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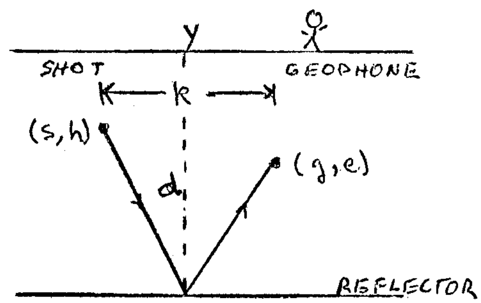
Velocity Analysis Based on the Wave Equation

by Steve Doherty & Jon Claerbout

Consider the coordinate frame defined by

$$z = e + h$$
$$w = e - h$$
$$y = (g + s)/2$$
$$k = (g - s)$$
$$d = t + (e + h)/c_m \qquad (0)$$



The downward continuation equation for this frame is

$$Q_{dz} = -\frac{c_m}{4} Q_{yy} - c_m Q_{kk} - c_m \left(\frac{1}{c_m^2} - \frac{1}{c_w^2} \right) Q_{dd} \qquad (1)$$

where c_m is the velocity in transformation (0) and c_w is the velocity in the wave equation.

Suppose now that we are not interested in arrival times, but instead are only interested in diffraction. With this in mind we will delete the Q_{dd} term from (1) since it deals predominately with changing arrival times.

$$Q_{dz} = -\frac{c_m}{4} Q_{yy} - c_m Q_{kk} \qquad (2)$$

Let us also assume that the reflectors have no dip, so that the Q_{yy} term is zero. Thus from (2) we get

$$Q_{dz} = - c_m Q_{kk} \qquad (3)$$

Equation (3) is useful for migrating hyperbolas in the (d,k) plane, (common depth point gathers). In terms of velocity error it is accurate to better than 2%, for waves propagating less than 15° from the vertical. Suppose we use (3) to migrate a CDP gather with arrivals resulting from a reflector at a depth corresponding to vertical travel time d_0 . If the velocity in (3) is correct and we restrict the data to 15° emergence angles, equation (3) will convert the hyperbola to a point (a focus really) when the data has been downward continued to a depth, $z_f = \frac{c_m d_0}{2}$. If the data were continued further, the focus will expand laterally into a concave upward hyperbola.

Let's study what happens when the material velocity, c_w , is different from the migration velocity c_m . As a first step, let's integrate (3) over d and express the z-derivative as a difference operator. The result is

$$Q(z + \Delta z) = Q(z) - c_m \Delta z \, Q^d_{kk} (z) \tag{4}$$

where we are using the superscript to indicate integration. Equation (4) shows how to compute $Q(z+\Delta z)$ from $Q(z)$. If $c_m = c_w(1+\Delta)$ equation (4) gives

$$Q(z + \Delta z) = Q(z) - c_w(1 + \Delta) \Delta z \, Q^d_{kk} \tag{5}$$

Thus, the value calculated for $Q(z+\Delta z)$ is value that corresponds to $Q(z+\Delta z(1+\Delta))$. This means that if one is migrating a wave form with $c_m > c_w$ one will get the same waveforms as he would if the migration were done with $c_m = c_w$. However, the waveforms occur shallower, in terms of z (the depth of the receivers), when $c_m > c_w$ than when $c_w = c_m$. Thus if we use $c_m = c_w(1+\Delta)$ to migrate the hyperbola generated by a reflector at d_o , instead focusing at $z_f = \frac{c_w d_o}{2}$ as before the hyperbola will focus at z_f given by

$$z_f = \frac{c_w^2 d_o}{2 c_m} \tag{6}$$

Equation (6) can be used as the basis of a velocity estimation procedure. Since d_o and c_m are known and z_f is determined by the migration, the material velocity, c_w , can be found by

$$c_w^2 = \frac{2 z_f c_m^2}{d_o} \tag{7}$$

Equation (3) is accurate only for waves propagating less than 15° from the vertical. If we estimate Q_{zz} instead of assuming it is zero, we get the more accurate equation

$$Q_{dz} = - c_m Q_{kk} - \frac{c_m^2}{4} Q_{kkz}^d \tag{8}$$

Equation (8) is accurate (less than 3% velocity error) for waves traveling up to 45° from the vertical. For the purposes of velocity estimation the main difference between (8) and (3) is the presence of the c_m^2 in (8). The fact that c_m^2 is present means that if $c_w \neq c_m$ complete focusing will never occur. We shall find that if $c_w \neq c_m$ the large offset data will tend to focus a little shallower or deeper (depends if $c_w > c_m$ or $c_m > c_w$) than the small offset data. The sharpness of focusing depends on the relative difference between c_w^2 and c_m^2 .

To see that we will still be able to handle quite large velocity errors we need to consider the relative importance of the two terms on the right side of (8).

Since (3) was accurate for angles less than 15° the correction term (in 8), Q_{kkz}^d is of necessity small for these angles.

For $\theta = 11.5°$, c_m must differ from c_w by at least an order of magnitude before results can be worse with (8) than with (3).

For larger angles, say 40°, equation (3) is moderately accurate and again reasonable errors in c_m do not significantly degrade the sharpness of the focus. If c_m is 20% bigger than c_w the rays propagating at angles near 40° will focus only about 4% shallower than near vertical rays.

ANGLE	VELOCITY ERROR OF (3)	VELOCITY ERROR OF (8)
12°	1%	.002%
26°	3.6%	.28 %
38°	10 %	1.1 %

The correction term $\frac{c_m^2}{4} Q^d_{kkz}$ implies that there is a limit to the emergence angle of data that can be used to estimate velocity. Since the differential equation which accurately treats data with larger emergence angles will probably contain terms proportional to c_m^2 and higher powers of c_m, we cannot expect to migrate these data to a focus unless our initial estimate of velocity is very close to the actual velocity. A practical limit for usable data is probably 45° .

A limitation on emergence angle will effect the resolution of wave equation velocity analysis in the same way it does on conventional procedures (summing along hyperbolas etc.). The velocity resolution of the wave method depends for the most part on the sharpness of the focus in the z-direction. That is, it is important to be able to tell exactly how far the data must be continued before it focuses.

We have already discussed the role of c_w and c_m in determining the focal point of near and far offset data. Limiting the emergence angle of the data has the effect of broadening focus in the z-direction because the k_z wave number spectrum bandwidth is narrowed by excluding the larger propagation angles. In general, the bandwidth increases proportional to $(\text{angle})^2$ (for moderate angles). Since the size of the focus is inversely proportional to the k_z bandwidth, velocity resolution also goes as $(\text{angle})^2$.

Estimation of the depth of focusing.

Since wave equation velocity analysis is based on depth of focusing we shall need a good focal depth estimator. A first approach might be to monitor the amplitude at $k = 0$. Then the focus would be picked at the depth where the amplitude on the zero offset trace is a maximum. There are two problems with this approach. First, we would like our velocity estimator to be uneffected by structure on the reflectors. Since focusing (and amplitude maximum at $k = 0$) can occur as the result of reflector curvature we will get spurious velocity estimates if we look only at amplitude build up on the zero offset trace. The second problem results from the fact that we are modeling a 3-dimensional process with a 2-dimensional algorithm. The major differences between 2-D and 3-D wave propagation are different rates of geometrical spreading and the fact that propagation through a 2-D focus causes a 90° phase shift while propagation through a 3-D focus does not. This 90° phase shift means that there will be some error associated with estimating focal depth by monitoring amplitudes, since the amplitude at a particular point on the $k = 0$ trace will depend on both the phase changes and the amplitude changes involved in focusing.

We first consider estimation of focal depth by monitoring the power of the envelope of Q on the $k = 0$ trace. That is we monitor

$$|P|^2 = |Q(z,d,k = 0) + i H * Q(z,d,k = 0)|^2 \qquad (9)$$

where H is a quadrature filter and $*$ denotes convolution on the d coordinate. Focusing occurs when

$$0 = \frac{d}{dz}|P|^2 = 2 P_z P^* = 2(Q_z + i H Q_z)(Q - i H Q) \qquad (10)$$

If we integrate equation (2) over d we can use the result to replace Q_z in (10). The criteria then becomes

$$0 = - (Q + i H * Q)((\frac{c}{4} Q^d_{yy} + c Q^d_{kk}) - i H * (\frac{c}{4} Q^d_{yy} + c Q^d_{kk})) \qquad (11)$$

where we have used the superscript to indicate integration.

Equation (11) is not quite what we need to indicate focal depths for velocity analysis since it has the Q^d_{yy} term which depends on reflector curvature. We are interested only in focusing due to curvature in the offset direction. Thus we are led to delete the Q^d_{yy} terms leaving

$$0 = (Q + i H * Q) (Q^d_{kk} + i H * Q^d_{kk}) \qquad (12)$$

Equation (12) is a criteria for focusing that can be used with equation (7) and (2) as the basis of an algorithm for estimating the velocity which collapses a CDP gather hyperbola to a focus. This velocity is essentially the same as what is conventionally called the normal moveout velocity.

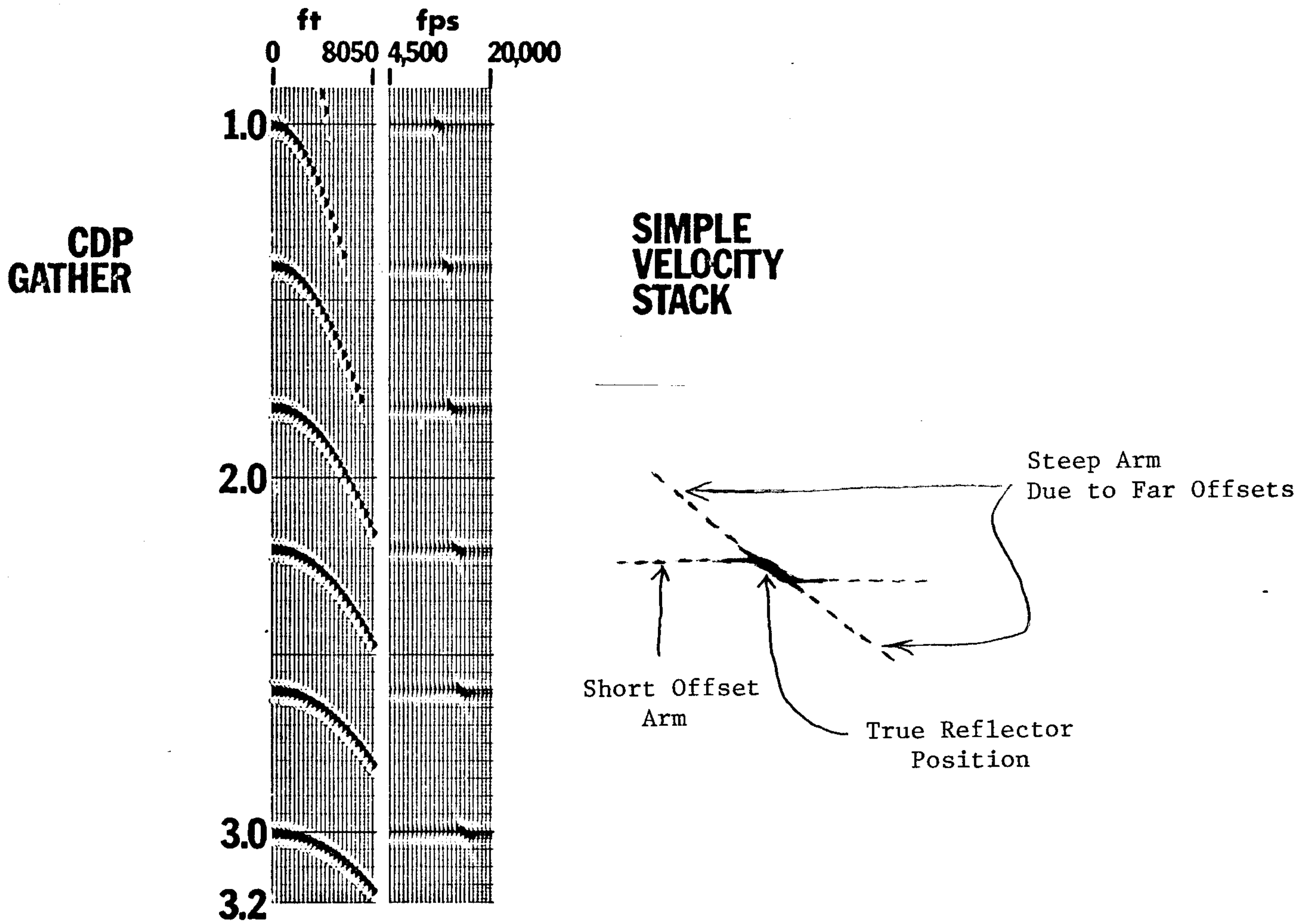


Figure 1. A conventional procedure.
After Continuous Velocity Estimation and Seismic Wavelet Processing (J. W. C. Sherwood and P. H. Poe, Geophysics Vol. 37 #5). The left frame displays a synthetic CDP gather. The right frame depicts the result of stacking the CDP gather with 24 different RMS velocities. The low amplitude horizontal arms on the right frame result from the fact that small offset data sum approximately in phase for the full range of velocities. Notice that there is a vertical displacement of the small offset arm in near the true reflection position. Later figures will indicate that this displacement is the result of the 90° phase shift associated with two dimensional focusing. (Since summing along hyperbolas is a type of integral solution to the wave equation the ideas of focusing and diffraction are relevant here.) A final thing to note is the steep event passing through the true reflection position. Since data corresponding to offsets larger than 8050' was not used in the analysis we should expect some truncation effects to appear in the rightmost frame. Later figures will indicate that the steep event is actually a diffraction resulting from truncation of the hyperbolas.

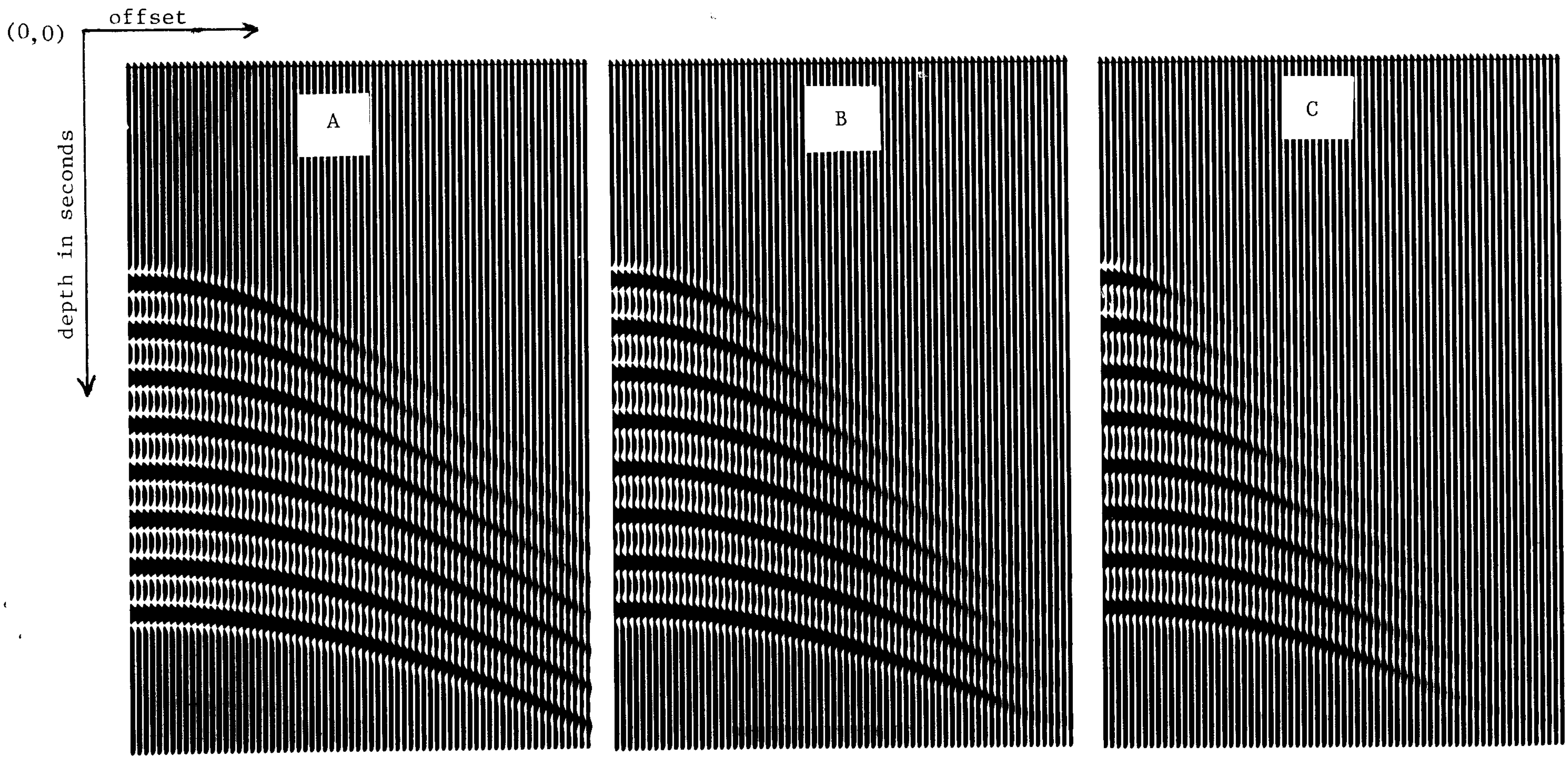
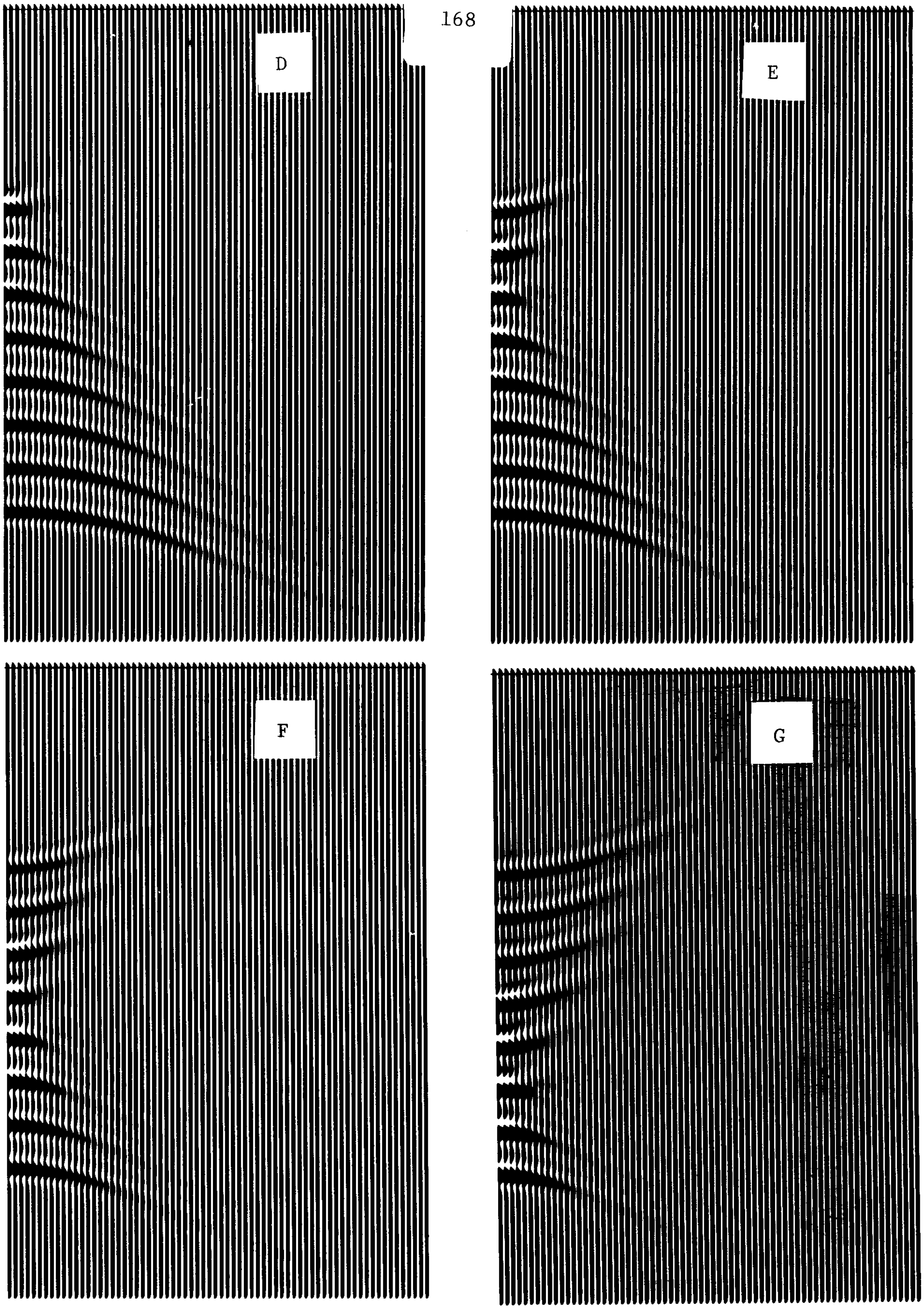


Figure 2. A sequence of 7 frames showing successive stages in the migration of a synthetic CDP gather. The left frame is a synthetic CDP gather. All of the hyperbolas were constructed with the same RMS velocity. Depth = 0 (equation 0) is at the top of each frame and the zero offset trace is on the left. The data has been tapered with a gausian function of propagation angles so that its amplitude is small on the right edge of the frame. The 1/e amplitude corresponds to a propagation angle of 28°. The other frames depict the result of migrating the synthetic gather with equation (8). We have used $c_w = c_m$. The amount of migration (the depth of the shots and receivers) increases linearly between frames B and G. In frame D we see that earliest hyperbola has been migrated to a focus. In later frames we note that the earliest hyperbolas have been over migrated. Also note the 90° phase shift which occurs as the data is migrated through the 2-D focus.



A Continuation of Figure 2.

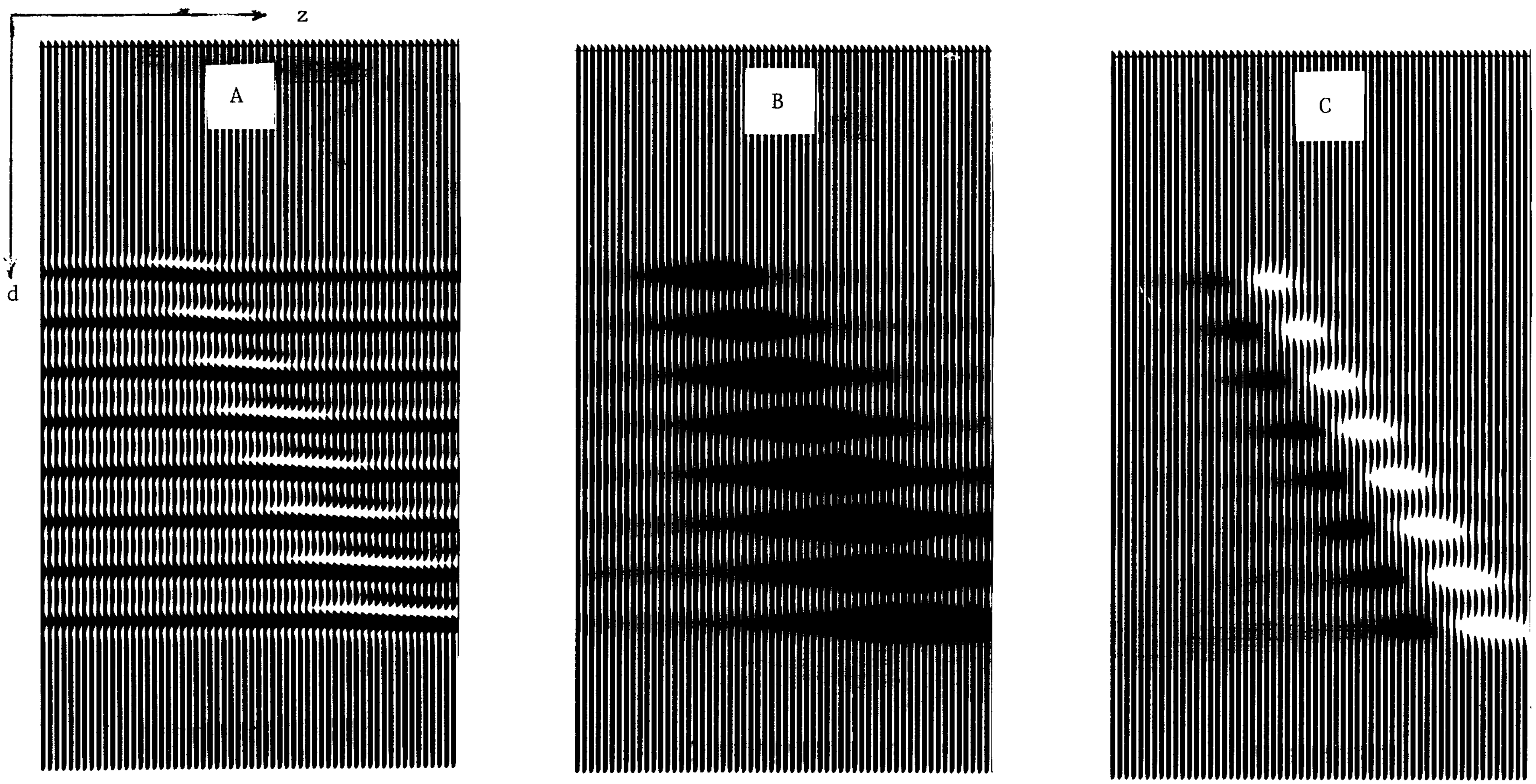


Figure 3. Three frames illustrating three different focal depth estimators. The coordinates are the same for all of the frames. The vertical axis is d of equation (0). The horizontal axis is z of equation (0), (the depth of the shots and receivers). The trace corresponding to surface shots and receivers is on the left. Frame A depicts the amplitude of the zero offset trace of the CDP gather of figure 2 at all stages of migration. This frame is somewhat similar to figure (1) in that the amplitude of the zero offset trace for a particular shot depth corresponds to the amplitude of a sum of the unmigrated gather along a hyperbola with a particular RMS velocity. Notice the phase shift and amplitude build-up associated with the 2-D focus. Frame B depicts the envelope of the zero offset trace as calculated with equation (9). The envelope shows no phase shifting during focusing. Notice that the width (in the z direction) of the focal amplitude build-up is larger for the later events. This is the result of the narrower angular bandwidth of the later hyperbolas in the un-migrated gather. Frame C depicts the z derivative of the envelope of the zero offset trace as calculated with equation (12). Using equation (7), the RMS velocity of the hyperbolas can be estimated from the zero crossings of frame C. We see that

Figure 3 Cont'd.
the zero crossings fall on a straight line passing through $d = o, z = o$. This is to be expected since all of the hyperbolas were constructed with the same RMS velocity.

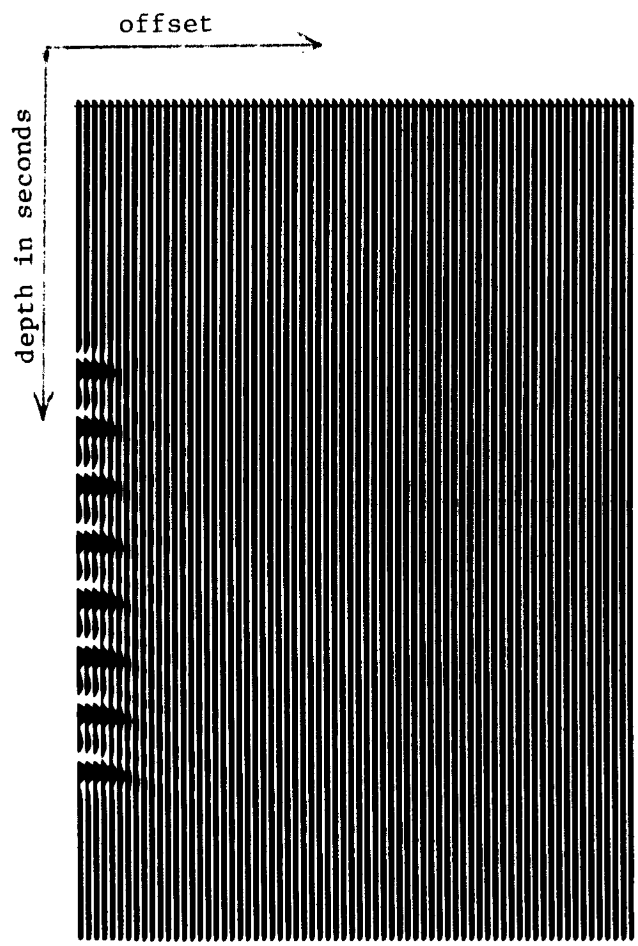


Figure 4. The properly migrated CDP gather of figure 2. Notice that the width of the focus in the offset coordinate direction increases slightly for the later events. This occurs because of the finite extent of the hyperbolas in the un-migrated gather. In spite of the angle dependent tapering, the angular extent of the data in the unmigrated gather decreases with increasing depth. This decrease in angular bandwidth causes the enlargement of the foci.

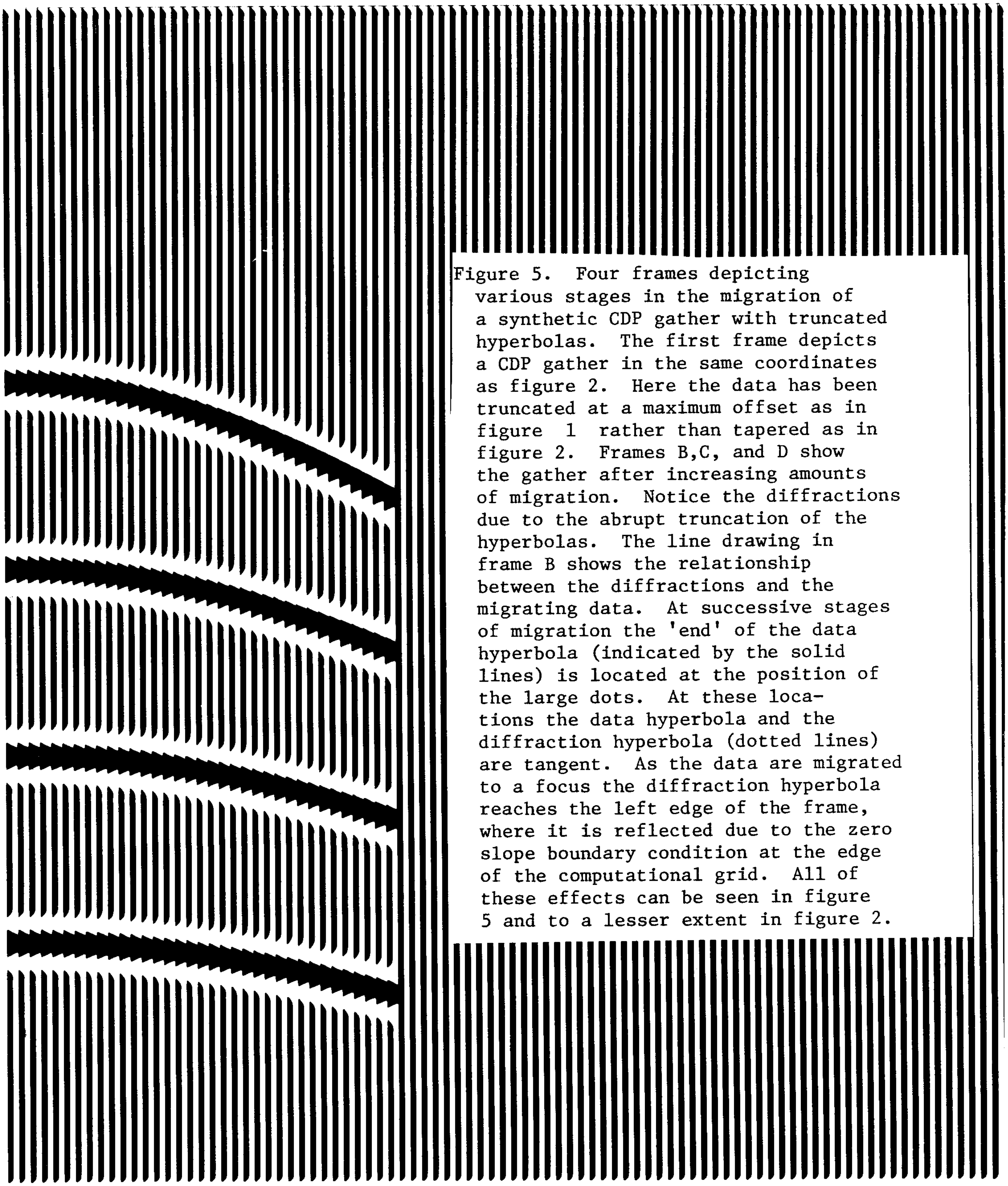
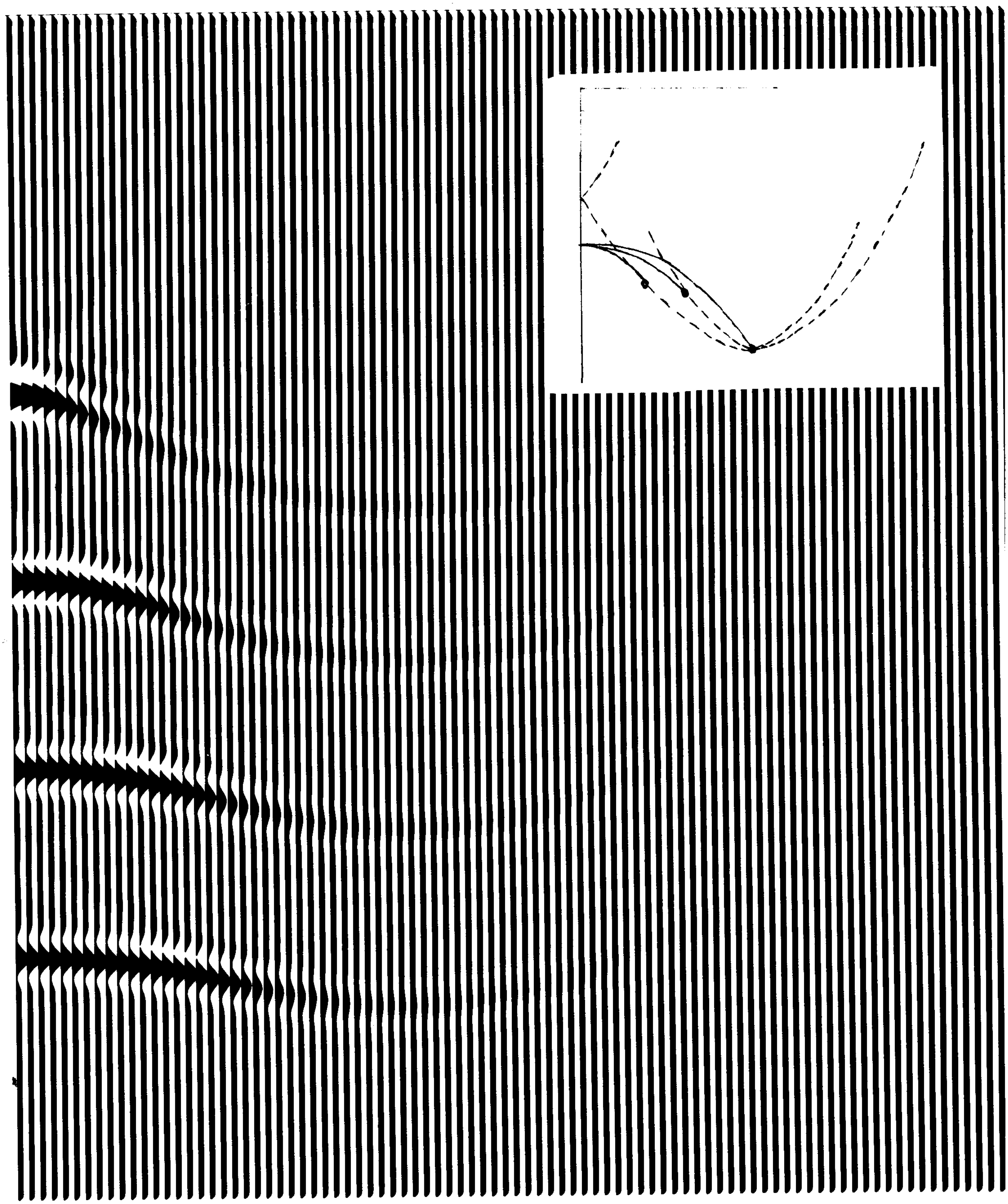
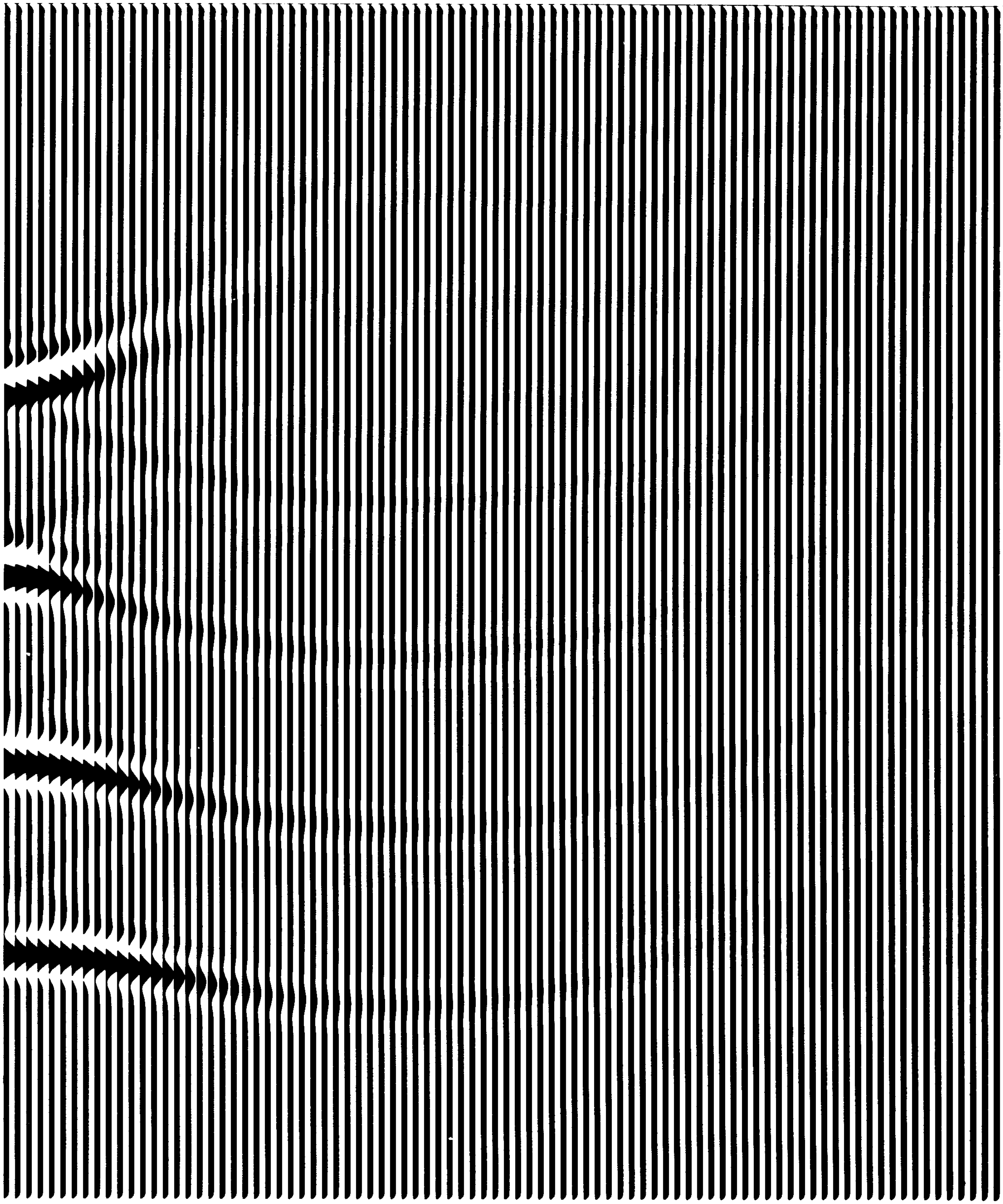


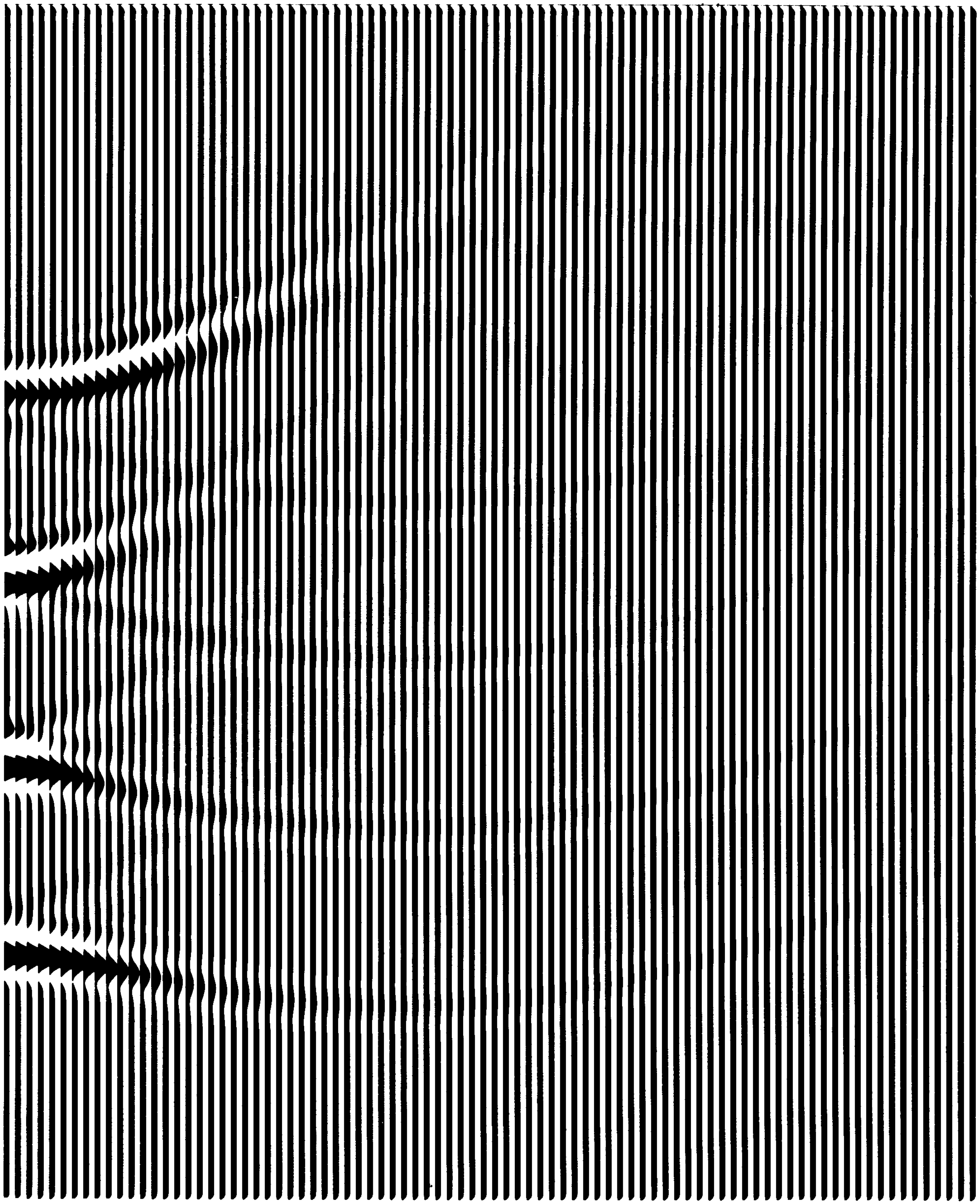
Figure 5. Four frames depicting various stages in the migration of a synthetic CDP gather with truncated hyperbolas. The first frame depicts a CDP gather in the same coordinates as figure 2. Here the data has been truncated at a maximum offset as in figure 1 rather than tapered as in figure 2. Frames B,C, and D show the gather after increasing amounts of migration. Notice the diffractions due to the abrupt truncation of the hyperbolas. The line drawing in frame B shows the relationship between the diffractions and the migrating data. At successive stages of migration the 'end' of the data hyperbola (indicated by the solid lines) is located at the position of the large dots. At these locations the data hyperbola and the diffraction hyperbola (dotted lines) are tangent. As the data are migrated to a focus the diffraction hyperbola reaches the left edge of the frame, where it is reflected due to the zero slope boundary condition at the edge of the computational grid. All of these effects can be seen in figure 5 and to a lesser extent in figure 2.



Frame B of Figure 5.



Frame C of Figure 5.



Frame D of Figure 5.

Frame A of Figure 6.

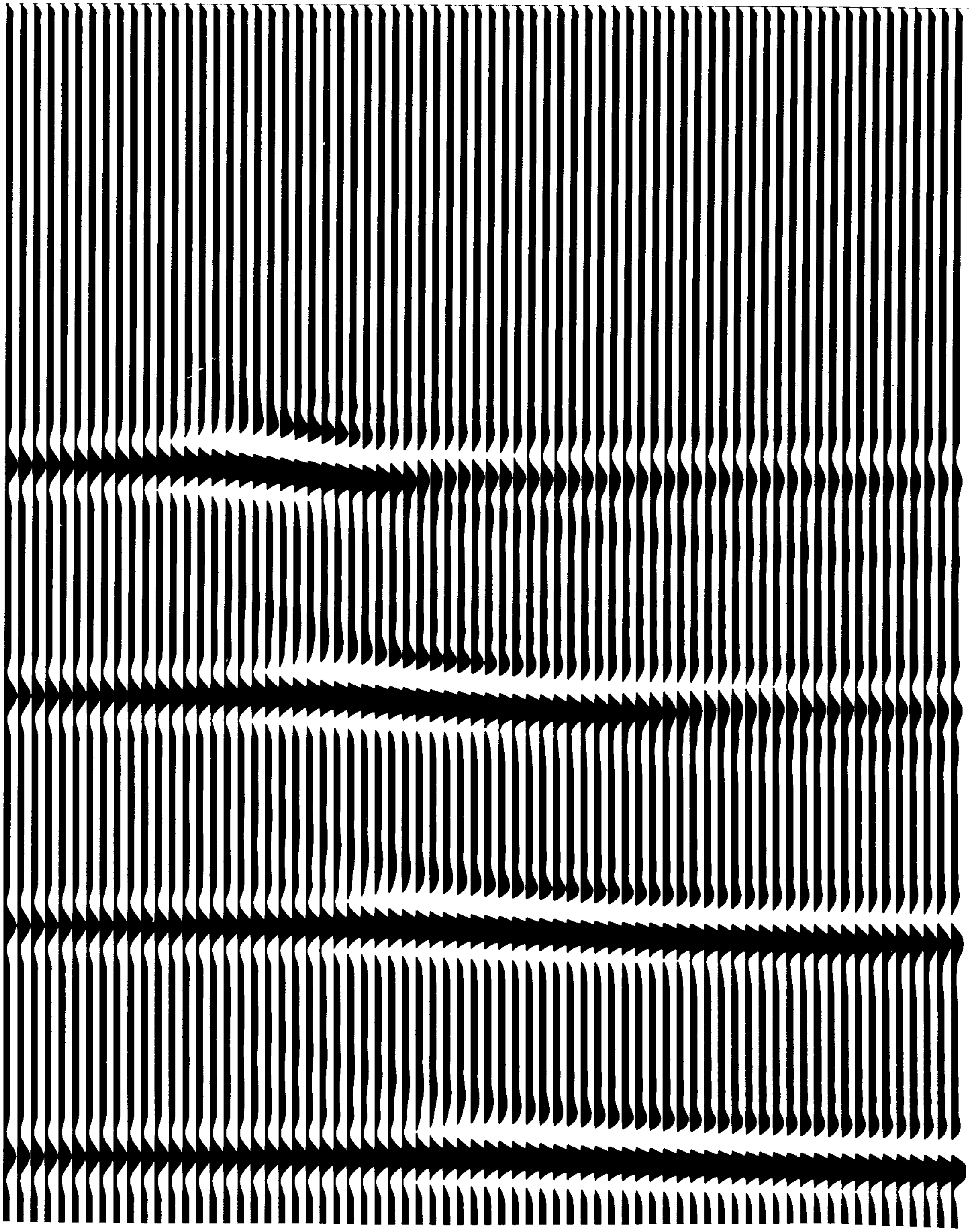


Figure 6 . Three frames illustrating three focal depth estimators. The frames depict the same estimates in the same sequence as in figure 3. The faint dipping events on the amplitude frame are due to the diffractions from the truncated hyperbolas of figure 5. Notice that these events have a character similar to the steep events of figure 1. We should not expect them to be exactly alike because the horizontal axis is not the same in both figures and because figure 1 does not have a reflective boundary at its left edge. Since the envelope

Frame B of Figure 6

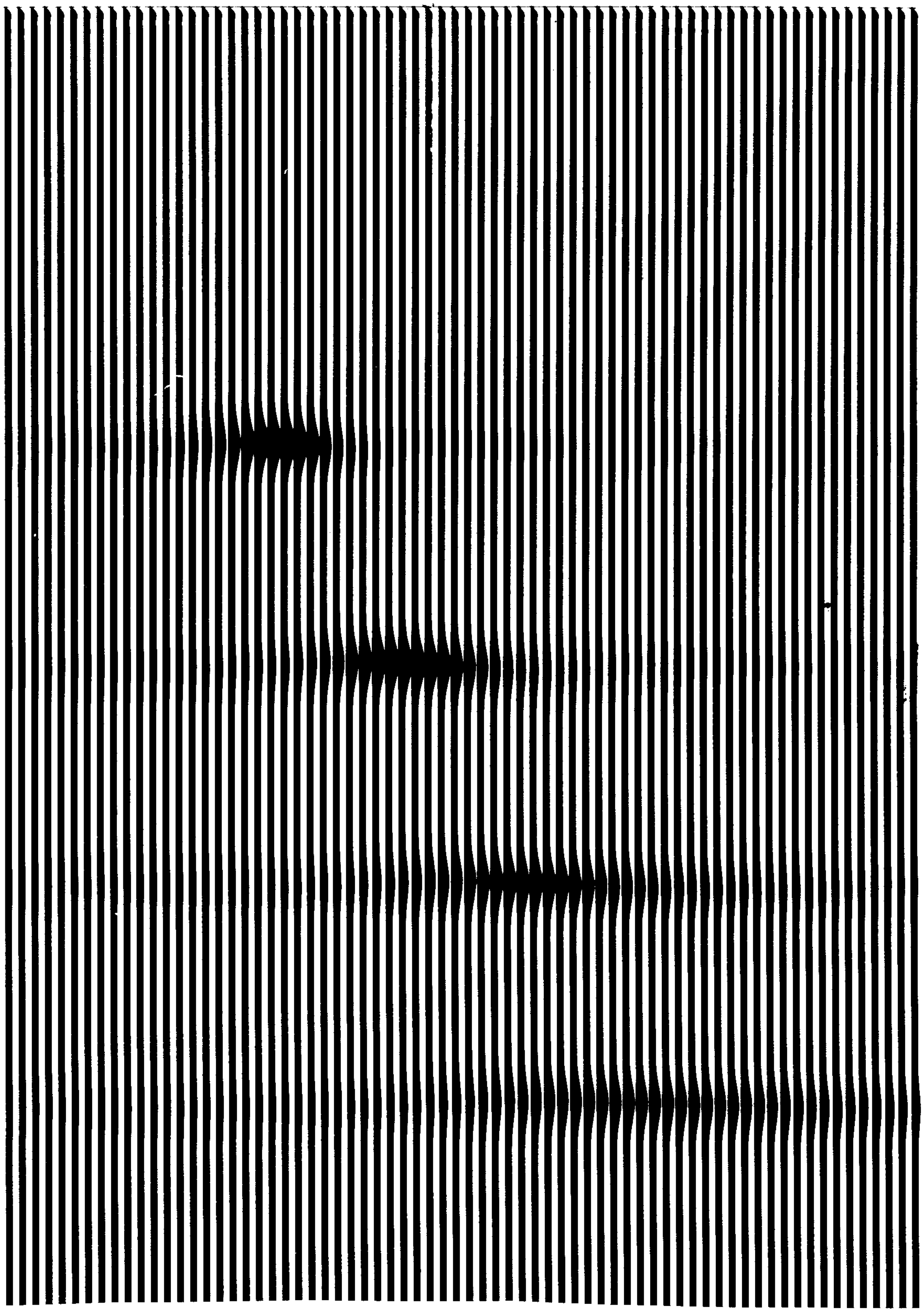
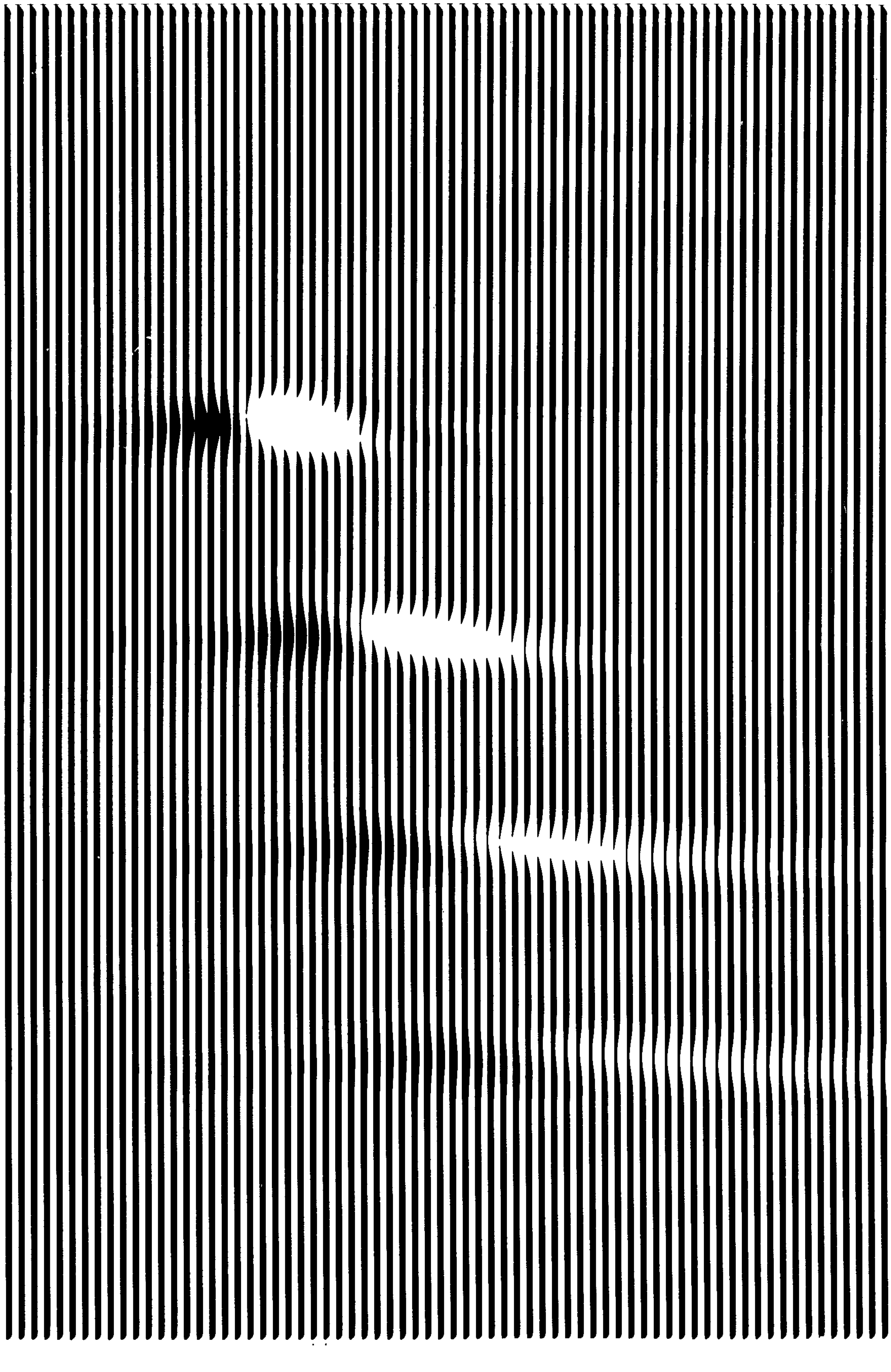


Figure 6 Cont'd.

of the zero offset trace and the z derivative of the envelope are quadratic in amplitude, the diffraction effects are not noticeable on those frames. However, the effect of the angular bandwidth reduction due to the truncation is noticeable. All the focal depth indicators tend to be more smeared out in z in figure 6 than in figure 3. This effect is particularly strong for the deep events where the bandwidth reduction is greatest.

Frame C of Figure 6



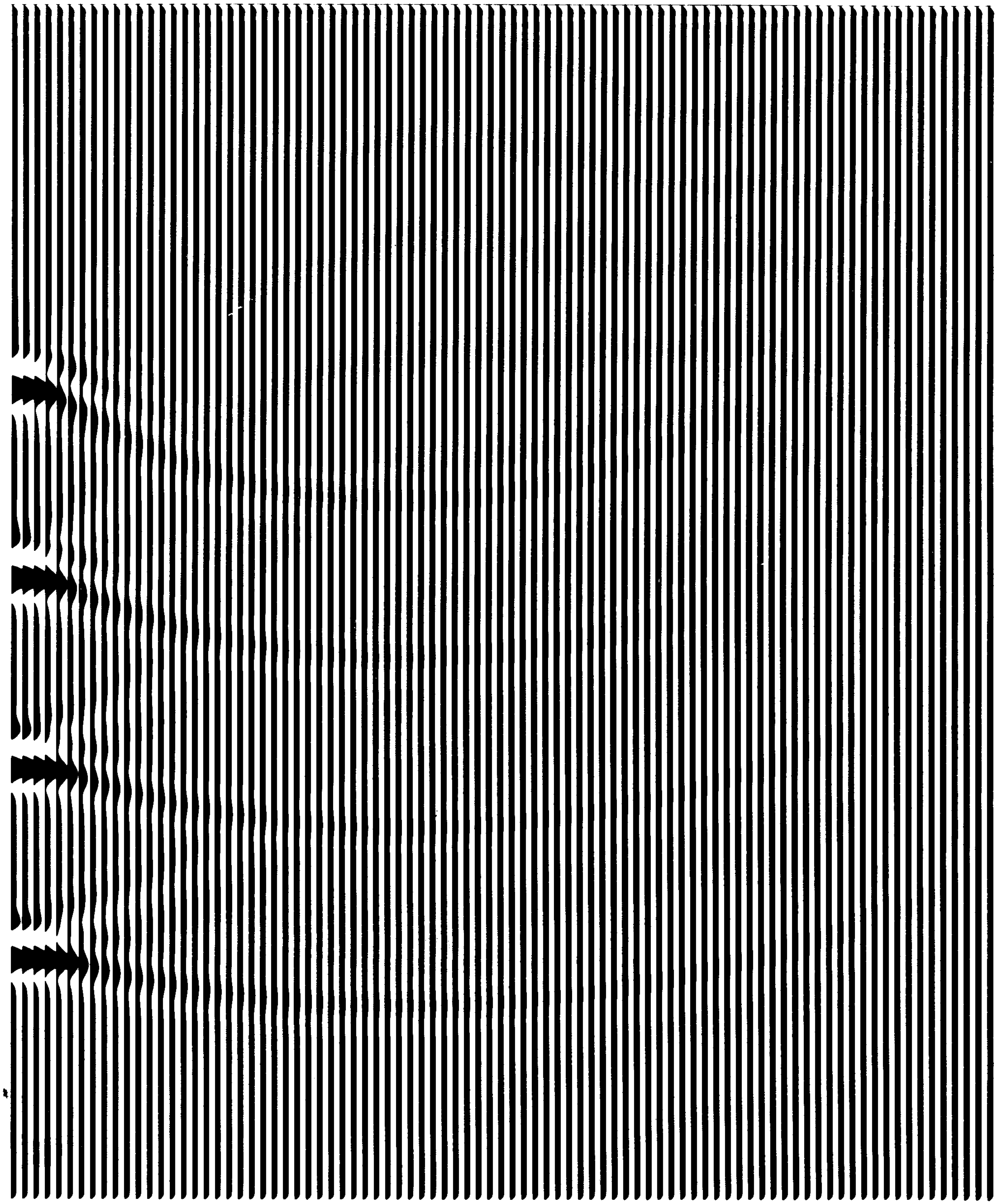


Figure 7. The properly migrated CDP gather of figure 5. The diffraction tails are obvious. The width of the focus increases with d due to the narrower bandwidth of the later events.