Migration Equations for Inhomogeneous Media

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In this paper we will consider the equation $P_z = i\left(\frac{\omega^2}{c^2} + \partial_x^{\,2}\right)^{1/2} P$, where the velocity c is a function of x. By introducing a constant $\bar{c}$ the equation can be rewritten as

$$P_z = \frac{i\omega}{\bar{c}} (1-L)^{1/2} P , \quad L = -\frac{\bar{c}^2}{\omega^2} \partial_x^{\,2} + 1 - \frac{\bar{c}^2}{c^2} . \qquad (1)$$

Now let the operator L, with zero slope boundary conditions at the endpoints of a finite interval, have the eigenvalues $\lambda_0 < \lambda_1 < \lambda_2 \ldots$ and corresponding eigenfunctions $y_0 , y_1 , y_2 , \ldots$ We recall that $\lambda_k = O(k^2)$ and that for every square integrable function f the following holds

$$\int \left(f - \sum_{k=0}^{N} \alpha_k y_k \right)^2 dx \to 0 , \text{ as } N \to +\infty$$

$$\alpha_k = \int y_k f \, dx \Big/ \int y_k^{\,2} \, dx ,$$

As is easily seen the linear operator $(1-L)^{1/2}$ can be defined by

$$(1-L)^{1/2} y_k = (1-\lambda_k)^{1/2} y_k , \quad \begin{array}{ll} (1-\lambda_k)^{1/2} \geq 0 , & \lambda_k \leq 1 \\ \operatorname{Im}(1-\lambda_k)^{1/2} > 0 , & \lambda_k > 1 \end{array}$$

Now assume $P(x,0) = y_k$, then $P(x,z) = \phi_k(z)\, y_k(x)$, where ϕ_k satisfies $\phi_k' = \frac{i\omega}{\bar{c}} (1-\lambda_k)^{1/2} \phi_k$. Since we do not care about exponentially decreasing solutions in z-direction, only eigenvalues $\lambda_k \leq 1$ are of interest. Therefore, we can choose a polynomial or a rational function $R(\lambda)$ such that

$$(1-\lambda)^{1/2} \simeq R(\lambda) , \quad \lambda_0 \leq \lambda \leq 1 \qquad (2)$$

and replace $(1-L)^{1/2}$ by $R(L)$. This is of course reasonable because of

$$[(1-L)^{1/2} - R(L)] y_k = [(1-\lambda_k)^{1/2} - R(\lambda_k)] y_k .$$

The monochromatic case. Assume ω in (1) has a definite value assigned to it and set $m = \omega/c$ and $\bar{m} = \omega/\bar{c}$ then the equation becomes

$$P_z = i\bar{m} (1-L)^{1/2} P , \quad L = -\frac{1}{\bar{m}^2} \partial_x^2 + 1 - \frac{m^2}{\bar{m}^2} . \qquad (3)$$

Let us now choose $\bar{m}$ such that the smallest eigenvalue λ_0 to L is equal to zero. This choice is convenient since it means that (2) shall be valid for $0 \leq \lambda \leq 1$. If we want to restrict ourselves to simple functions $R(\lambda)$, rational approximations of first degree,for example, it may be necessary to decrease the interval to $0 \leq \lambda \leq A$, say, in order to get a reasonably good fit. This means that components of the form

$$y_k(x) \cdot \exp(i\bar{m}(1-\lambda_k)^{1/2} z) , \quad \lambda_k > A$$

in the solution P of (3) may not be very well approximated when $(1-L)^{1/2}$ is replaced by $R(L)$. For constant velocity $A = \sin^2\theta$, where θ is the largest propagation angle from the vectical which is well treated. Now let μ_0 be the smallest eigenvalue to $-\partial_x^2 - m^2$, then the smallest eigenvalue to L is $1 + \mu_0 / \bar{m}^2$ and thus $\bar{m}^2 = - \mu_0$ or at least $\bar{m}^2 \geq - \mu_0$ which guarantees that L has no negative eigenvalues.

The eigenvalue μ_0 to $-\partial_x^2 - m^2$ can easily and effectively be determined in the following way. Add $\max m^2$ to the operator above and replace the new operator by a matrix T , say. The smallest eigenvalue τ of T can be determined by using inverse iteration

$$T\, y_{n+1} = y_n \;,\quad y_0 \;\; \text{arbitrary}$$

$$\frac{y_{n+1}^{*}\, y_n}{y_n^{*}\, y_n} = \tau_n \to \tau \;,\quad \text{as} \quad n \to \infty$$

The rate of convergence can be speeded up considerably by making so-called shifts. That is after a few iterations T is replaced by $T - \tau_n I$ and then the process is repeated. The approximation to μ_0 we get in this way is the last τ value plus the sum of the shifts minus $\max m^2$. A detailed description and analysis of the method indicated above is given in a book by Wilkinson "The Algebraic Eigenvalue Problem".

Let us first choose $R(\lambda)$ as

$$R(\lambda) = \frac{\alpha - \beta\lambda}{1 - \gamma\lambda} \;,$$

where α , β and γ are given for different values of θ in (Internal report, June 17, 74, G. Starius). Equation (3) shall of course now be replaced by

$$(1 - \gamma L)\, P_z = i\bar{m}\,(\alpha - \beta L)\, P$$

or after making the variable transformation $P = Q\, e^{i\bar{m}z}$ by

$$(1 - \gamma L)\, Q_z = i\bar{m}\,(\alpha - 1 - (\beta - \gamma) L)\, Q \;. \qquad (4)$$

A discretization T of L can be obtainey by simply replacing $\partial_x^{\,2}$ by $(\Delta x)^{-2}\, \delta_{xx}$. Even if the velocity is discontinuous the approximation $(T + \varepsilon I)\, v = f$ of the boundary value problem $L\, y + \varepsilon\, y = f$ is of second order, at least in L_2-norm. See for example "A Mathematical

Analysis of the Finite Element Method" by Fix and Strang. The quantity $\varepsilon > 0$ is inserted only in order to guarantee existence and uniqueness of a solution (the smallest eigenvalue of L is zero). The best way to discretize (4) in z-direction is probably by using a Crank-Nicholson scheme.

We will now consider a second degree rational approximation that is

$$(1-\lambda)^{1/2} \simeq \frac{A - B\lambda + C\lambda^2}{1 - D\lambda + E\lambda^2}, \quad 0 \leq \lambda \leq \sin^2 80° \simeq 0.97$$

Suitable values of the parameters are given below together with the maximal error.

A	B	C	D	E	maximal error
0.9993	1.6613	0.6674	1.1714	0.2417	9×10^{-4}

The partial differential equation we now get instead of (3) is

$$(1 - DL + EL^2)\, P_z = i\bar{m}(A - BL + CL^2)\, P$$

which discretized in z-direction by using Crank-Nicholson's method becomes

$$[1 - i\alpha A + (-D + i\alpha B)L + (E - i\alpha C)L^2]P^{n+1} = [1 + i\alpha A + (-D - i\alpha B)L + (E + i\alpha C)L^2]P^n,$$

where $\alpha = \bar{m} \cdot \Delta z / 2$. Now let β and γ be the roots of the algebraic equation

$$(E - i\alpha C)\mu^2 + (-D + i\alpha B)\mu + 1 - i\alpha A = 0,$$

then the scheme can be written

$$(T-\beta)(T-\gamma)\; P^{n+1} = \rho(T-\bar{\beta})(T-\bar{\gamma})\; P^n, \quad \rho = \frac{E + i\alpha C}{E - i\alpha C}$$

where we have replaced L by a discretization T . To make one step in the z-direction in the scheme above is equivalent to solving two tridiagonal systems of linear equations, namely

$$(T-\beta)Q^{n+1} = \rho(T-\bar{\beta})P^{n}$$

$$(T-\gamma)P^{n+1} = (T-\bar{\gamma})Q^{n+1} \quad .$$

Finally we point out that it is possible to use the 1/12 trick in the scheme above.

The non-monochromatic case. The equation (1) will now be considered without any restriction on ω . Since $(1-L)^{1/2}$ shall be replaced by $R(L)$ we must require that (2) is valid for λ such that

$$\inf_{\omega} \lambda_0 \leq \lambda \leq A \leq 1 \ ,$$

where A is a constant whose meaning has been indicated in the previous section. By using the well-known variational principle

$$\inf_{y} \frac{\int [\frac{\bar{c}^2}{\omega^2} (y')^2 + (1-\frac{\bar{c}^2}{c^2})\, y^2]dx}{\int y^2\, dx} = \lambda_0(\omega^2) \ , \qquad (5)$$

where y satisfies the zero slope boundary conditions, we get

$$\lambda_0 \geq \min_{x} \left(1 - \frac{\bar{c}^2}{c^2} \right) \ , \text{ for all } \omega \ .$$

From (5) it is also easily seen that λ_0 decreases in ω^2 and that

$$\lambda_0(+\infty) = \inf_{y} \frac{\int (1-\frac{\bar{c}^2}{c^2})y^2\, dx}{\int y^2\, dx} = \min_{x}\left(1 - \frac{\bar{c}^2}{c^2} \right)$$

A convenient choice of $\bar{c}$ is therefore $\bar{c} = c_m = \min c(x)$ because then $\lambda_0(+\infty) = 0$ and (2) shall be valid for $0 \leq \lambda \leq A$.

Now let $R(\lambda)$ be a first degree rational function then (1) shall be replaced by

$$[\, 1 + \gamma(\frac{c_m^2}{\omega^2} \partial_x + \eta(x))]P_z = \frac{i\omega}{c_m}(\alpha + \beta\, (\frac{c_m^2}{\omega^2} \partial_x^{\,2} + \eta(x)))P$$

$$\eta(x) = c_m^{\,2} / c^2(x) - 1$$

and after making the variable transformation $P = Q\, e^{\frac{i\omega\alpha}{\bar{c}} z}$, where $\bar{c}$ is a new constant we can choose later, and by replacing ω by $i\,\partial_t$ we get

$$(1+\eta(x))Q_{ttz} - \gamma\, c_m^2 Q_{xxz} = c_m(\beta - \alpha\gamma \frac{c_m}{\bar{c}})\, Q_{xxt} - \varepsilon(x)\, Q_{ttt}$$

$$\varepsilon(x) = \alpha(\frac{1}{c_m} - \frac{1}{\bar{c}}) + (\frac{\beta}{c_m} - \frac{\alpha\gamma}{\bar{c}})\, \eta(x) \,. \tag{6}$$

Parenthetically we point out that if c is a constant and $\bar{c} = c$ then (6) is identical to the equation considered in (Internal report, June 17, 74, G. Starius). In the present paper we will only consider a simplification of (6), good for lower frequencies in x-direction only. Set $\gamma = 0$ and integrate with respect to t then we get

$$Q_{tz} = \beta\, c_m\, Q_{xx} - \varepsilon(x)\, Q_{tt} \,, \tag{7}$$

which corresponds to $R(\lambda) = \alpha - \beta\lambda$. The parameters α and β are chosen such that

$$E(\alpha\, ,\beta\,) = \max \left|\sqrt{1-\lambda} - (\,\alpha - \beta\lambda\,)\right|$$

$$0 \leq \lambda \leq \sin^2\theta$$

is minimized. If we are particularly interested in getting a good fit for small values of λ (small frequencies in x-direction) we instead minimize $E(1, \beta)$.

θ	α	β	min $E(\alpha,\beta)$	β	min $E(1,\beta)$
20	1.00023	0.51555	2×10^{-4}	0.51281	3×10^{-4}
30	1.00120	0.53590	1×10^{-3}	0.52938	2×10^{-3}
40	1.00387	0.56624	4×10^{-3}	0.55365	5×10^{-3}
50	1.00971	0.60872	1×10^{-2}	0.58683	1×10^{-2}
60	1.02083	0.66667	2×10^{-2}	0.63060	3×10^{-2}

As was pointed out by Claerbout equation (7) can be solved approximately by using a splitting method (Russian method) and then no restriction on $\varepsilon(x)$ is needed. As an alternative we will here derive an explicit scheme for (7) under the assumption that $\varepsilon(x) \geq 0$. We have

$$\min \varepsilon = 0 \Rightarrow \frac{1}{c} = \left(1 - \frac{\beta}{\alpha}\right)\frac{1}{c_m} + \frac{\beta}{\alpha}\frac{c_m}{c_M^2} , \quad c_M = \max c$$

$$\varepsilon(x) = \beta c_m \left(\frac{1}{c^2} - \frac{1}{c_M^2}\right) \leq \frac{\beta}{c_m} .$$

Now let $t_n = n \cdot \Delta t$, $z_k = k \cdot \Delta z$ and let $Q_k^n(x)$ correspond to $Q(x, z_k, t_n)$ and consider the family of schemes

$$(I - \delta T)(Q_{k+1}^{n+2} + Q_k^n - Q_k^{n+2} - Q_{k+1}^n) + 2aT[d(Q_{k+1}^{n+2} + Q_k^{n+2} + Q_{k+1}^n + Q_k^n) + (1-2d)(Q_{k+1}^{n+1} + Q_k^{n+1})] + b(I - \delta T)[Q_{k+1}^{n+2} + Q_k^{n+2} - 2(Q_{k+1}^{n+1} + Q_k^{n+1}) + Q_{k+1}^n + Q_k^n] = 0 \tag{8}$$

$$a = \frac{\Delta t \, \Delta z \, c_m \, \beta}{2 \cdot (\Delta x)^2} \quad \text{and} \quad b = \frac{\varepsilon(x) \, \Delta z}{\Delta t} ,$$

where T is now a matrix corresponding to δ_{xx} . Since we want to solve for Q_{k+1}^{n+2} the scheme is explicit if $2ad(x) = (1 + b(x))\delta$ and by using this condition we get

$$(1+b)(Q_{k+1}^{n+2} + Q_k^n) + (-2b + 2(a-\delta)T)(Q_{k+1}^{n+1} + Q_k^{n+1}) + \\ + (-1+b+2\delta T)(Q_{k+1}^n + Q_k^{n+2}) = 0 \tag{9}$$

The stability investigation can be done in exactly the same way as in (Internal report, June 17, 74, G. Starius). Stability in the t-direction means stability of the difference equation

$$(1+b)\,Q_{k+1}^{n+2} - 2\,(b - 4(a-\delta))\,Q_{k+1}^{n+1} + (-1 + b + 8\delta)\,Q_{k+1}^{n} = 0$$

which is stable $(a, b, \delta \geq 0)$ if and only if

$$|a - \delta| \leq \delta \leq \frac{1}{4} \tag{10}$$

The overall stability in the z-direction is unconditional as in the paper referred to above. Therefore (9) is stable if (10) holds, where a is given in (8) . In the stability analysis above we have assumed that b was a constant.

Finally we want to point out that a desirable continuation of this paper seems to be to derive a good scheme for equation (6).