Non-Seismic Prospecting Methods

We have continued our studies in non-seismic methods. Magneto-tellurics and resistivity surveys are not only valid exploration techniques by themselves but they also provide slanted views of the problems of seismic data analysis. For example we had problems fully grasping all the implications of downward continuation of seismic sources. This led us to consider the more concrete example of an electrical resister mesh being probed in the fashion of exploration surveys.

J.F.C. 24 September 1974

Final Report on Magnetotelluric Field Extrapolation

by Robert H. Brune

I. Introduction

The use of first order techniques for extrapolation of the wave equation involves several approximations. Here we will consider the effect of some of these approximations when extrapolating electromagnetic fields in media with strong contrasts in conductivity.

As a particular case of electromagnetic propagation in the earth, we will consider transverse magnetic propagation in a conductive two dimensional half space. The horizontal magnetic component is taken parallel to strike. If we neglect displacement currents, then from Maxwell's equations we have:

$$\left(\frac{1}{\sigma} B^y_z \right)_z + \left(\frac{1}{\sigma} B^y_x \right)_x = -i\omega\mu \; B^y \tag{1}$$

where subscripts denote partial differentiation and superscripts denote components. A geometry of some interest for which there are some analytic solutions and analog models is that of a vertical contact. This is the model considered here; its general nature is indicated by the sketches of figure 1.

In geophysical practice, impedances or apparent resistivities are the parameters used to describe the results of measurements. Apparent resistivity is calculated from the wave field as:

$$\rho_a = \frac{1}{\omega\mu\sigma^2} \left| \frac{B^y_z}{B^y} \right|^2_{z=o} \tag{2}$$

We use a two point star (z=0 and z=1) to estimate both numerator and denominator.

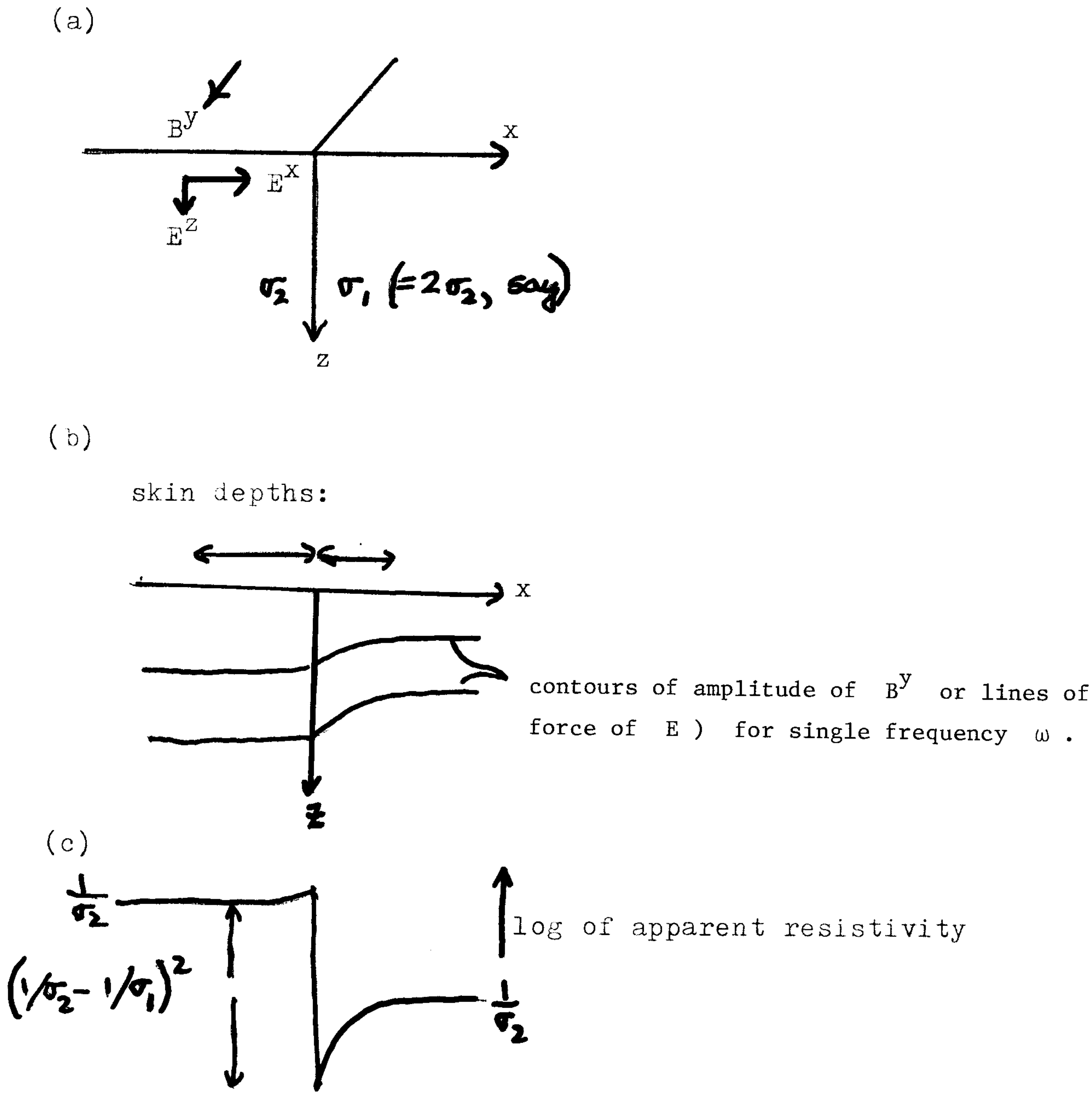

Figure 1. Transverse magnetic mode in two dimensional earth with vertical contact. At the earth's surface, H^y is taken constant, since $\partial H^y / \partial x = \partial H^y / \partial z = 0$ in the non-conducting region above the earth's surface. (This is not so for the T.E. case since $\sigma = 0$ does not guarantee the invariance of E^y above the earth's surface).

II. Up and Downgoing Waves

The second order equation (1) has both upgoing and downgoing solutions. Our first order extrapolation has only one solution, downgoing, when we start at the earth's surface.

For the transverse magnetic mode we can eliminate the vertical component of E to get the matrizant equation:

$$\partial_z \begin{bmatrix} E^x \\ H^y \end{bmatrix} = \begin{bmatrix} 0 & \partial_x(\frac{1}{\sigma}\partial_x) + i\omega\mu \\ -\sigma & 0 \end{bmatrix} \begin{bmatrix} E^x \\ H^y \end{bmatrix} \tag{3}$$

We want to diagonalize the matrix to get an equation in two uncoupled quantities that we wish to represent upgoing and downgoing waves. If we assume no variation with x, then we can get a diagonalized equation in two new variables, D, U:

$$\begin{bmatrix} D \\ U \end{bmatrix} = \begin{bmatrix} \frac{1}{2\sqrt{i\omega\mu}} & \frac{1}{2i\sqrt{\sigma}} \\ \frac{-1}{2\sqrt{i\omega\mu}} & \frac{1}{2i\sqrt{\sigma}} \end{bmatrix} \begin{bmatrix} E^x \\ H^y \end{bmatrix}$$

From Maxwell's equations:

$$E^x = -\frac{1}{\sigma} H^y_z$$

We further simplify things by considering σ constant, so that:

$$H^y_z = i\sqrt{i\omega\mu\sigma}\; H^y$$

Thus we get for homogeneous media:

$$D = -i\; H^y / \sqrt{\sigma} \tag{4}$$

The real case of interest is when the matrizant equation (3) involves ∂_x operations. But when we start to diagonalize

$$\begin{bmatrix} 0 & \partial_x(\frac{1}{\sigma}\partial_x) + i\omega\mu \\ -\sigma & 0 \end{bmatrix}$$

we note that the two off diagonal elements don't commute; in the acoustics case they do. I don't see a way around this seemingly fundamental problem. We want to deal with strong x variations so it doesn't seem wise to neglect the commutivity problem. Additionally, it is not clear just what equation (4) means for our extrapolation. Can we put H^y on the grid at $z=o$ and extrapolate it with the first order methods, or is it actually some other quantity (like D of (4)) that can be extrapolated?

III. Extrapolation Techniques

Let's briefly compare some approaches to the T.M. problem. The physics is given by equation (1) with appropriate boundary conditions. Perturbation about $e^{i\bar{m}z}$ gives:

$$2 i \bar{m} Q_z + Q_{xx} - (\sigma_x / \sigma) Q_x + (m^2 - \bar{m}^2) Q = 0 \quad (5)$$

We can then substitute $Q = B e^{-i\bar{m}z}$; if we then set $\bar{m}$ to m ("strong contrast" techniques) we get:

$$2 i m B_z + B_{xx} - (\sigma_x / \sigma) B_x + 2 m^2 B = 0$$

or since: $m = \sqrt{i \omega \mu} \sqrt{\sigma}$

$$B_z - (i / 2m) B_{xx} + (i m_x / m^2) B_x - i m B = 0 \quad (6)$$

The essence of wanting to extrapolate second order equations by use of first order methods is to find the square root of an operator. For the wave equation with material properties that are functions of x only, one such square root approximation is (Claerbout, 1971):

$$- i \partial_z \approx m + (1 / 2m) \partial_{xx} - (m_x / 2m^2) \partial_x \equiv 0p \quad (7)$$

We then get $(i 0p = \partial_z)$:

$$B_z - (i / 2m) B_{xx} + (1/2)(i m_x / m^2) B_x - i m B = 0 \quad (8)$$

Note that the only difference between equations (6) and (8) is the 1/2 in the m_x term of (8) . These terms are like secondary sources at the changes in conductivity. The 1/2 might be interpreted as an obliquity factor for the sources; a value of 1/2 makes some intuitive sense since we are dealing with one way extrapolation.

We are trying to get a dispersion relation for one directional propagation that is a best approximation to the second order dispersion relation. The operator (8) or the inhomogeneous equation (6) each add another term to the usual parabolic approximation; this extra term gives an asymmetric dispersion relation. The dispersion relations are of course functions of conductivity.

The usual approach taken in the "45 degree equation" development is to estimate Q_{zz} from Q_z as in equation (5) and plug back in for the dropped Q_{zz} term. This was done here for equation (5), transformed back to B , and then $\bar{m}$ was set to m . This gives equation (7), p.200, SEP report, March, 1974. Unfortunately the results were no better than those from equation (6) here, and in some cases, worse.

Figure 2 shows an extrapolated field using equation (8). Also shown is the apparent resistivity from equation (2). These are not correct.

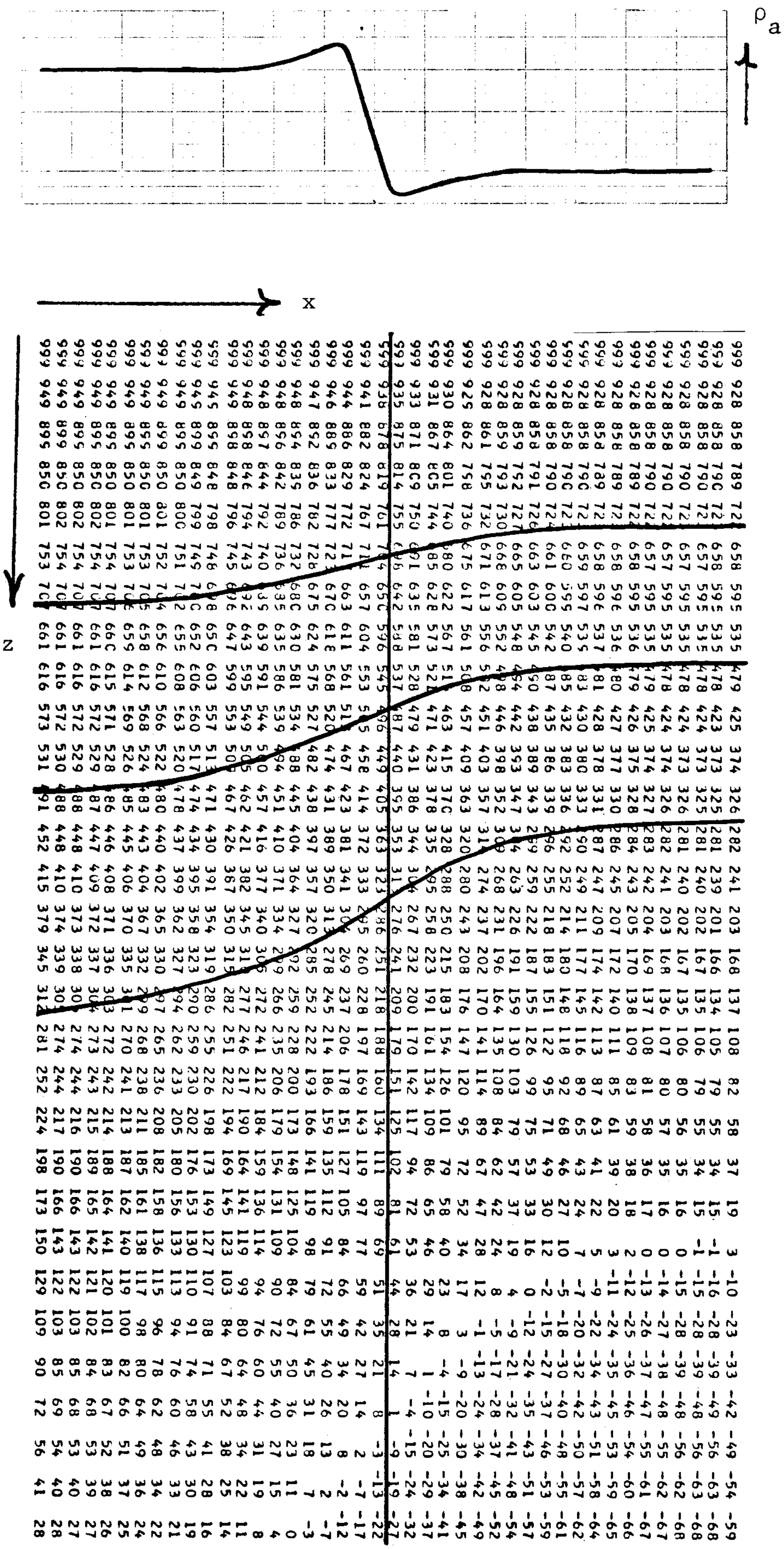

Figure 2. Use of first order extrapolation as in equation (8) gives this wave field. Compare with figure 1(b). $\Delta x = \Delta z$ in this plot, but the calculation was done with $\Delta x = .2$. Conductivity was varied with a cos over a range of 4.0 in x rather than with a step. Conductivity on the right is .01; on the left, .005 . Angular frequency, ω , = 1.0 ; magnetic permeability, μ , = 1.0. Skin depth on the right is 14.14; on the left, 20.0 units.

IV. Relaxation

The correct version of figure 2 can be calculated from equation (1) by relaxation. The results are given in figure 3. Figure 4 indicates the differences between figures 2 and 3; it is the error in the first order extrapolation.

The second order equation (1) was solved by Gauss-Seidel technique with a constant value at the top edge, known 1 dimensional solutions on the sides, and zero slope on the bottom (where the field is attenuated to small values). The theory to determine convergence of successive overrelaxation does not apply here: it requires a symmetric matrix. Several ad hoc trials indicated that it converges slowly, if at all. Gauss-Seidel, however, converges nicely, albeit slowly. The program to generate figures 2-4 is given at the end of this report. It is of interest that Jones and Price (1970) get similar results using Gauss-Seidel with the homogeneous version of equation (1).

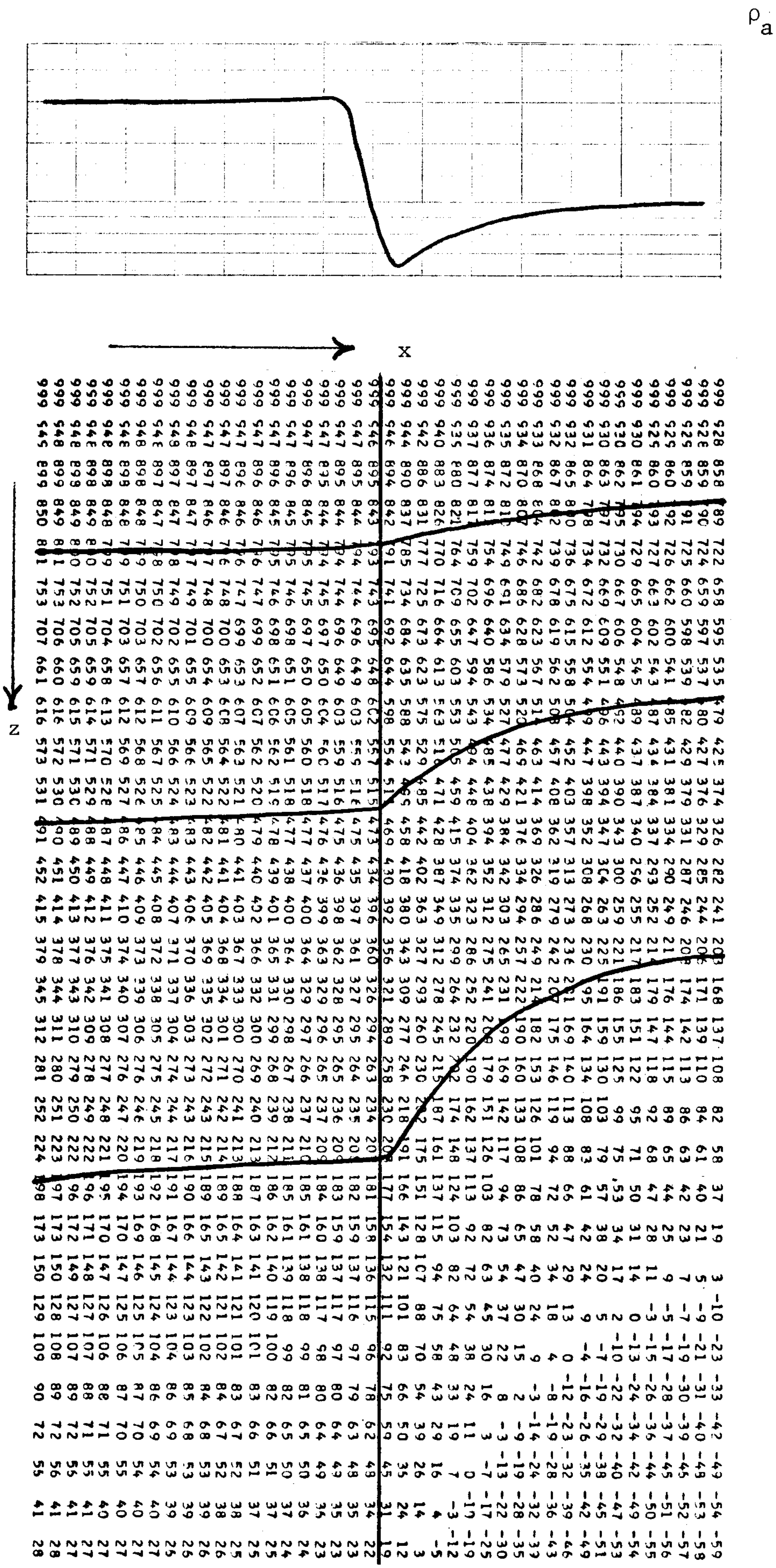

Figure 3. This is the same physical problem as in figure 2, but using a Gauss-Seidel relaxation scheme to solve equation (1). This plot is after 140 iterations, using figure 2 as an initial solution.

$\Delta x = \Delta z = 1$

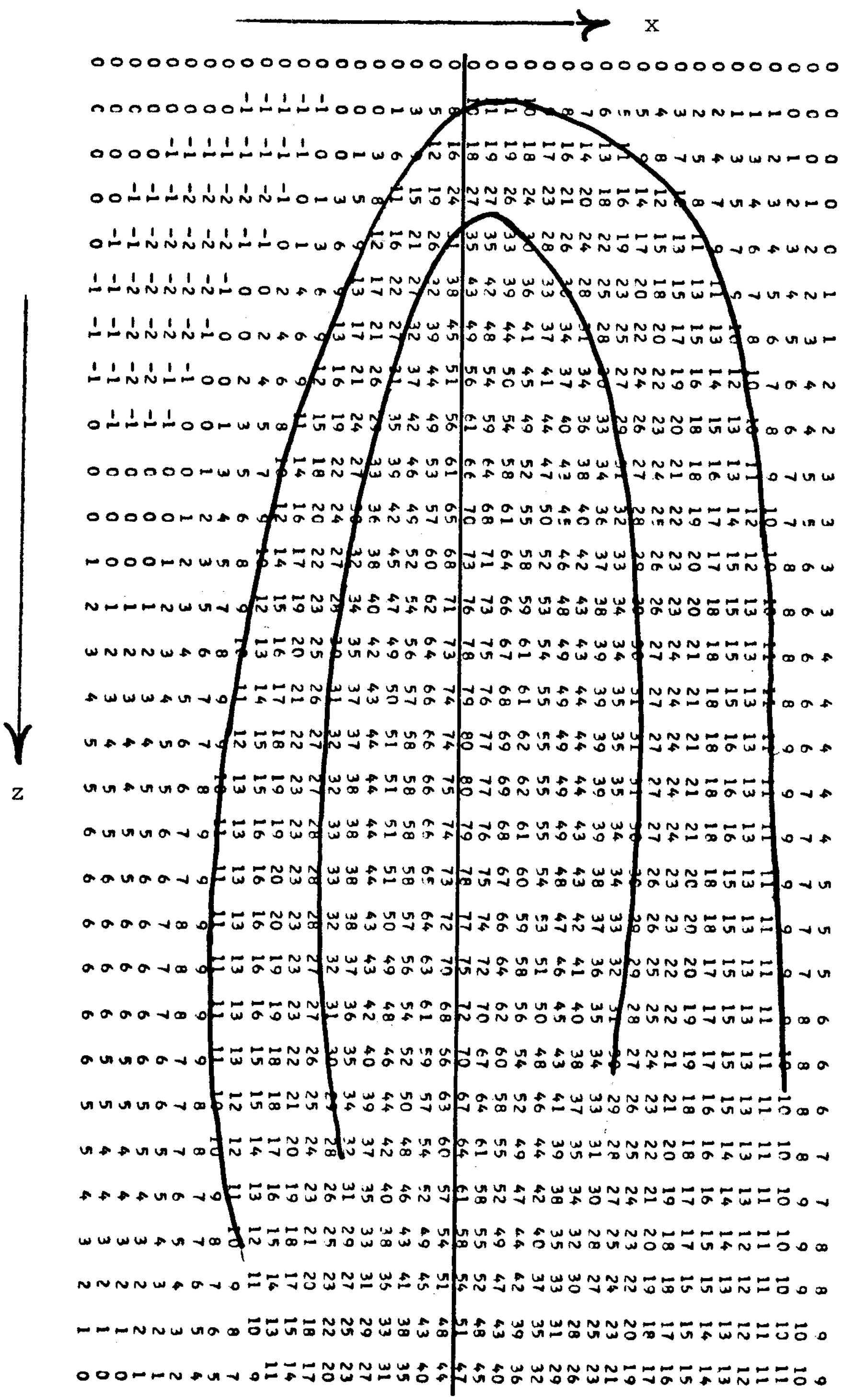

Figure 4. The difference between first order extrapolation, equation (8), and iterative solution of the second order equation (1) is displayed here. The errors are, obviously enough, systematic and related to the conductivity contrast.

V. Improved Operators

The operator of equation (7) is an approximate square root of the ∂_{zz} operation in the homogeneous version of equation (1):

$$- \partial_{zz} B = (m^2 + \partial_{xx}) B$$

Thus:

$$- \partial_{zz} \approx Op^2 = (m^2 + \partial_{xx}) + \text{Error terms}$$

The error terms are like (left hand column):

(1)	$(m_{xx} / 2m) B$	1
(2)	$- (m_x^2 / 2 m^2) B$	1
(3)	$- (2 m_x^2 / m^5) B_x$	$\sin\theta / m^2$
(4)	$- (m_{xxx} / 4 m^3) B_x$	$\sin\theta / m$
(5)	$(3 m_{xx} m_x / 2 m^4) B_x$	$\sin\theta / m$
(6)	$(m_{xx} / 4 m^4) B_x$	$\sin\theta / m^2$
(7)	$(2 m_x^2 / m^4) B_{xx}$	$\sin^2\theta$
(8)	$- (3 m_{xx} / 4 m^3) B_{xx}$	$\sin^2\theta$
(9)	$- (m_x / m^3) B_{xxx}$	$m \sin^3\theta$
(10)	$(1 / 4 m^2) B_{xxxx}$	$m^2 \sin^4\theta$

Now, what's the point of writing these terms out? The iterative development of a "better" operator could proceed by using these terms; we want to examine which are significant. The right hand column on the preceding page shows the significance of the error terms. We have first assumed m_x is of the order of m ; we note that an x derivative of B may be represented by the wave number k_x and that in turn k_x / m is like $\sin\theta$ where θ is the angle off vertical of the propagation. The usual optics, ray theoretical, high frequency approach would say to drop all but the first two error terms above.

The above error terms were calculated on the wave field of figure 2 for the first several z levels. In fact, the third and sixth terms were largest. The reason is that instead of having large m , as for high frequencies, we have $|m| < 1$.

With the above thoughts in mind concerning the significance of these error terms, we will consider how a "better" operator might be developed. For example, let a first guess at an operator be:

$$O\,p_1 = m$$

then:

$$O\,p_1^2 = \underbrace{(m^2 + \partial_{xx})}_{\text{wave eq.}} - \underbrace{(\partial_{xx})}_{\text{error, } E_1}$$

From the i^{th} operator, we can get an $i + 1$ operator that includes the error terms of the i^{th} operator; we write:

$$O\,p_{i+1}^2 = (O\,p_i)^2 \,(1 - E_i / O\,p_i^2) \qquad (9)$$

If we assume the commutivity of the product on the right hand side, then we can use the approximation $\sqrt{1+b} \approx 1+b/2$ for the right most term:

$$0\,p_{i+1} = 0\,p_i - E_1 / (2\,0\,p_i)$$

Thus, the second operator is:

$$0\,p_2 = m + \partial_{xx} / 2\,m$$

If we now square this second operator and collect the error terms as before:

$$0\,p_2^2 = (m^2 + \partial_{xx}) + E_2$$

The next "better" operator can then be written (using equation (9)):

$$0\,p_3 = (m + \partial_{xx} / 2\,m) - E_2 / (2\,m + \partial_{xx} / m)$$

The operator of equation (7) then turns out to be $0\,p_3$ if we make these assumptions: assume only $2\,m$ in the denominator of the right-most term is significant, drop terms that are like $\sin^3\theta$ and $\sin^4\theta$ as insignificant for angles near vertical, and make the high frequency approximation by dropping terms that are small for large m . We then pick up only one additional term which gives us the operator of equation (7):

$$0\,p_3 = (m + \partial_{xx} / 2\,m) - (m_x\,\partial_x / 2\,m^2)$$

The development of "better" first order operators in an iterative manner has some questions to be answered before it can be carried through. In particular we need to be able to decide which terms to keep; the above doesn't answer this very decisively.

Another alternative that is possibly more straightforward is to develop successive operators in matrix form, keeping all terms. Successive operators would have bandwidths of 1, 3, 5, 9, ... This approach seemingly would circumvent the problem of deciding which error terms to include, but at added cost.

VI. Conclusions

First order extrapolation techniques have been applied to electromagnetic wave propagation in the presence of strong material contrasts. The results have been shown to be incorrect for the techniques used here, by comparison with solutions to the second order equations. The use of improvements ("45 degree equation" and iteratively developed operators) has a number of problems and/or questions.

The physical problem considered here exceeds the limits of validity for existing extrapolation techniques and new, suitable techniques have not yet been discovered.

Program listings. The main program sets up the conductivity configuration (by a call to SIGIN), then does the first order extrapolation using equation (8) and an implicit difference scheme (with a call to TR13). Finally a call to RELAXD causes desampling of the wave field, Gauss-Seidel relaxation with equation (1), and the calculation of apparent resistivities by equation (2). The desampling was necessary because of the extremely slow convergence with the finer grid.

```
      IMPLICIT COMPLEX (C)
      REAL CABS
      DIMENSION CP( 200, 50), SIG ( 200), CA ( 200), CB ( 200),
     * CC ( 200), CD ( 200), CE ( 200), CF ( 200), CM ( 200),
     * CMX ( 200), IPLOT ( 30)
      S1 = .01
      S2 = .005
      NX= 200
      NX1 = NX-1
      DX = .2
      NZ = 50
      NDISP = 30
      DZ = 1.0
      CALL SIGIN (NX,S1,S2,SIG)
      CI = CMPLX(0.0,1.0)
      CIWM = CSQRT (CI)
         DO 10 IX=1,NX
         CP(IX,1) = CMPLX(999.0,0.0)
10       CM(IX) = CIWM*SQRT(SIG(IX) )
      CMX(1) = CMPLX(0.0,0.0)
      CMX(NX) = CMPLX(0.0,0.0)
         DO 20 IX=2,NX1
20       CMX(IX) = (.5/DX)*(CM(IX+1)-CM(IX-1)  )
         DO 30 IZ=2,NZ
            DO 40 IX=1,NX
            C1 = CI*CM(IX)/2.0
            C2 = CI/(4.0*CM(IX)*DX*DX)
            C3 = CI*CMX(IX)/(8.0*CM(IX)**2*DX)
            CA(IX) = -C2-C3
            CB(IX) = 1.C/DZ-C1+C2+C2
            CC(IX) = -C2+C3
            IF(IX.NE.1 .AND. IX.NE.NX) CD(IX) = CP(IX-1,IZ-1)*(C2+C3)
     *      +CP(IX,IZ-1)*(1.0/DZ+C1-C2-C2) + CP(IX+1,IZ-1)*(C2-C3)
            IF(IX.EQ.1) CD(1) = CP(1,IZ-1)*(1.0/DZ+C1-C2-C2)
     *      + CP(IX+1,IZ-1)*(C2-C3)
40          IF(IX.EQ.NX) CD(NX) = CP(NX-1,IZ-1)*(C2+C3)
     *      + CP(NX,IZ-1)*(1.0/DZ+C1-C2-C2)
30       CALL TRI3 (CA,CB,CC,NX,CP(1,IZ),CD,CE,CF)
```

```
C
C      WRITE (6,2000)
C2000 FORMAT (1H1)
C         DO 50 IX=1,NX
C            DO 60 IZ=1,NDISP
C  60        IPLOT(IZ) = REAL(CP(IX,IZ) )
C  50     WRITE (6,1000) IPLCT
C1000 FORMAT (1X,30I4)
C      WRITE (6,2000)
C         DO 150 IX=1,NX
C            DO 160 IZ=1,NDISP
C 160        IPLOT(IZ) = CABS(CP(IX,IZ) )
C 150     WRITE (6,1000) IPLCT
C
      CALL RELAXD (S1,S2,NDISP,NX,NZ,DX,DZ,CM,SIG,CP)
      RETURN
      END

      SUBROUTINE TRI3 (A,B,C,N,T,D,E,F)
C
C     TRIDIAGONAL SOLUTION WITH ZERO SLOPE BOUNDARY CONDITIONS AND
C     NONCONSTANT ENTRIES ALONG THE DIAGONALS.
C
      COMPLEX A, B, C, T, D, E, F, DEN
      DIMENSION T ( N), D ( N), E ( N), F ( N), A ( N), B ( N), C ( N)
      N1 = N-1
      E(1) = 1.0
      F(1) = 0.0
         DO 10 I=2,N1
         DEN = B(I) + C(I)*E(I-1)
         E(I) = -A(I)/DEN
   10    F(I) = (D(I) - C(I)*F(I-1) ) / DEN
      T(N) = F(N1) / (1.0-E(N1) )
         DO 20 J=1,N1
         I = N-J
   20    T(I) = E(I)*T(I+1) + F(I)
      RETURN
      END
      SUBROUTINE SIGIN (NX,S1,S2,SIG)
      DIMENSION SIG(NX)
      PI20 = .050*3.14159265
         DO 10 IX=1,90
   10    SIG(IX) = S1
         DO 20 IX=91,110
   20    SIG(IX) = (S1+S2)*.5+(S1-S2)*.5*COS(PI20*(IX-90.5) )
         DO 30 IX=111,NX
   30    SIG(IX) = S2
      RETURN
      END
```

```
      SUBROUTINE RELAXD (S1,S2,NDISP,NX,NZ,DX,DZ,CM,SIG,CP)
C     DESAMPLE X AXIS BEFORE RELAXATION
      IMPLICIT COMPLEX(C)
      REAL CABS
      DIMENSION SIGX (  40), IPLOT (  30), CGRID (  42, 51),
     * CP ( 200, 50), CM ( 200), SIG ( 200)
     * , CRESID ( 40, 50), RHOA ( 40)
      NDES = 5
      IOFF = NDES/2
      MX = NX/NDES
      MX1 = MX+1
      MX2 = MX+2
      MXM1 = MX-1
      DXD = DX*NDES
      NZ1 = NZ+1
      A = 1.0/(DZ**2)
      B = 1.0/(DXD**2)
      D = .5/DX
      CI = CMPLX(0.0,1.0)
      CI3 = CI*CSQRT(CI)
      CIM1 = CI3*SQRT(S1)
      CIM2 = CI3*SQRT(S2)
         DO 10 IX=1,MX
            DO 10 IZ=1,NZ
            CRESID(IX,IZ) = 0.0
   10       CGRID(IX+1,IZ) = CP(IX*NDES-IOFF,IZ)
      SIGX(1) = 0.0
      SIGX(MX) = 0.0
         DO 40 IX=2,MXM1
   40    SIGX(IX) = D*(SIG(IX*NDES-IOFF+1)-SIG(IX*NDES-IOFF-1) )
     *    / SIG(IX*NDES-IOFF)
C
         DO 30 IZ=1,NZ1
         CGRID(1,IZ) = 999.0*CEXP(CIM1*(IZ-1.) )
   30    CGRID(MX2,IZ) = 999.0*CEXP(CIM2*(IZ-1.) )
      WRITE (6,1000)
         DO 20 IX=1,MX2
            DO 25 IZ = 1,NDISP
   25       IPLOT(IZ) = REAL(CGRID(IX,IZ) )
   20    WRITE (6,2000) IPLOT
C
      DO 5 I=1,140
         DO 50 IX=2,MX1
         CGRID(IX,NZ1) = CGRID(IX,NZ)
            DO 50 IZ=2,NZ
            CDUM = ( 1.0/(2.*A+2.*B-CM((IX-1)*NDES-IOFF)**2) ) *
     *      (A*CGRID(IX,IZ-1)+A*CGRID(IX,IZ+1)+(B-D*SIGX(IX-1))*
     *      CGRID(IX+1,IZ)+(B+D*SIGX(IX-1))*CGRID(IX-1,IZ) )
            CRESID(IX-1,IZ) = (CDUM-CGRID(IX,IZ))+CRESID(IX-1,IZ)
   50       CGRID(IX,IZ) = CDUM
      IF (MOD(I,140) .NE. 0) GO TO 5
      WRITE (6,1000)
 1000 FORMAT (1H1)
         DO 150 IX=1,MX2
```

```
        DO 160 IZ=1,NDISP
160     IPLOT(IZ) = REAL(COSTID(IX,IZ) )
150   WRITE(6,2000) IPLOT
2000 FORMAT (1X,20I4)
     WRITE (6,1000)
      DO 70 IX=1,NX
        DO 80 IZ=1,NDISP
80      IPLOT(IZ) = REAL(COSSTD(IX,IZ) )
70    WRITE(6,2000) IPLOT
      TITLE
      DO 100 IX=1,NX
      FDUM = ( (COSTID(IX+1,2)-COSTID(IX+1,1) ) / DZ/
     & (.5*(COSTID(IX+1,1)+COSTID(IX+1,2) ) ) ) ** 2
100    HCF(IX) = (1.0/SIN(IX*NDEG-IOFF)**2 ) * CABS(FDUM)
     WRITE (6,1000)
     WRITE (6,3000) HCF
3000 FORMAT (1X,10F12.6)

     RETURN
     END
```

Bibliography

Claerbout, J. F., 1971, Toward a Unified Theory of Reflector Mapping: Geophysics, v. 36, p. 467-481.

Jones, F. W. and A. T. Price, 1970, The Perturbation of Alternating Geomagnetic Fields by Conductivity Anomalies: Geophysical J. of R. Astr. Sco., v. 20, p. 317-334.