The Emergent Angle Frame for Sections

by Steve Doherty

Following the emergent angle frame for profiles we are lead to the 4 coordinate frame for sections.

$$p = (g-s) / t \tag{1a}$$

$$y = \frac{(g+s)}{2} + \frac{z(g-s)}{2(t^2-(g-s)^2)^{1/2}} \tag{1b}$$

$$d = \frac{1}{2} ((t^2 - (g-s)^2)^{1/2} + z) \tag{1c}$$

$$r = 2z / ((t^2 - (g-s)^2)^{1/2} + z) \tag{1d}$$

The inverse transform is

$$s = y - d\,p/(1-p^2)^{1/2} \tag{2a}$$

$$g = y + d(1-r)\ p/(1-p^2)^{1/2} \tag{2b}$$

$$t = d(2-r) / (1-p^2)^{1/2} \tag{2c}$$

$$z = r\ d \tag{2d}$$

We have assumed the velocity v is 1. Figure 1 gives a geometric view.

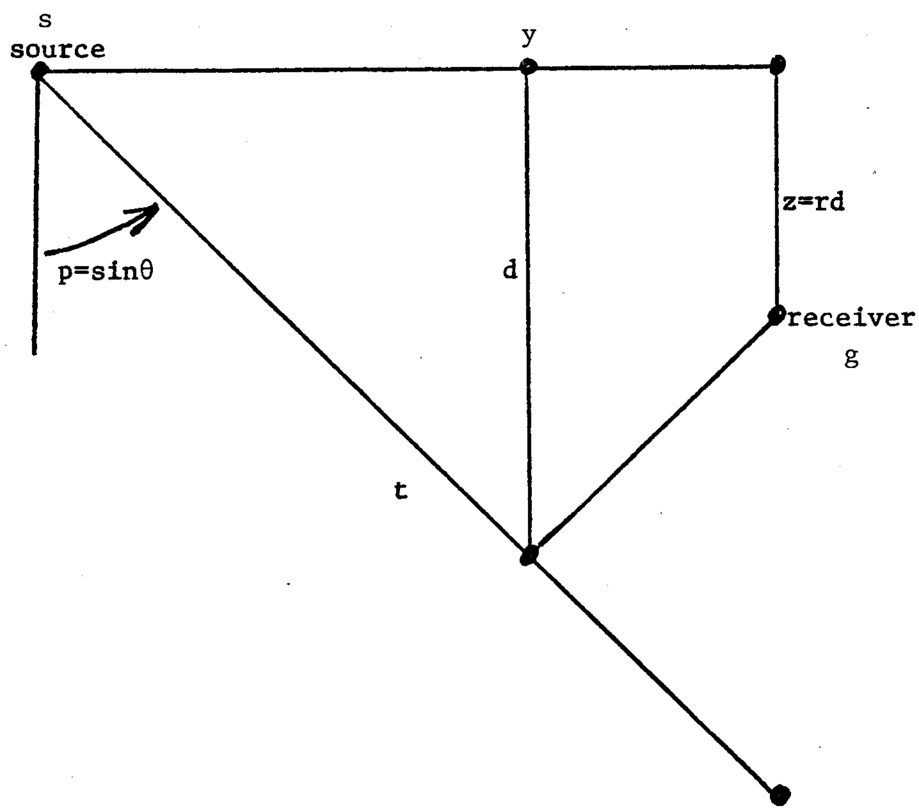

Figure 1. The geometry of the transformation. A complete set of descriptions is either (y, p, r, d) or (g, s, z, t) .

We will need the partial derivatives

$$\frac{\partial y}{\partial (g,t,z)} = \frac{1}{2} + \frac{z}{2\beta} + \frac{(g-s)^2 z}{2\,\beta^3} \quad ; \quad \frac{-(g-s)\,t\,z}{2\,\beta^3} \quad ; \quad \frac{(g-s)}{2\,\beta} \tag{3a}$$

$$\frac{\partial d}{\partial (g,t,z)} = \frac{-\,(g-s)}{2\,\beta} \quad ; \quad \frac{t}{2\,\beta} \quad ; \quad \frac{1}{2} \tag{3b}$$

$$\frac{\partial p}{\partial (g,t,z)} = \frac{1}{t} \quad ; \quad \frac{-\,(g-s)}{t^2} \quad ; \quad 0 \tag{3c}$$

$$\frac{\partial r}{\partial (g,t,z)} = \frac{2(g-s)z}{\beta\,(\beta+z)^2} \quad ; \quad \frac{-\,2\,z\,t}{(\beta+z)^2\beta} \quad ; \quad 2\left(\frac{1}{(\beta+z)} - \frac{z}{(\beta+z)^2}\right) \tag{3d}$$

where

$$\beta = (t^2 - (g-s)^2)^{+1/2}$$

These will be inserted into an equation of the form

$$[(p_g \partial p + r_g \partial r + y_g \partial y + d_g \partial d)^2$$

$$+ (p_z \partial p + r_z \partial r + y_z \partial y + d_z \partial d)^2$$

$$- (p_t \partial p + r_t \partial r + y_t \partial y + d_t \partial d)^2] Q = 0 \quad (4)$$

The definitions of p, d, r are the same as for the emergent angle profile frame. Thus, we have identically zero coefficients for Q_{dd} , Q_{rp} and Q_{dp} .

The coefficient of Q_{pp} is

$$(p_g^2 + p_z^2 - p_t^2) = \frac{(1-p^2)^2}{(2-r)^2 d^2} \quad (5)$$

The coefficient of Q_{rr} is

$$(r_g^2 + r_z^2 - r_t^2) = \frac{(1-r)}{4 d^2} \quad (6)$$

The coefficient of Q_{rd} is

$$2 (r_g d_g + r_z d_z - r_t d_t) = \frac{1}{d} \quad (7)$$

The new terms are Q_{yd} , Q_{yz} , Q_{yp} , Q_{yy} .

The coefficient of Q_{yd} is

$$2(d_g y_g + d_z y_z - d_t y_t) = \frac{z(g-s)}{2 \beta^2} \{ -1 + \frac{(g-s)^2}{\beta^2} + \frac{t^2}{\beta^2} \} = 0 \quad (8)$$

The coefficient of Q_{yr} is

$$2\,(y_g r_g + y_z r_z - y_t r_t) = \frac{2(g-s)}{\beta(\beta+z)} = \frac{p(1-p^2)^{-1/2}}{d} \tag{9}$$

The coefficient of Q_{yp} is

$$2\,(y_g p_g - y_z p_t) = \frac{1}{t}\left(1+\frac{z}{\beta}\right) = \frac{(1-p^2)^{1/2}}{d(2-r)}\left(1+\frac{r}{2-r}\right) \tag{10}$$

The coefficient of Q_{yy} is

$$y_g^{\,2} + y_z^{\,2} - y_t^{\,2} = \frac{1}{4}\left(\left(1+\frac{z}{\beta}\right)^2\left(1+\frac{(g-s)^2}{\beta^2}\right)\right) =$$

$$= \frac{1}{4}\left(1+\frac{r}{2-r}\right)^2\left(1+\frac{p^2}{1-p^2}\right) \tag{11}$$

The resulting equation including the paraxial approximation is

$$\frac{1}{4}\left(1+\frac{r}{2-r}\right)^2\left(1+\frac{p^2}{(1-p^2)}\right)Q_{yy} + \frac{(1-p^2)^{1/2}}{d(2-r)}\left(1+\frac{r}{2-r}\right)Q_{yp}$$

$$+ \frac{p}{d(1-p^2)^{1/2}}\,Q_{yr} + \frac{(1-p^2)^2}{(2-r)^2 d^2}\,Q_{pp} + \frac{1}{d}\,Q_{dr} = 0 \tag{12}$$

which simplifies to

$$Q_{dr} = \frac{-d}{(2-r)^2}\left(1+\frac{p^2}{1-p^2}\right)Q_{yy} - \frac{(1-p^2)^2}{d(2-r)^2}\,Q_{pp}$$

$$-\frac{2(1-p^2)^{1/2}}{(2-r)^2}\,Q_{yp} - \frac{p}{(1-p^2)^{1/2}}\,Q_{yr} \tag{13}$$

This compares well with equations (11) in the emergence angle frame of 12 June and equation (12) in the June 10 opus.