (x,t) or (x). A linear combination of these functions is then used to form P:

$$P = \sum_{i=1}^{n} a_i f_i \tag{8}$$

where the a_i are generalized coordinates or generalized displacement amplitudes. If $f_i \in (x)$, P is a function of time. Equation (8) is substituted into (7) which is then made stationary with respect to a_i. This results in a set of linear equations for a_i which are then used to find P. Hopefully, this is a good approximation to P.

However, it may be difficult to select trial functions that satisfy the boundary and initial conditions, (2) - (4), of the original problem. This is especially true for problems with complex geometry. To extend the Rayleigh-Ritz method so that it could handle more general problems, the finite element method was developed.

The key to this method lies in relating the unknown function to an individual element or subregion of the domains, (x,t) or (x), rather than to the total domain. The choice of approximating functions, f_i, is then made independently of the boundary and initial conditions of the total problem. The only conditions to be met are:

(1) the number of coefficients a_i must at least equal the number of nodes associated with the element. In the one dimensional wave propagation example, if $f_i \in (x)$, there must be two nodes and two coefficients. See figure 1.

(2) the functions must be continuous and differentiable.

(3) the functions must provide compatibility across element interfaces, i.e., in the one dimensional case the approximations to P over

each element must equal each other at adjacent nodes.

Once the functions are chosen and the (local) system of equations is derived for a particular element, the local equations for each of the N elements are then "assembled" to give a larger (global) system of equations for the total problem. This merely involves superimposing the local equations into the appropriate position of the global matrix. The global matrix is typically sparse and banded.

The difference between the finite element method and the finite difference method can now be understood. The latter approximates the differential operator on a finite point set, whereas the former uses a continuous approximation. Although the finite element method is usually associated with a variational principle, it stands independently as a sophisticated mathematical concept concerning the approximation of surfaces.

Now to get to some concrete examples. We can choose to approximate $P(x,t)$ in (x) by the linear function

$$\begin{aligned} P(x,t) &= a_1(t) + a_2(t)\, x \\ &= f^T(x)\, a(t) \end{aligned} \tag{9}$$

where $f^T(x) = [1,x]$ and $a^T(t) = [\, a_1(t),\, a_2(t)\,]$. To find a we identify two nodal points of an element and evaluate (9) at each node. This gives a matrix equation

$$P = \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix} = C\, a$$

This can be inverted to give

$$a = \begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix} = \frac{1}{x_2-x_1} \begin{bmatrix} x_2 & -x_1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} = C^{-1} P$$

Substituting into (9)

$$P(x,t) = \frac{1}{x_2-x_1} \begin{bmatrix} x_2-x , & -x_1+x \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix}$$

$$= \phi^T(x)\, P(t) \qquad (10)$$

where

$$\phi_1(x) = \frac{x_2-x}{x_2-x_1} \qquad \phi_2(x) = \frac{-x_1+x}{x_2-x_1}$$

Since the displacement at a node i depends only on x and t and not on any other displacement, the compatibility condition is satisfied.

We now substitute (10) into (7) to give

$$F(P) = \frac{1}{2} \int_0^T \int_x \left\{ \dot{P}^T \phi\, \phi^T \dot{P} - c^2 P^T \frac{d\phi}{dx} \frac{d\phi^T}{dx} P \right\} dx\, dt$$

The Euler-Lagrange equation is (for no x derivatives)

$$\frac{\partial L}{\partial P} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{P}} \right) = 0 = \left[\int_x \phi\, \phi^T dx \right] \ddot{P} + \left[c^2 \int_x \frac{d\phi}{dx} \frac{d\phi^T}{dx} \right] P \qquad (11)$$

which may be recognized as the Newtonian equation of motion of the element. The local "mass matrices"

$$M^e = \int_x \phi\, \phi^T dx \qquad e = 1, 2, \ldots, N$$

and "stiffness matrices:

$$K^e = c^2 \int_x \frac{d\phi}{dx} \frac{d\phi^T}{dx} dx \qquad e = 1, 2, \ldots, N$$

are then assembled to give a global system of differential equations. There are several methods of solving this initial valve problem [2, p. 24], but they are somewhat cumbersome.

Difference equations could be established by substituting approximate expressions for $P(t)$ into (11). These would be functions that would satisfy the initial and end conditions (3) and (6). Equation (11) is then made stationary with respect to the constants of the expression for $P(t)$. This results in a set of difference equations for P (which may be tridiagonal if a low order expression is used for $P(t)$). However, the end condition $P(T)$ has to be specified in order to solve this system. By using (4), the difference equations may be arranged to connect $P(0)$ and $\dot{P}(0)$ with the unknown $P(T)$.

Rather than go through the details of finding difference equations for (11), we can exploit a property of finite elements. Finite elements can be developed in non-Euclidean spaces [3]. This means that we can approximate $P(x,t)$ by trial functions $f_i \varepsilon(x,t)$, i.e., finite elements in space and time. This has been proposed for the wave equation but no literature seems to exist on its application.

We can choose the linear polynomial

$$\begin{aligned} P(x,t) &= a_1 + a_2 x + a_3 t \\ &= f^T(x,t)\, a \end{aligned} \qquad (12)$$

By identifying three nodal points (see Figure (2)), we can evaluate (12) at each node and eventually obtain a set of equations similar to (10):

$$P(x,t) = \Psi^T(x,t)\, P$$

where

$$\Psi_1(x,t) = \frac{1}{2A}\left[(x_2\, t_3 - x_3\, t_2) + (t_2 - t_3)\, x + (x_3 - x_2)\, t \right]$$

$$\Psi_2(x,t) = \frac{1}{2A}\left[(x_3\, t_1 - x_1\, t_3) + (t_3 - t_1)\, x + (x_1 - x_3)\, t \right]$$

$$\Psi_3(x,t) = \frac{1}{2A}\left[(x_1\, t_2 - x_2\, t_1) + (t_1 - t_2)\, x + (x_2 - x_1)\, t \right]$$

and where A is the area of the element in the x,t plane.

Substituting (12) into (7) we get

$$F(P) = \frac{1}{2}\int_0^T \int_x \left\{ P^T \frac{\partial \Psi}{\partial t}\frac{\partial \Psi^T}{\partial t} P - c^2\, P^T \frac{\partial \Psi}{\partial x}\frac{\partial \Psi^T}{\partial x} P \right\} dx\, dt \qquad (13)$$

The Euler-Lagrange equation is

$$\frac{\partial L}{\partial P} - \frac{d}{dt}\left(\frac{\partial L}{\partial P}\right) = 0 = \left[\int_0^T \int_x \left\{ \frac{\partial \Psi}{\partial t}\frac{\partial \Psi^T}{\partial t} - c^2 \frac{\partial \Psi}{\partial x}\frac{\partial \Psi^T}{\partial x} \right\} dx\, dt \right] P \qquad (14)$$

This is the finite element matrix analog of the one dimensional wave equation. It differs significantly from (11) in that it is not a functional of velocity. By assembling the local matrices into a global matrix, a system of equations results. Although it may not be obvious, the system is explicit for any degree of trial functions used. We have used the simplest trial functions.

Suppose that we are given the (one dimensional) boundary and initial conditions (2) - (4). For the finite element network of Figure (3), the solution would proceed as follows:

(1) The first tier of elements $0 \leqslant t \leqslant k$ is considered. The boundary conditions (2) are satisfied by setting $P_1 = P_6 = P_{11} = \ldots = 0$ and $P_5 = P_{10} = P_{15} = \ldots = 0$. The initial condition (3) is satisfied by setting $P_2 = f(h)$, $P_3 = f(2h)$, $P_4 = f(3h)$.

(2) Since P_2 , P_3 , and P_4 are prescribed, (14) may be rearranged to solve for P_7 , P_8 and P_9 . This is the displaced profile after k sec.

(3) The next tier of elements $k \leqslant t \leqslant 2k$ is considered and the solution is marched along in time.

Stability criteria would be similar to those of finite difference schemes [4].

There is another variational principle for the wave equation. It is the time derivative of a more complicated variational principle found by Gurtin [5] for initial value problems. The intriguing aspect of this principle is that it involves the time reversed wave equation in $Q(x,t) = P(x, T-t)$. The principle is

$$F(P) = \int_0^T \int_x \left\{ \frac{\partial Q}{\partial t} \frac{\partial P}{\partial t} - c^2 \frac{\partial Q}{\partial x} \frac{\partial P}{\partial x} \right\} dx\, dt + 2 \int_x g(x)\, P(x,T)\, dx \tag{15}$$

where $g(x)$ is the initial velocity condition. Although it does not necessarily concern us, the advantage of this principle is that it does not explicitly require the end condition $h(x) = P(x,T)$ in the Euler-Lagrange equations. Q and P can be approximated by space-time finite

elements as follows

$$P(x,t) = \Psi^T(x,t)\, P$$

$$Q(x,t) = \phi^T(x,T-t)\, Q$$

to give

$$F(P) = \frac{1}{2}\int_0^T \int_x \left\{ Q^T \frac{\partial \phi}{\partial t}\frac{\partial \Psi^T}{\partial t} P - c^2 Q^T \frac{\partial \phi}{\partial x}\frac{\partial \Psi^T}{\partial x} P \right\} dx\, dt$$

$$+ 2\int_x g(x)\, \Psi^T(x,T)\, P\, dx$$

Taking variations with respect to Q results in a different set of equations for P. Variations with respect to P will result in a set of equations for Q (which need $g(x) = \partial P/\partial t\,|t=0$) .

So far, we have spoken only of the "forward problem" of wave propagation. The "inverse problem" of using seismic sections as initial conditions and pushing the waves back into the earth is quite another game, made more difficult by the fact that solutions in the z direction are decaying and growing exponentials. The possibility of using variational principles to separate the up and downgoing components of the solution of the wave equation has not been investigated. After this article was finished, R. S. Anderssen, a visitor to Stanford from Australia, showed how such a separation could be achieved for the one dimensional wave equation. A successful separation depends not only on the choice of an appropriate variational principle, but on the correct choice of trial functions. His approach is in this report.

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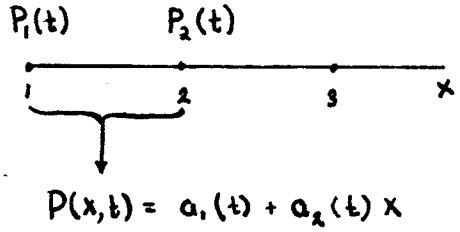

FIGURE 1

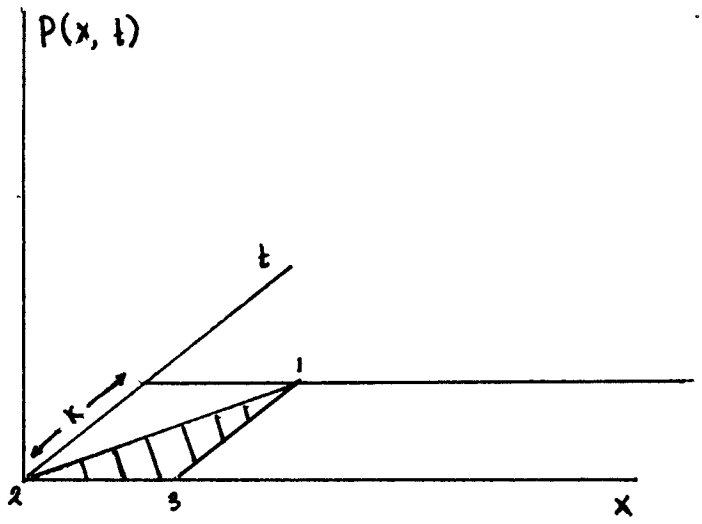

FIGURE 2

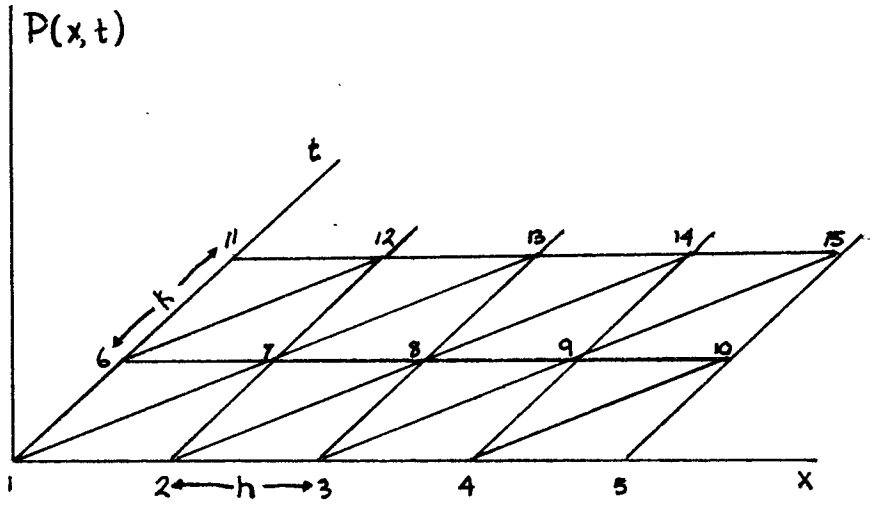

FIGURE 3