Slant Frames in Layered Media

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1. Introduction

The integral transformations and integration techniques developed in my last two papers on stratified media, give the clue for getting an equation for slant frames in stratified media. The problem was already discussed by Claerbout in his paper "Shifting Frames" (April 16), but some questions still remained open due to the presence in the proposed transformation of integrals in relation to travel time. This would imply the knowledge of velocity and other parameters (angles) as functions of travel time instead of depth.

At this point, I would like to call attention to the fact that we already know how to express travel time t as a function of the ray parameter p and the depth z (first paper in layered media):

$$t(p,z) = \int_0^z \frac{1}{v(w)} \left[1 - (pv(w))^2 \right]^{-1/2} dw \quad . \qquad (1\text{-}1)$$

This eventually could allow us to transform the unwanted integrals in travel time into integrals in relation to depth. Thus, for example, in the original transformation proposed by Claerbout ("Shifting Frames"):

$$x' = x - \int_{t_0}^{t} v(t) \sin\theta(t)\, dt \quad , \qquad (1\text{-}2)$$

using (1-1) and the fact that $\sin\theta = pv$, we could reexpress this integral in the following way:

$$\int_{t_0}^{t} v(t) \sin\theta(t)\, dt = \int_{z_0}^{z} v(w) \sin\theta(w) \frac{dt}{dw}\, dw = p \int_{z_0}^{z} v(w)\left[1-(pv(w))^2\right]^{-1/2} dw \quad . \qquad (1\text{-}3)$$

But, as we will show later in this paper, this last transformation can be interpreted as a generalization of the more simple one (for a velocity constant medium):

$$x' = x - z \tan\theta \ , \tag{1-4}$$

rather than as a generalization of the original slant frame transformation:

$$x' = x - vt \sin\theta \ . \tag{1-5}$$

As we shall see, this fact will imply an equation that differs from the one obtained by Claerbout in his paper.

We encountered a similar situation when considering the h-frame in stratified media and saw how this problem could be solved if in the definition of our transformations we avoid having to express the new transformed variables as functions of travel time. Fortunately, it's our experience that most of the already defined transformations, especially those of the slant-frame type, are non-unique, and almost always we can find different transformations (mathematically) that will do approximately the same job. So, hopefully this non-uniqueness may help us to find other sets of transformations mathematically more suitable to our particular requirements. These different types of transformations can be obtained either starting from some mainly physical requirements and assumptions or following a more rigorous mathematical approach. In order to understand better how this can be done as well as the genesis itself of many of the transformations we have been using and might use in the future, I feel it would be of interest to review very briefly both procedures.

2. The two approaches to defining transformations

The physical approach

Let's start our discussion with what I call the physical approach, by reviewing once more the two basic physical ideas behind these transformations.

1 - If we regard the wave equation as a physical operator that translates and diffracts energy in time and space, what we want is a transformation that keeps only the diffraction part of this operator. This means that we want a moving frame that incorporates in itself the simple translation of the energy in space and produces a new equation where only diffraction is left. At this point we may want to express the diffraction as a function of the travel time or of the depth. As we learned from the Claerbout-Johnson transformation, the first case implies a transformation where the new x' and z' remain still and time flows, while the second case implies a transformation where x' and t' are frozen and z flows. For obvious reasons, the second possibility is preferred.

2 - We want a transformation that allows us to get separate equations for downgoing and upcoming waves respectively.

Usually this second condition is fulfilled in two steps: a) by defining two different frames, moving in opposite directions in correspondence with the two types of waves and b) by dropping some terms in the transformed wave equation.

The non-uniqueness of the transformation is rather a consequence of the first condition. As we may notice from Figure 1, for a wave propagating at a fixed direction θ , we could define x' and t' in many different ways and still satisfy this condition.

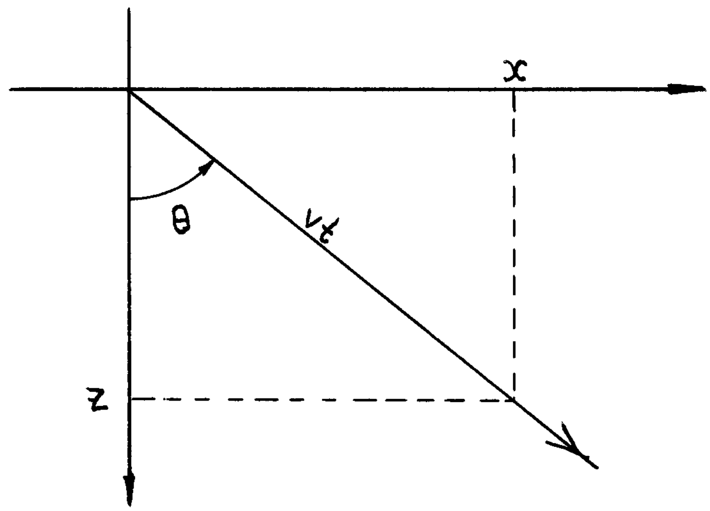


FIGURE 1.

For example, we could make $x' = x - vt \sin\theta$, or $x' = x - z\tan\theta$, or $x' = x \cos^2\theta - z \sin\theta \cos\theta$, etc., and all these transformations would make x' remain still in the new frame. In the same way we could define t' through several linear combinations of x , z and t in such a way that t' was also frozen. For example, $t' = t - (x \sin\theta)/v - (z \cos\theta)/v$, $t' = t - x \operatorname{cosec}\theta / 2v - z \sec\theta / 2v$ etc., will do the job. Among this class of transformations, which one should we use? At least three extra criteria could be pointed out:

1 - We want a transformed equation numerically computable and as simple as possible. In particular, we would like several coefficients of the new equation, such as $Q_{t't'}$ and $Q_{x't'}$ to vanish.

2 - We want the best possible approximation, after neglecting some terms of the new equation (such as $Q_{z'z'}$). This approximation could be thought of in terms of how well the approximated equation fits the exact solution (circle) in the dispersion plane k_z , k_x (see "Dispersion relationship for the slant frames ... 3/12/74).

3 - Finally we may want transformations or equations suitable for the particular problems we want to solve. For example, in our case (layered media) we do not want new space variables to be expressed, if possible, as functions of the travel time.

A better illustration of these last three criteria will be found immediately, through the discussion of the mathematical approach.

The mathematical approach

According to this approach, the basic idea is to start by expressing the new coordinates x' , z' , t' through a generalized transformation of the old ones x , z , t :

$$\begin{aligned} x' &= f_1(x,z,t) \\ z' &= f_2(x,z,t) \\ t' &= f_3(x,z,t) \end{aligned} \tag{2-1}$$

The most simple case to be considered is the case when x' , z' and t' are just linear combinations of x , z and t . Further, we would transform the wave equation with these still unknown functions and try to give a mathematical form to all the conditions previously stated ($Q_{t't'} = 0$, $Q_{x't'} = 0$, fit to the circle, etc.). These last mathematical conditions would act as additional constraint equations for the generalized functions in the transformation and, hopefully, would allow us to define them. This was, more or less, the technique used by Claerbout in his paper on "shifting frames".

Since we are less familiar with this approach, I will try to illustrate it in more detail through a very simple example that we might as well use later when considering stratified media. According to what was said before, for the particular problem we are considering in this paper, we need transformations where x' and z' are not functions of t. In this case one of the simplest ways to write (2-1) would be:

$$x'(x,z) = a(\theta)x + b(\theta)z \tag{2-2a}$$

$$z'(x,z) = c(\theta)x + d(\theta)z \tag{2-2b}$$

$$t'(x,z,t) = e(\theta)x + f(\theta)z + t \quad , \tag{2-2c}$$

the Jacobian of this transformation being:

$$x'_{x,z,t} = a,\ b,\ 0 \tag{2-3a}$$

$$z'_{x,z,t} = c,\ d,\ 0 \tag{2-3b}$$

$$t'_{x,z,t} = e,\ f,\ 1 \tag{2-3c}$$

If now we transform the wave equation:

$$(\partial_{xx} + \partial_{zz} - \frac{1}{v^2} \partial_{tt}) P = 0 \tag{2-4}$$

according to (2-2), we obtain in the new coordinate system the following equation:

$$(a^2+b^2)Q_{x'x'} + (c^2+d^2)Q_{z'z'} + (e^2+f^2-1/v^2)Q_{t't'} + 2(ac+bd)Q_{x'z'} +$$

$$+ 2(ae+bf)Q_{x't'} + 2(ce+df)Q_{z't'} = 0 \ . \tag{2-5}$$

Further, we would like the coefficients of $Q_{t't'}$, $Q_{x'z'}$ and $Q_{x't'}$ to vanish. This condition will give us the first three constraint equations:

$$e^2 + f^2 = \frac{1}{v^2} \tag{2-6a}$$

$$ac + bd = 0 \tag{2-6b}$$

$$ae + bf = 0 \tag{2-6c}$$

After neglecting, as usual, the $Q_{z'z'}$ term, we are left with the simpler equation:

$$(a^2 + b^2)Q_{x'x'} + 2\ (ce + df)Q_{z't'} \cong 0\ . \tag{2-7}$$

The next requirement that we must satisfy in order to separate both down and upcoming waves is that, after transforming this approximate equation back to the original coordinate system x, z, t, the coefficient of P_{zz} must vanish (this would give us a single defined function in the dispersion plane k_x, k_z). To do that let us notice that the inverse transformation to (2-2) is:

$$x = -\frac{d}{\gamma}x' + \frac{b}{\gamma}z' \tag{2-8a}$$

$$z = \frac{c}{\gamma}x' - \frac{a}{\gamma}z' \tag{2-8b}$$

$$t = \frac{de - cf}{\gamma}x' + \frac{af - be}{\gamma}z' + t'\ , \tag{2-8c}$$

with its Jacobian being

$$x_{x',z',t'} = \frac{d}{\gamma}\ ;\ \frac{b}{\gamma}\ ;\ 0 \tag{2-9a}$$

$$z_{x',z',t'} = \frac{c}{\gamma}\ ;-\frac{a}{\gamma}\ ;\ 0 \tag{2-9b}$$

$$t_{x',z',t'} = \frac{de - cf}{\gamma}\ ;\ \frac{af - be}{\gamma}\ ;\ 1\ , \tag{2-9c}$$

where

$$\gamma = bc - ad\ . \tag{2-10}$$

A simple algebra shows then that the coefficient of P_{zz} comes out to be $c^2(a^2+b^2)/\gamma^2$. As we see, requiring this coefficient to be zero is equivalent to assuming that $c=0$. But this result is compatible with equations (2-6b) and (2-6c) only in the trivial case of b and e also being zero, which would simply imply a vertical direction of propagation (along z). Since we are interested in other directions of propagation too, the only way to go further seems to be discarding equation (2-6b). This implies that within the range of the proposed transformations (2-2), we cannot get rid of the $Q_{x'z'}$ term in the transformed equation. We shall discuss later the implications of this fact.

Let's then assume that $ac + bd \neq 0$ and, again try to transform back the more complex equation:

$$(a^2 + b^2)\, Q_{x'x'} + 2(ac + bd)\, Q_{x'z'} + 2(ce + df)\, Q_{z't'} = 0 \qquad (2\text{-}11)$$

In this case we obtain for the coefficient of P_{zz}; $(b^2 - a^2 - 2abd/c)\, c^2/\gamma^2$. Making this coefficient vanish leaves us with two possibilities: either $c = 0$ or $b^2 - a^2 - 2abc/c = 0$. Notice that these conditions exclude a rotation of the axis as a possible transformation to be considered. Furthermore, if we wanted to consider a stronger (worse) approximation of equation (2-5) by neglecting both $Q_{z't'}$ and $Q_{x'z'}$ terms, this last result, plus the previous one, indicates that we would be left only with the possibility of c being zero.

Using equation (2-6c) to reexpress the second possibility through e and f, we can then write our three constraint equations as:

$$e^2 + f^2 = 1/v^2 \qquad (2\text{-}12a)$$

$$a/b = -f/e \qquad (2\text{-}12b)$$

$$c = 0 \qquad (2\text{-}12c)$$

or

$$c/d = 2\,ef/(f^2-e^2) \qquad (2\text{-}12d)$$

We could go farther and try to get additional constraints by satisfying finer conditions (a good fit to the circle in the dispersion plane etc.), but these cosiderations at the present moment would take us too far from our main objective of including layered media in our scheme. I shall simply point out that when picking out a given transformation, besides the mathematical conditions discussed above, we must see that the chosen transformation has an inverse and that the resulting expressions and equations converge to the previous results for vertical propagation.

Thus, we will stop here and try to define our six functions only with these three equations (2-12). In doing that, for simplicity, we will choose the most obvious and straightforward alternatives. That means for equation (2-12a):

$$e = \mp \frac{\sin\theta}{v} \tag{2-13a}$$

$$f = \mp \frac{\cos\theta}{v} \, , \tag{2-13b}$$

the "-" sign being for downgoing waves (x,z-positive) and the "+" sign for the upcoming ones. From now on we will refer only to the downgoing waves, so that we will keep only the minus sign in (2-13). Notice that defining e and f in the opposite way wouldn't give us the right results in the limiting case of $\theta = 0°$. The definitions (2-13) imply, among others, two simple possibilities for a and b according to equation (2-12b):

either,

$$a = \cos\theta \tag{2-14a}$$

$$b = -\sin\theta \tag{2-14b}$$

or

or

$$a = 1 \tag{2-15a}$$

$$b = -\tan\theta \tag{2-15b}$$

Further we shall consider separately each of the two possibilities (2-12c) and 2-12d):

<u>c = 0</u>

If we start by assuming that (2-12c) holds, then d will remain undefined and the simplest choice is to make it equal 1 (although we could use it to modify the coefficient of $Q_{x'x'}$):

$$c = 0 \tag{2-16a}$$

$$d = 1 \; . \tag{2-16b}$$

Therefore, in this case we get two solutions. Replacing (2-13), (2-14) and (2-16) into equation (2-11) will result in:

$$Q_{z't'} = \frac{v}{2} \sec\theta \, Q_{x'x'} - v \tan\theta \, Q_{x'z'} \; ; \tag{2-17a}$$

$$\left(\text{or} \quad Q_{z't'} = \frac{v}{2} Q_{x'x'} - v \tan\theta \, Q_{x'z'} \quad \text{for} \quad d = \sec\theta \right) \tag{2-17b}$$

while replacing (2-13), (2-15) and (2-16) into (2-11) will produce:

$$Q_{z't'} = \frac{v}{2} \sec^3\theta \, Q_{x'x'} - v \sec\theta \tan\theta \, Q_{x'z'} \tag{2-18a}$$

$$\left(\text{or} \quad Q_{z't'} = \frac{v}{2} Q_{x'x'} - v \sec\theta \tan\theta \, Q_{x'z'} \quad \text{for} \quad d = \sec^3\theta \right) . \tag{2-18b}$$

<u>c ≠ 0</u>

If now we assume that (2-12d) holds, then this relation plus the definition of e and f in (2-13) allows us to define c and d in various ways, among which we may choose:

$$c = 2 \sin\theta \cos\theta \tag{2-19a}$$

$$d = \cos^2\theta - \sin^2\theta \tag{2-19b}$$

or

$$c = 2 \tan\theta \tag{2-20a}$$

$$d = 1 - \tan^2\theta \tag{2-20b}$$

In this case we are left with four possible combinations: Replacing (2-13),(2-14) and (2-19) into (2-11) we get:

$$Q_{z't'} = \frac{v}{2} \sec\theta \, Q_{x'x'} + v \tan\theta \, Q_{x'z'} \tag{2-21}$$

Replacing (2-13), (2-15) and (2-19):

$$Q_{z't'} = \frac{v}{2} \sec^3\theta \, Q_{x'x'} + v \sec\theta \tan\theta \, Q_{x'z'} \tag{2-22}$$

Replacement of (2-13), (2-14) and (2-20) produces:

$$Q_{z't'} = \frac{v}{2} \cos\theta \, Q_{x'x'} + v \tan\theta \, Q_{x'z'} \tag{2-23}$$

and, finally, replacing (2-13), (2-15) and (2-20):

$$Q_{z't'} = \frac{v}{2} \sec\theta \, Q_{x'x'} + v \sec\theta \tan\theta \, Q_{x'z'} \; . \tag{2-24}$$

As we might see, a very superficial consideration has already produced us six different equations for our particular problem. If now we replace for each of these solutions the chosen values of a, b, c, d, e and f into the inverse transformation (2-8) and transform back these applied equations to the original frame x, z, t, all of them will transform into exactly the same equation:

$$[(1-\tan^2\theta)\partial_{xx} - \frac{1}{v^2}(1+\sec^2\theta)\partial_{tt} - 2\tan\theta\partial_{xz} - \frac{2}{v}\sec\theta\tan\theta \, \partial_{xt} - \frac{2}{v}\sec\theta \, \partial_{zt}] P = 0 \; . \tag{2-25}$$

As we shall show later in a simpler case, this behavior is probably due to the fact that, seemingly all the coefficients of equation (12-25) can be expressed in terms of e and f only, which are the same for all the six considered equations [(2-17) through (2-24)]. In other words, it could be said also that the neglected term $Q_{z'z'}$ transforms back equally for all these six equations. But (2-25) is the already well known equation corresponding to the hyperbolic approximation, extensively discussed in our paper of SEP, March 1974 ("Dispersion relationship for the slant frames approximation"). What this says is that all the found solutions fit equally well the circle in the dispersion plane $k_x k_z$.

3. The generalization to layered media

In order to illustrate how we can generalize any of the obtained equations to include stratified media, we will consider only the first two equations (2-17) and (2-18). Replacing (2-13), (2-14) and (2-16) into (2-2), we will then have the direct transformation corresponding to the first of these equations for a velocity-constant model:

$$x' = x \cos\theta - z \sin\theta \qquad (3\text{-}1a)$$

$$z' = z \quad (\text{or} \quad z' = z \sec\theta \quad \text{for} \quad d = \sec\theta\ ) \qquad (3\text{-}1b)$$

$$t' = - x \sin\theta/v - z \cos\theta/v + t' \qquad (3\text{-}1c)$$

Now let's assume that, instead, we have a layered medium [$v = v(z)$] , but that all the rays leave the surface at the same angle θ_i , so that the ray parameter $p = \sin\theta_i/v_i$ remains constant. Then, recalling that

$$\sin\theta(z) = pv(z) \tag{3-2a}$$

$$\cos\theta(z) = [1 - (pv(z))^2]^{1/2} \tag{3-2b}$$

and considering a differential "cake" model of the type we had at the beginning of our two previous papers on layered media, it is not difficult to obtain the proper generalization for (3-1) (or any other transformation of this type):

$$x' = [1 - (pv(z))^2]^{1/2}\, x - p\int_0^z v(w)\, dw \tag{3-3a}$$

$$z' = z \quad \left(\text{or } z' = \int_0^z [1 - (pv(w))^2]^{1/2} dw \quad \text{for } d=\sec\theta \right) \tag{3-3b}$$

$$t' = -px - \int_0^z \frac{1}{v(w)}[1 - (pv(w))^2]^{1/2}\, dw + t \tag{3-3c}$$

To show that this is the right transformation notice that the corresponding Jacobian is

$$x'_{x,z,t} = [1 - (pv(z))^2]^{1/2} \; ; \; -pv(z) \; ; \; 0 \tag{3-4a}$$

$$z'_{x,z,t} = 0 \; ; \; 1 \; ; \; 0 \quad (\text{or} \quad 0 \; ; \; [1 - (pv(z))^2]^{-1/2} \; ; \; 0 \;) \tag{3-4b}$$

$$t'_{x,z,t} = -p \; ; \; -\frac{1}{v(z)}[1 - (pv(z))^2]^{1/2} \; ; \; 1 \quad . \tag{3-4c}$$

Hence, the coefficients of the transformed equations are:

$$Q_{x'x'} : \quad x'^2_x + x'^2_z - x'^2_t/v^2 = 1 - (pv)^2 + (pv)^2 = 1 \tag{3-5a}$$

$$Q_{z'z'} : \quad z'^2_x + z'^2_{z'} - z'^2_t/v^2 = 1 \quad (\text{ or } [1 - (pv)^2]^{-1}) \tag{3-5b}$$

$$Q_{t't'} : \quad t'^2_2 + t'^2_z - \frac{t'^2_t}{v^2} = p^2 + \frac{1 - (pv)^2}{v^2} - \frac{1}{v^2} = 0 \tag{3-5c}$$

$$Q_{x'z'} : \quad 2(x'_x z'_x + x'_z z'_z - x'_t z'_t/v^2) = -2pv \quad (\text{or } -2pv[1 - (pv)^2]^{-1/2}) \tag{3-5d}$$

$$Q_{x't'} : 2(x'_x\, t'_x + x'_z\, t'_z - x'_t\, t'_t / v^2) = 2\{-p[1-(pv)^2]^{1/2} + + p[1-(pv)^2]^{1/2}\} = 0 \qquad (3\text{-}5e)$$

$$Q_{z't'} : 2(z'_x\, t'_x + z'_z\, t'_z - z'_t\, t'_t / v^2) = -\frac{2}{v}[1-(pv)^2]^{1/2} \quad (\text{ or } -\frac{2}{v}) \qquad (3\text{-}5f)$$

The final transformed equation is then going to be:

$$Q_{x'x'} + Q_{z'z'} - 2\, pv(z)\, Q_{x'z'} - \frac{2}{v}[1-(pv(z))^2]^{1/2} Q_{z't'} = 0 \qquad (3\text{-}6)$$

Or, written in the more familiar way, after neglecting the $Q_{z'z'}$ term:

$$Q_{z't'} \cong \frac{v(z)}{2}\, [1-(pv(z))^2]^{-1/2} Q_{x'x'} - pv^2(z)[1-(pv(z))^2]^{-1/2} Q_{x'z'} \qquad (3\text{-}7a)$$

$$(\text{ or } Q_{z't'} \cong \frac{v(z)}{2} Q_{x'x'} - pv^2(z)[1-(pv(z))^2]^{-1/2} Q_{x'z'} \text{ for } d=\sec\theta\ ) \qquad (3\text{-}7b)$$

Using relations (3-2), this equation could be written rather in terms of angles and we would obtain equation (2-17):

$$Q_{z't'} \cong \frac{v(z)}{2} \sec\theta(z)\, Q_{x'x'} - v(z)\, \tan\theta(z)\, Q_{x'z'} \quad . \qquad (3\text{-}8a)$$

$$(\text{or } Q_{z't'} \cong \frac{v(z)}{2} Q_{x'x'} - v(z)\, \tan\theta(z)\, Q_{x'z'}\ ) \qquad (3\text{-}8b)$$

We could repeat exactly the same procedure in relation to any other of the considered equations. Thus, if for example, we replace (2-13), (2-15) and (2-16) in (2-2), we would obtain the transformation corresponding to equation (2-18):

$$x' = x - z \tan\theta \qquad (3\text{-}9a)$$

$$z' = z \qquad (3\text{-}9b)$$

$$t' = x \sin\theta/v - z \cos\theta/v + t' \quad . \qquad (3\text{-}9c)$$

Therefore, using again (3-2), the obvious generalization to stratified media would be:

$$x' = x - p\int_0^z v(w)[1-(pv(w))^2]^{-1/2}\,dw \qquad (3\text{-}10a)$$

$$z' = z \qquad (3\text{-}10b)$$

$$t' = -px - \int_0^z \frac{1}{v(w)}[1-(pv(w))^2]^{1/2}\,dw + t \qquad (3\text{-}10c)$$

(Compare with result (1-3) and the corresponding discussion).

If in this case we compute the Jacobian as well as the coefficients of the transformed equation, we would end rather with (after dropping the $Q_{z'z'}$ term):

$$Q_{z't'} = \frac{v(z)}{2}[1-(pv(z))^2]^{-3/2}Q_{x'x'} - \frac{p\,v(z)^2}{1-(pv(z))^2}Q_{x'z'}\ , \qquad (3\text{-}11)$$

which in terms of angles is exactly equation (2-18):

$$Q_{z't'} = \frac{v(z)}{2}\sec^3\theta(z)\ Q_{x'x'} - v(z)\sec\theta(z)\tan\theta(z)\ Q_{x'z'}\ . \qquad (3\text{-}12)$$

4. Significance of the presence of the $Q_{x'z'}$ term

As we pointed out before, the price that we have to pay for suppressing t in the transformation of the spatial coordinates, is the presence in the transformed equations of a $Q_{x'z'}$ term, which can by no means be avoided. I would like here to partially evaluate how high this price can be. The first thing to notice is that, computationally speaking, the presence of this term doesn't represent any special difficulty. As a matter of fact, we have already computed it before in other problems. Economically speaking, it does represent a difference since, to start with, we can no longer use an explicit scheme. As shown previously, in computer time this means, at least, a factor of 2 .

On the other hand, studying a different situation where this term also arises, Steve Doherty has shown that its presence doesn't seem to be of great significance within the problems considered by him. I will focus attention on the effect that the suppression of this term has in fitting the circle (exact solution) in the dispersion plane $k_x k_z$. In doing so, let's start by recalling that, out of the six obtained equations, only the first two (c=0) allow such an approximation. If we try to neglect the $Q_{x't'}$ in the last four equations ($c \neq 0$), we would come out with non-vanishing P_{zz} terms when transforming them back to the original frame. Thus, we will have to restrict our further analysis only to equations (2-17) and (2-18).

Neglecting both the $Q_{z'z'}$ and the $Q_{x'z'}$ terms and assuming that c=0 , the transformed equation (2-5) becomes:

$$(a^2 + b^2)\, Q_{x'x'} + 2\, d\, f\, Q_{z't'} = 0 \quad . \tag{4-1}$$

Replacing c=0 into (2-9), the Jacobian corresponding to the related inverse transformation is then:

$$x_{x',z',t'} = \frac{1}{a} \; ; \; -\frac{b}{ad} \; ; \; 0 \tag{4-2a}$$

$$z_{x',z',t'} = 0 \; ; \; \frac{1}{d} \; ; \; 0 \tag{4-2b}$$

$$t_{x',z',t'} = -\frac{e}{a} \; ; \; \frac{be-af}{ad} \; ; \; 1 \quad . \tag{4-2c}$$

Now let's transform back equation (4-1) to the original frame:

$$[\ (a^2+b^2)(x_{x'}\partial_x + t_{x'}\partial_t)^2 + 2df(x_{z'}\partial_x + z_{z'}\partial_z + t_{z'}\partial_t)\partial_t\ ]\ P = 0 \qquad (4\text{-}3)$$

Further we will compute each coefficient separately and, with the help of relations (2-12), we will express the results in terms of e and f :

$$P_{xx} : (a^2 + b^2)x_{x'}^2 = (a^2 + b^2)/a^2 = 1 + (e/f)^2 = 1/(fv)^2 \qquad (4\text{-}4a)$$

$$P_{zz} : 0 \qquad (4\text{-}4b)$$

$$P_{tt} : (a^2+b^2)t_{x'}^2 + 2dft_{z'} = e^2(a^2+b^2/a^2 + 2df(be-af)/ad = [(e/f)^2 - 2]/v^2 \qquad (4\text{-}4c)$$

$$P_{xz} : 0 \qquad (4\text{-}4d)$$

$$P_{xt} : 2(a^2+b^2)x_{x'}t_{x'} + 2df\,x_{z'} = -2e(a^2+b^2)/a^2 - 2bdf/ad = 2e(e/f)^2 \qquad (4\text{-}4e)$$

$$P_{zt} : 2\,d\,f\,z_{z'} = 2\,f \qquad (4\text{-}4f)$$

Placing all these expressions in equation (4-3) we obtain:

$$\left\{ \frac{1}{(fv)^2}\,\partial_{xx} + \frac{1}{v^2}[(\frac{e}{f})^2 - 2]\partial_{tt} - 2e(\frac{e}{f})^2\,\partial_{xt} + 2f\,\partial_{zt} \right\} P = 0 \qquad (4\text{-}5)$$

Notice that all the coefficients of this equation depend only on e and f. As mentioned before, this may explain why we get the same result in all the different cases previously studied. If now we use relations (2-13) for e and f , we finally get:

$$[\ \sec^2\theta\ \partial_{xx} + \frac{1}{v^2}(\tan^2\theta - 2)\partial_{tt} + \frac{1}{v}\sin\theta\tan^2\theta\ \partial_{xt} - \frac{2}{v}\cos\theta\ \partial_{zt}\ ]\ P = 0\ . \qquad (4\text{-}6)$$

To get the corresponding dispersion relation, we now insert the trial solution for a plane wave of unit magnitude:

$$P = \exp(i k_x x + i k_z z - iwt) . \qquad (4-7)$$

This produces the dispersion equation:

$$k_x^2 - 2 m \sin^3\theta k_x + 2 m \cos^3\theta k_z +(1 - 3 \cos^2\theta)m^2 = 0 . \qquad (4-8)$$

By comparison with our previous hyperbolic approximation, this equation corresponds simply to a parabola, with its axis being parallel to k_z (as we might have already expected, since the function has to be singly defined). Let's briefly review the two extreme cases when $\theta=0°$ and $\theta=90°$.

If $\theta=0°$, equation (4-8) becomes:

$$k_x^2 + 2 m k_z - 2 m^2 = 0 , \qquad (4-9)$$

If $\theta=90°$, we get instead:

$$k_x = m , \qquad (4-10)$$

which again coincides with the result previously obtained when considering the hyperbolic approximation. In between ($0° < \theta < 90°$) , equation (4-8) produces, as said before, a parabola that I computed and plotted for several cases ($\theta=15°$, 45° and 75°). As we did before, for reference, all the plots include the exact solution (circle). The relative error in fitting the circle is also defined through the absolute value of the difference between the radius-vectors of the circle and the approximation (parabola or hyperbola) for a fixed direction of propagation. In order to compare better both approximations in this region, look at Table 1, where I compare the range of angles around the main direction of propagation for which this relative error doesn't exceed 5% :

θ	$\Delta\theta$ Hyperb. app.	$\Delta\theta$ Parab. app.
15°	47°	40°
45°	42°	24°
75°	23°	8°

TABLE 1

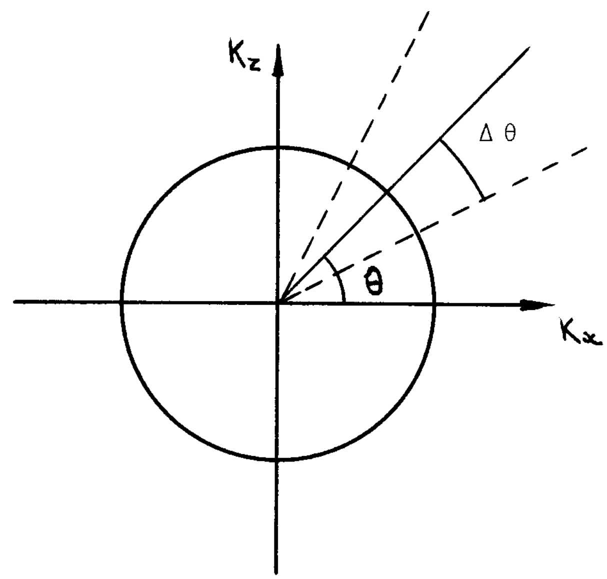


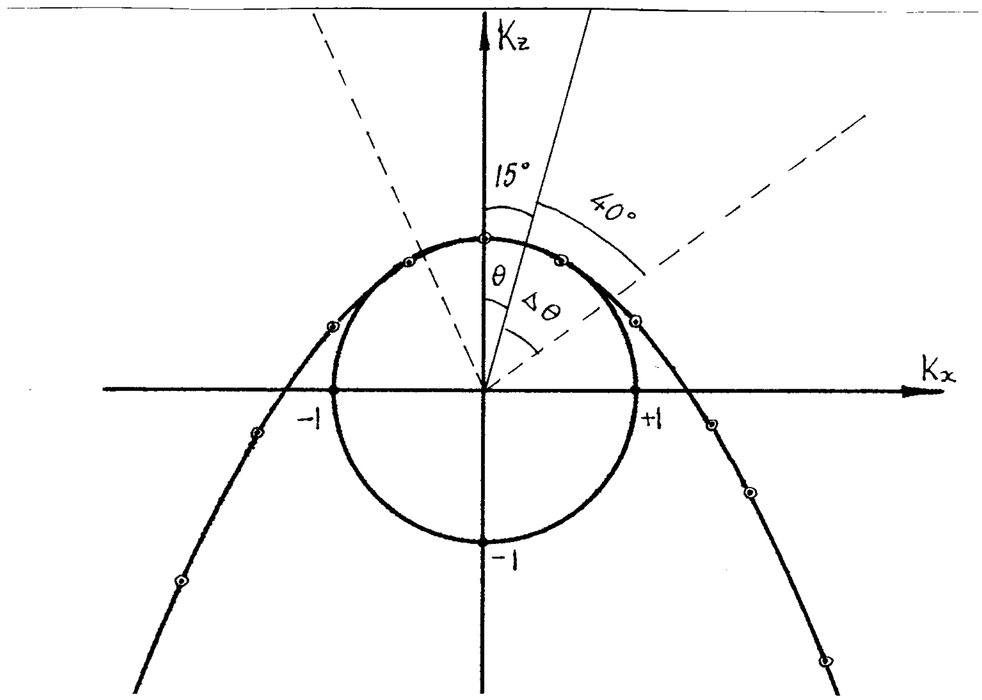
FIG. 1

As we might see from Table 1 and Figures 2 through 7, for small angles (up to 30° app.) both approximations behave more or less the same, but as the angle increases, the situation deteriorates fast with the parabolic approximation. We shall remember at this point that in case it was necessary, we could improve considerably these values by evaluating $Q_{z'z'}$ and $Q_{x'z'}$ instead of neglecting them. This technique has been extensively described in several papers of our previous reports.

Finally, I would like to call attention to the fact that the existence of this approximation with an internal to the circle parabola, suggests the existence of a similar approximation, but with the parabola external to the circle. It is not difficult to realize that such an approximation should work better when moving toward steeper angles. Without trying very hard, I couldn't get it within the frame of the discussed transformations (2-2). What I suspect is that this approximation belongs rather to the class of transformations where t enters explicitly in the transformation of the spatial coordinates.

DISPERSION RELATION FOR M=1 AND DIP=15 DEG.

PARABOLIC APPROXIMATION



HYPERBOLIC APPROXIMATION

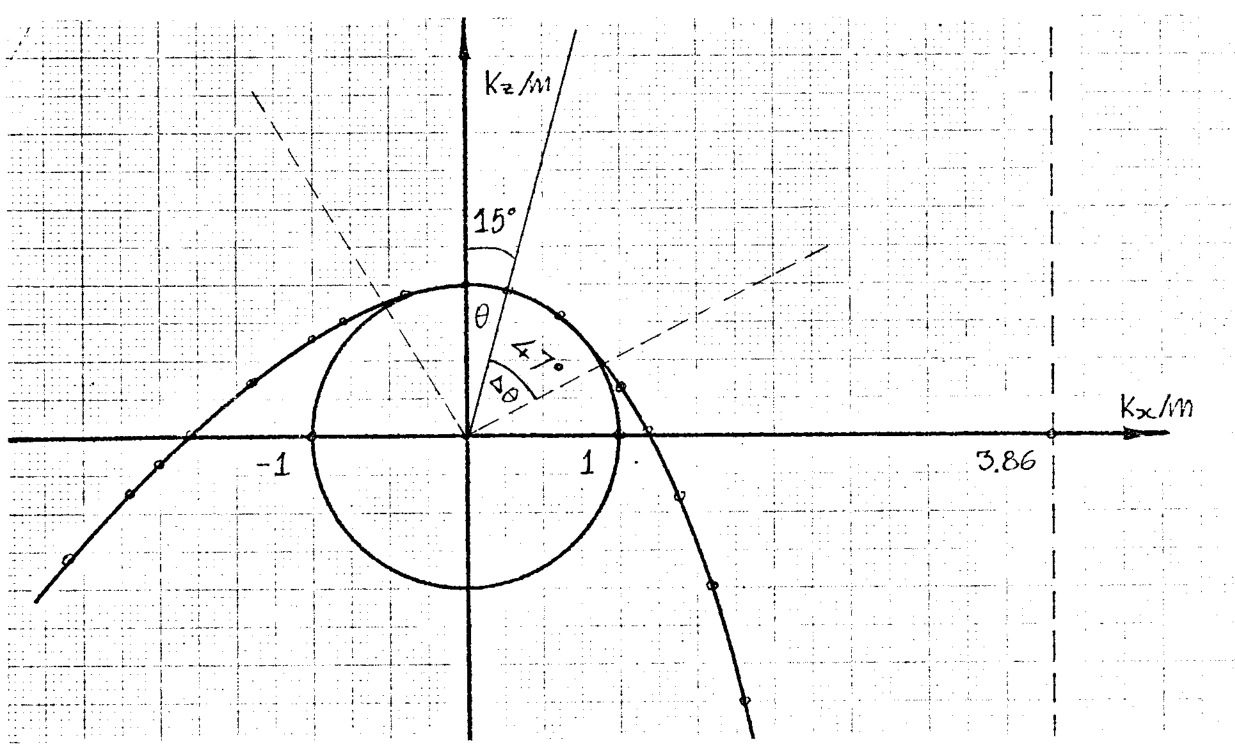
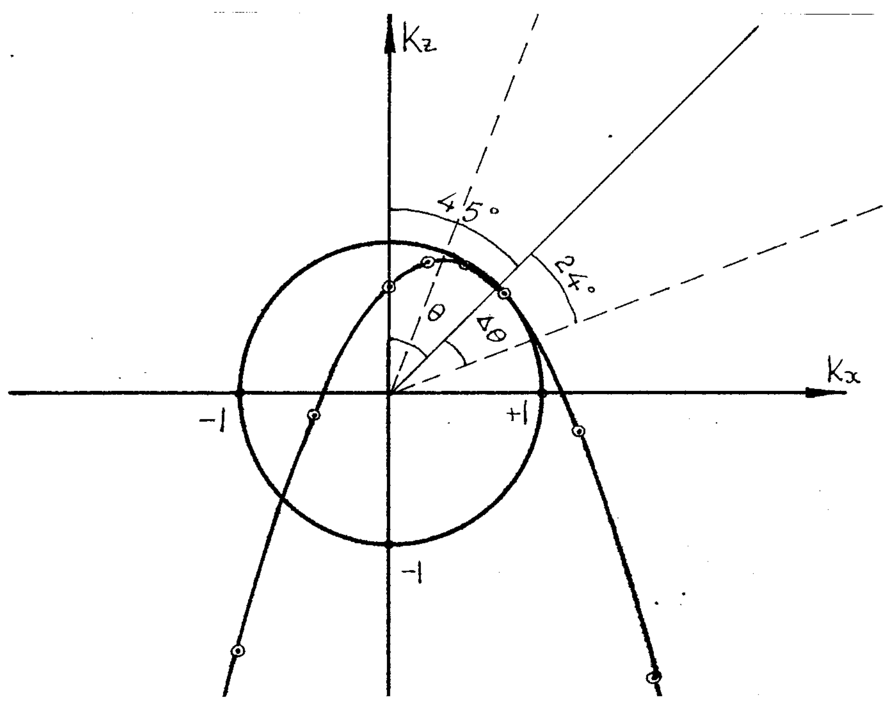


FIG. 3

DISPERSION RELATION FOR M=1 AND DIP=45 DEG .

PARABOLIC APPROXIMATION



HYPERBOLIC APPROXIMATION

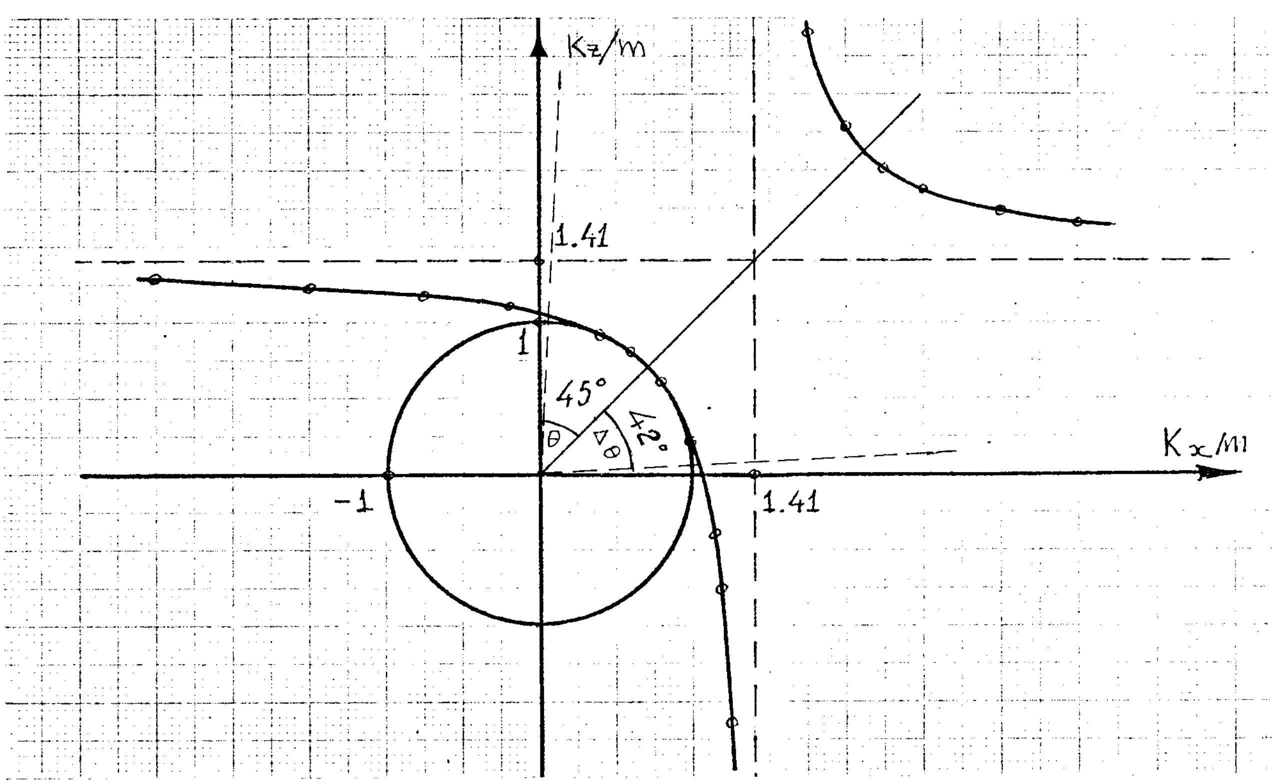
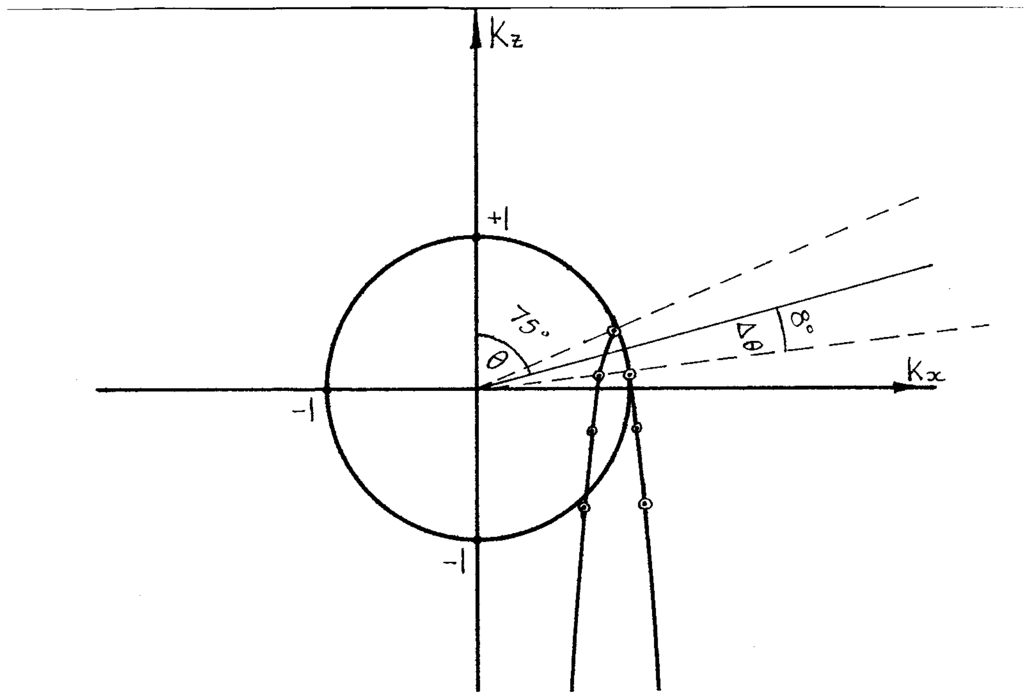


FIG. 4

DISPERSION RELATION FOR M=1 AND DIP=75 DEG.

PARABOLIC APPROXIMATION



HYPERBOLIC APPROXIMATION

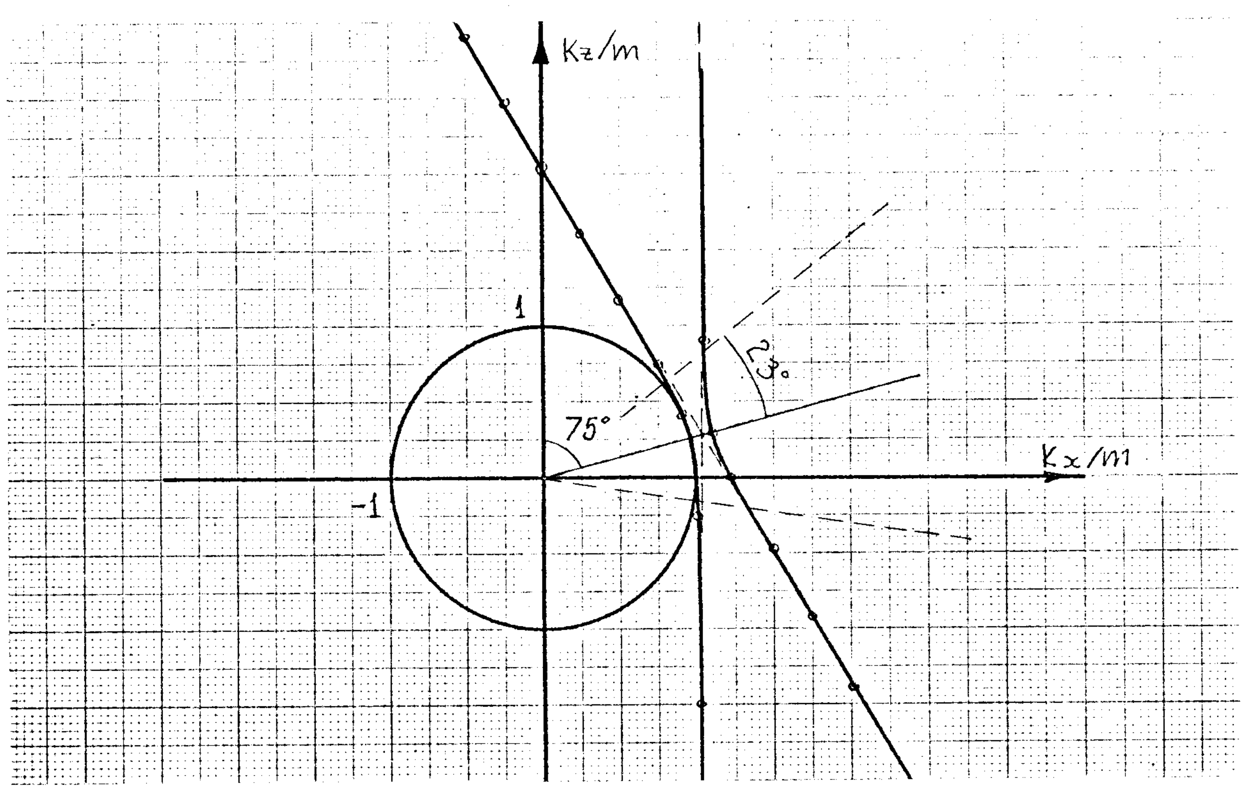


FIG. 5