Computing Diffracted Multiples

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Continuation Equations

In this section we will relax the one-dimensional assumption made earlier and consider a two-dimensional model of the subsurface. And, as before, we'll choose to neglect shear waves and attenuation of the waves other than that associated with 2D geometrical spreading and losses at the reflecting interfaces. Under these conditions the two-dimensional scalar wave equation completely describes the behavior of the seismic wavefields. In addition to primary reflections, implicit in the wave equation are the predicted effects of spherical divergence, moveout, diffractions, and multiple reflections. Usually a processing sequence attempts to invert each of these effects, one at a time, in order to arrive at a depth section. In reality these effects occur simultaneously.

In this paper the only "processor" we will be using will be approximations to the wave equation. For the most part we will be concerned with calculation of the forward problem, i.e. computing a reflection seismogram given a depth model. Finally we will consider the inversion of the seismograms back to the depth model in view of techniques developed in doing the forward problem. In both cases our success will depend on how closely we can simulate the relevant features of the wave equation.

To begin with we will split the wave equation into two parts: one which describes the propagation of upcoming waves and another for the downgoing waves. We do this for two quite different reasons. The first is that by splitting we may define local coordinate frames which propagate with the two wavefields. In this way we reduce much of the work the partial differential equation has to do. It makes little sense to use a wave equation to move energy to a place which can be predicted by a well-chosen coordinate frame. The pay off of this practice comes in the ability to propagate a relatively large distance for the same cost as a small distance.

The second reason for separating the waves is that we then have complete control over the coupling of the two. This coupling, defined by the reflection coefficients of the model, may be modified in order to selectively synthesize all or any of the classes of multiple reflections. As we shall see, this will be analagous to the gating technique used in the one-dimensional algorithm. Of far more importance, however, is that separation of the wavefields is the fundamental principle behind the inversion technique.

Our choice of coordinate transformations for both the upcoming and downgoing wavefields is essentially a description of ray paths for each in some model. We may try to build in a large amount of information we may have in the hopes of arriving at a simpler transformed wave equation. Unfortunately, the more constrained the transformation gets to actually modelling the rays the more complicated the resulting equation. The simplest constraint is to say the upcoming waves go up and the downgoing waves go down along vertical paths; in which case we get the simplest form of the continuation equations.

Although these would be adequate for our purposes, we may wish to examine the effect of offset in our problem.

Figure 1 illustrates the transformation in terms of the recording geometry. For the transformed upcoming wave U(y,f,d,z) the coordinate d takes on the meaning of two-way vertical travel time to reflection point 0 . In the downgoing wave D(y,f,d',z) frame the d' coordinate is referenced with respect to the first arrival at the reflection point.

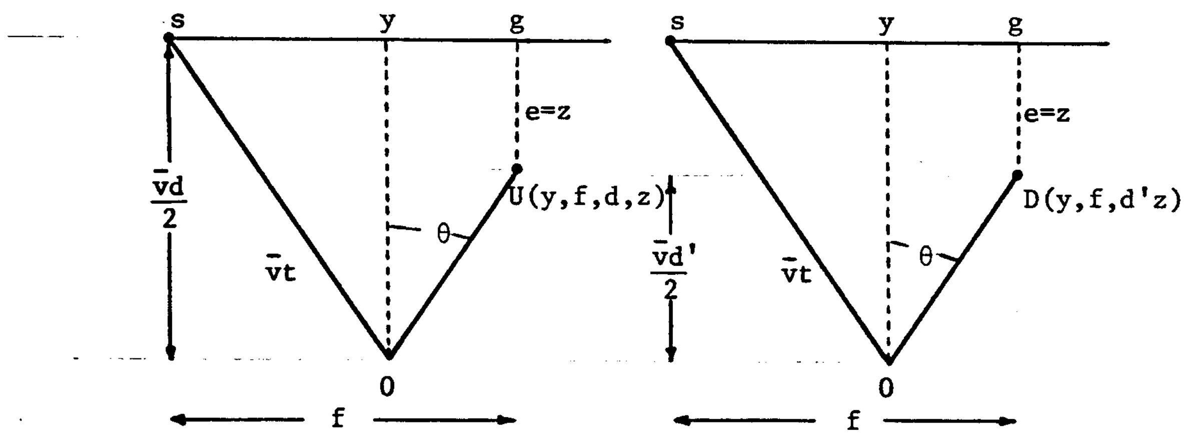

Figure 1. Geometry for upcoming and downgoing transformations. s is the location of the shot, g is the geophone location, y the midpoint, f offset, e the depth of the wavefields, t is the ray path travel time. $\bar{v}$ is a velocity taken to be a spatial average of the velocity of the medium.

The two dimensional scalar wave equation in our original coordinate system is

$$P_{ee} + P_{gg} - \frac{1}{\tilde{v}^2} P_{tt} = 0 \qquad (1)$$

where $\tilde{v}(e,g)$ is the compressional velocity. The upward and downward transformations are

$$y = \frac{(g+s)}{2} + \frac{e}{2}\tan\theta \tag{2a}$$

$$f = g - s \tag{2b}$$

$$z = e$$

upcoming

$$(g - s)^2 + (\bar{v}d - e)^2 = (\bar{v}t)^2 \tag{2d}$$

downgoing

$$(g - s)^2 + (\bar{v}d' + e)^2 = (\bar{v}t)^2 \tag{2e}$$

We'll also find it convenient to define a zero-offset two-way reflector depth term as

$$r \triangleq \sqrt{(\bar{v}t)^2 - (g-s)^2} = \bar{v}\,d' + e = \bar{v}\,d - e \tag{3}$$

Thus (2a) may be rewritten as

$$y = \frac{(g+s)}{2}(1 + e/r) - \frac{es}{r} \tag{4}$$

Now we make the statement that the wavefields are invariant under a coordinate transformation.

$$P(g, s, t, e) = U(y, f, d, z) = D(y, f, d', z) \tag{5}$$

Although equation (5) may seem paradoxical at first since we are trying to separate the waves, recall that at this point U and D merely represent different dependent transform variables. Shortly we will make the approximation which will do the separation. We'll first compute the necessary derivatives using the chain rule. First the upcoming transformation

$$P_{gg} = y_g^2 U_{yy} + 2y_g U_{yf} + 2 d_g y_g U_{dy} + 2 d_g U_{fd} + U_{ff} + d_g^2 U_{dd} + y_{gg} U_y + d_{gg} U_d \quad (6a)$$

$$P_{ee} = y_e^2 U_{yy} + 2y_e U_{yz} + 2y_e d_e U_{dy} + 2d_e U_{zd} + U_{zz} + d_e^2 U_{dd} + y_{ee} U_y + d_{ee} U_d \quad (6b)$$

$$P_{tt} = y_t^2 U_{yy} + 2 y_{td} d_t U_{dy} + d_t^2 U_{dd} + d_{tt} U_d + y_{tt} U_y \quad (6c)$$

Computing the necessary transform derivatives we have

$$\frac{\partial y}{\partial(g,t,e)} = 1/2 + \frac{e}{2r} + \frac{(g-s)^2 e}{2\, r^3} \;,\; -\frac{(g-s)e\, \bar{v}^{-2} t}{2\, r^3} \;,\; \frac{(g-s)}{2\, r}$$

$$\frac{\partial^2 y}{\partial(g,t,e)^2} = \frac{3\, e}{2} \left[\frac{(g-s)}{r^3} + \frac{(g-s)^3}{r^5} \right],\; -\frac{(g-s)e}{2} \left[\frac{\bar{v}^{-2}}{r^3} - \frac{3\bar{v}^{-4}\, t^2}{r^5} \right],\; 0$$

$$\frac{\partial f}{\partial(g,t,e)} = 1,\; 0,\; 0$$

$$\frac{\partial d}{\partial(g,t,e)} = -\frac{(g-s)}{\bar{v}\, r} \;,\; \frac{\bar{v}t}{r} \;,\; -\frac{1}{\bar{v}}$$

$$\frac{\partial^2 d}{\partial(g,t,e)^2} = -\frac{(g-s)^2}{v\, r^3} - \frac{1}{\bar{v}r} \;,\; \frac{\bar{v}}{r} - \frac{\bar{v}^3 t^2}{r^3} \;,\; 0$$

$$\frac{\partial z}{\partial(g,t,e)} = 0,\; 0,\; 1$$

If we consider the case where the ratio of offset to reflector depth is small, i.e. $(g-s)/r \to 0$ these derivatives become in the limit

$$\lim \frac{\partial y}{\partial(g,t,e)} = 1/2 + e/2r \;,\; 0 \;,\; 0$$

$$\lim \frac{\partial^2 y}{\partial(g,t,e)^2} = 0 \;,\; 0 \;,\; 0$$

$$\lim \frac{\partial d}{\partial(g,t,e)} = 0 \;,\; \bar{v}t/r \;,\; +\frac{1}{\bar{v}}$$

$$\lim \frac{\partial^2 d}{\partial (g,t,e)^2} = -\frac{1}{\bar{v}r} \quad , \quad \frac{\bar{v}}{r} - \frac{\bar{v}^3 t^2}{r^3} \quad , \quad 0$$

Using these small offset/large depth limits in equations (6) and inserting into the wave equation (1) we have

$$-U_{zd} = \frac{\bar{v}}{2}(1/2 + e/2r)^2 U_{yy} + \frac{\bar{v}}{2}\left(\frac{1}{\bar{v}^2} - \frac{1}{\tilde{v}^2}\right)U_{dd} - \frac{1}{2r}U_d + \frac{\bar{v}}{2}(1+e/r)U_{yf} + \frac{\bar{v}}{2}U_{ff} + \frac{\bar{v}}{2}U_{zz} \tag{8}$$

In going to the small offset limit we have lost the term that does normal moveout correction, namely the $d_g\ U_{fd}$ term. The residual terms U_{yf} and U_{ff} account for shifting moveout in toward the midpoint. For negligible dip and offset we may consider dropping these terms as small.

At this point we make the parabolic approximation which will insure separation of the upcoming waves in this transformation. Specifically we drop the U_{zz} term. This is the usual approximation we make and amounts to fitting one half of the original circular dispersion relation with a parabolic dispersion law. The $\frac{1}{2r}\ U_d$ term models two-dimensional geometrical spreading of the wavefronts. As this correction can easily be done prior to processing we will also drop this term.

Dropping all these equation (8) collapses down to the familiar continuation equation with the only differences being a depth variable coefficient which ranges from $\bar{v}/8$ at the surface to $\bar{v}/2$ at the reflecting point and a term dependent on excursions of the medium velocity from the frame velocity.

$$U_{zd} = -\bar{v}/2(\ 1/2 + e/2r\ )^2\ U_{yy} - \bar{v}/2\left(\frac{1}{\bar{v}^2} - \frac{1}{\tilde{v}^2}\right) U_{dd} \tag{9}$$

Now let's return to consideration of the downgoing transformation. Note that the only difference here is replacing D for U , d' for d and computing the additional coordinate derivatives

$$\frac{\partial d'}{\partial(g,t,e)} = -\frac{(g-s)}{\bar{v}\ r}\ ,\ \frac{\bar{v}t}{r}\ ,\ -\frac{1}{\bar{v}}$$

$$\frac{\partial^2 d'}{\partial(g,t,e)} = -\frac{(g-s)^2}{\bar{v}\ r^3} - \frac{1}{\bar{v}r}\ ,\ \frac{\bar{v}}{r} - \frac{\bar{v}^3 t^2}{r^3}\ ,\ 0$$

Which, again taking the small offset/large depth limit, reduce to

$$\lim \frac{\partial d'}{\partial(g,t,e)} = 0\ ,\ \frac{\bar{v}t}{r}\ ,\ -\frac{1}{\bar{v}}$$

$$\lim \frac{\partial^2 d'}{\partial(g,t,e)^2} = -\frac{1}{\bar{v}r}\ ,\ \frac{\bar{v}}{r} - \frac{\bar{v}^3 t^2}{r^3}\ ,\ 0 \tag{10}$$

Inserting everything into the wave equation we arrive at an equation similar to equation (8)

$$D_{zd'} = \frac{\bar{v}}{2}(1/2 + e/2r)^2 D_{yy} + \frac{\bar{v}}{2}\left(\frac{1}{\bar{v}^2} - \frac{1}{\tilde{v}^2}\right) D_{d'd'} - \frac{1}{2r} D_{d'} + \frac{\bar{v}}{2}(1+e/r) D_{yf} + \frac{\bar{v}}{2} D_{yy} + \frac{\bar{v}}{2} D_{zz} \tag{11}$$

As before, we assume corrections for normal moveout and geometrical spreading to be previously applied. In addition we again drop the D_{zz} term making the parabolic approximation

$$D_{zd'} = \frac{\bar{v}}{2}\left(1/2 + e/2r\right)^2 D_{yy} + \frac{\bar{v}}{2}\left(\frac{1}{\bar{v}^2} - \frac{1}{\tilde{v}^2}\right) D_{d'd'} \tag{12}$$

Define $\quad \varepsilon(y,z) = \left(\dfrac{\bar{v}^2}{\tilde{v}^2(y,z)} - 1\right)$

in which case equations (9) and (12) may be rewritten as

$$U_{zd} = -\frac{\bar{v}}{2}\left(1/2 + e/2r\right)^2 U_{yy} + \frac{\varepsilon}{2\bar{v}} U_{dd} \tag{13a}$$

$$D_{zd'} = \frac{\bar{v}}{2}(1/2 + e/2r)^2 D_{yy} - \frac{\varepsilon}{z\bar{v}} D_{d'd'} \tag{13b}$$

Reflected Waves

Now we wish to go back and pick up the reflected waves. In arriving at equations (13) we deliberately suppressed reflections by making the parabolic approximation in our effort to separate the up and downgoing waves. Recall that if the waves are coupled, then their superposition at every point in time and space should equal the waves in our original wave equation. That is, now making the statement

$$P(s,g,t,e) = U(y,f,d,z) + D(y,f,d',z) \tag{14}$$

is equivalent to coupling U and D. Thus, to regenerate the reflection we again crank (14) into both the transformations. Consider the upcoming transformation. Using $d' = d - 2z/\bar{v}$ we transform

$$P(s,g,t,e) = U(y,f,d,z) + D(y,f,d - 2z/\bar{v} ,z)$$

which after carefully keeping track of the independent variables, and making the small offset and geometrical spreading assumptions, but without the parabolic approximation results in

$$[U_{zd}(d) + D_{zd}(d- 2z/\bar{v})] = - \frac{\bar{v}}{2}(1/2+e/2r)^2[U_{yy}(d) - D_{yy}(d-2z/\bar{v})] \tag{15}$$

$$+ \frac{\varepsilon}{2\bar{v}}[U_{dd}(d)-D_{dd}(d-2z/\bar{v})] - \frac{\bar{v}}{2}[U_{zz}(d) - D_{zz}(d-2z/\bar{v})]$$

where the time variables have been explicitly written for emphasis. Since we have not yet suppressed the reflections, equation (15) has both up and downgoing reflected and transmitted waves. If we simply subtract off the transmitted downgoing wave (equation (13b)) we will leave the reflected portion of the downgoing energy. Doing this we have

$$U_{zd} = -\frac{\bar{v}}{2}(1/2 + e/2r)^2\, U_{yy} + \frac{\varepsilon}{2\bar{v}} U_{dd} - \frac{\bar{v}}{2}[\, U_{zz} - D_{zz}(d - 2z/\bar{v})]$$

Again in order to suppress upward reflections we drop U_{zz}

$$U_{zd} = -\frac{\bar{v}}{2}(\, 1/2+e/2r\,)^2\, U_{yy} + \frac{\varepsilon}{2\bar{v}} U_{dd} + \frac{\bar{v}}{2} D_{zz}(\, d-2z/\bar{v}\,) \tag{16}$$

Similarly, if we insert

$$P(\, s,g,t,e\,) = U(\, y,f,d' + 2z/\bar{v},\, z\,) + D(\, y,f,d',z\,)$$

into the downgoing wave transformation and subtract off the transmitted portion of the upcoming wave (13a) we have, dropping D_{zz}

$$D_{zd'} = \frac{\bar{v}}{2}(\, 1/2+e/2r\,)^2\, D_{yy} - \frac{\varepsilon}{2\bar{v}} D_{d'd'} + \frac{\bar{v}}{2} U_{zz}(\, d'+2z/\bar{v}\,) \tag{17}$$

Rather than calculate D_{zz} in (16) or U_{zz} in (17) directly we wish to make the connection with reflection coefficients more explicit. Furthermore, we have both theoretical problems resulting from making the parabolic approximation in the first place and numerical problems associated with the mixed differencing. Let us estimate D_{zz} and U_{zz} from our original continuation equations (13). For the moment, let's assume we have a one-dimensional situation, in which case (13b) becomes

$$D_{zd'} \cong -\varepsilon/2\bar{v}\, D_{d'd'}$$

or

$$D_z \cong -\varepsilon/2\bar{v}\, D_{d'}$$

and once more differentiating in the depth coordinate

$$D_{zz} \cong -1/2\bar{v}(\ \varepsilon_z D_{d'} + \varepsilon\ D_{d'z}\ ) \tag{18a}$$

and the estimate for U_{zz} from (13a) is

$$U_{zz} \cong 1/2\bar{v}(\ \varepsilon_z\ U_d + \varepsilon\ U_{dz}\ ) \tag{18b}$$

Inserting these into (16) and (17) and using $D_{d'}(d') = D_d(d-2z/\bar{v})$ and $U_d(d) = U_{d'}(d'+2z/\bar{v})$ we have

$$U_{zd} = -\frac{\bar{v}}{2}(1/2+e/2r)^2 U_{yy} + \frac{\varepsilon}{2\bar{v}} U_{dd} - \frac{1}{4}[\varepsilon_z D_d(d\ -2z/\bar{v}) + \varepsilon\ D_{dz}(d\ -2z/\bar{v})] \tag{19a}$$

$$D_{zd'} = \frac{\bar{v}}{2}(1/2+e/2r)^2 D_{yy} - \frac{\varepsilon}{2\bar{v}} D_{d'd'} + \frac{1}{4}[\ \varepsilon_z U_{d'}(d'+2z/\bar{v}) + \varepsilon U_{d'z}(d'+2z/\bar{v})] \tag{19b}$$

In the case where $\bar{v}$ is some well chosen spatial average of $\tilde{v}(\ y,z\ )$ we simplify things one more step by letting $\varepsilon \to 0$

$$U_{zd} = -\frac{\bar{v}}{2}(1/2 + e/2r)^2\ U_{yy} - \frac{\varepsilon_z}{4} D_d(\ d\ - 2z/\bar{v}\ ) \tag{20a}$$

$$D_{zd'} = \frac{\bar{v}}{2}(\ 1/2 + e/2r\ )^2\ D_{yy} + \frac{\varepsilon_z}{4} U_{d'}(\ d' + 2z/\bar{v}\ ) \tag{20b}$$

Thus we have a coupled pair of continuation equations, the coupling being due to a source term which is the product of the gradient of material properties and the time derivative of the shifted, opposite travelling wave.

The Noah Approximation - Relation to 1D Algorithm

Recall that in the discussion of the one-dimensional algorithm interbed reflections were not modelled. This we did not view as a serious drawback, arguing that such reflected energy was $O(\varepsilon_2^3)$ where ε_2 was a representative subsurface reflection coefficient. This is equivalent to suppressing all reflections generated by the upcoming wave and appearing on the downgoing wave. If we were to make the Noah Approximation in the present study we would simply drop the source term in equation (20b).

Our source term is defined in terms of ε_z .

$$-\frac{\varepsilon_z}{4} = \frac{1}{4}\frac{\partial}{\partial z}(1 - \bar{v}^2/\tilde{v}^2(y,t)) = \frac{\tilde{v}_z}{2\tilde{v}}$$

which is very good definition of a reflection coefficient for a constant density material. Therefore we define a reflection coefficient as

$$C(y,z) \triangleq -\varepsilon_z/4 = \tilde{v}_z/2\tilde{v} \tag{21}$$

The only thing we have left in order to specify the forward algorithm is to account for the reflection at the free surface. Thus, making the Noah Approximation and denoting the reflection seismogram as R we have the following initial-boundary value problem

D.E.

$$U_{zd} = -\frac{\bar{v}}{8}\left(1 - \frac{z}{\bar{v}d} \right)^{-2} U_{yy} + C(y,z)\, D_d(d -2z/\bar{v}) \tag{22a}$$

$$D_{zd'} = \bar{v}/z(1/2+e/2r)^2 D_{yy} \tag{22b}$$

B.C.

$$D(y,d',z=0) = - U(y,d,z=0) \tag{22c}$$

$$R(y,d) = U(y,d,z=0) \tag{22d}$$

I.C.

$$D(y,d'=0,z=0) = 1 \tag{22e}$$

$$U(y,d=0,z=0) = 0 \tag{22f}$$

This is the complete mathematical description of the forward procedure, that is, given a reflection coefficient map C(y,z) it shows how to develop a reflection seismogram R(y,d) inclusive of a U diffracted and focused sea floor, pegleg and structure multiple reflections.

It is perhaps important, if not at least interesting, to examine these equations for the case of a one-dimensional, layered medium. We merely have to set all y derivatives of the waves to zero and integrate.

$$U_{zd} = C(y,z) D_d(d -2z/\bar{v})$$

$$D_{zd'} = 0 \rightarrow D(y,d',z) = D(y,d',z=0) = - U(y,d,z=0) = -R(y,d')$$

$$U_z = C(y,z) D(d -2z/\bar{v}) = - C(y,z) R(y,d-2z/\bar{v})$$

$$U(y,d,z=0) = \int_0^{\bar{v}d/2} C(y,z') R(y,d-2z'/\bar{v}) dz'$$

$$R(y,d) = \int_0^{\bar{v}d/2} C(y,z') R(y,d-2z'/\bar{v}) dz$$

If we discretize time $j \Delta t = d$, and depth $k \Delta z = z$; and as usual define $\Delta z = \bar{v} \Delta t/2$ we have, writing the above eqn. as a discrete summation,

$$r_j = \sum_{k=o}^{j} c_k r_{j-k} = \sum_{k=o}^{j} c_{j-k} r_k$$

however, with the constraint that $c_0 = 0$ and the assumption that $r_0 = 1$ it may be rewritten as

$$r_j = c_j + \sum_{k=1}^{j-1} r_k c_{j-k} \tag{23}$$

which is identical with the one-dimensional Noah Algorithm of equation (3b) in the section entitled "One Dimensional Noah's Deconvolution."

Doing the Forward Calculation

Equations (22) is what we wish to program on the computer. In many respects it is simply the migration/diffraction equation described in detail elsewhere in this report. The difference lies in the source term included in the equation (22a) governing upcoming transmitted and reflected waves. Let us discretize the coordinates as follows: $y = j\Delta y$, $z = k\Delta z$, $d = n\,\Delta d$, and $d' = n'\Delta d$. We denote approximations to $U(y,d,z)$ as $U^{n}_{k,j}$ and to $D(y,d',z)$ as $D^{n'}_{k,j}$. We will use the finite difference scheme of Crank and Nicolson. In this method we center all differencing at the location $(n + 1/2 , k + 1/2 , j)$. Let us consider the solution for the upcoming waves since the case of a source-free downgoing equation has been discussed in other sections. Writing the Crank-Nicolson formulation of equation (22a), with $b = \bar{v}/2\ (1/2+e/2r)^2$, we have the lengthy expression

$$
\begin{aligned}
U^{n+1}_{k+1,j} - U^{n}_{k+1,j} - U^{n+1}_{k,j} + U^{n}_{k,j} &= \\
= \frac{b\ \Delta z\ \Delta d}{4}\,\delta^2_y[\,U^{n+1}_{k+1,j} + U^{n}_{k+1,j} + U^{n+1}_{k,j} + U^{n}_{k,j}\,] \\
+ \frac{1}{2}[\,C_{k+1,j}\,(D^{n+1-2k\Delta z/\bar{v}}_{k+1,j} - D^{n-2k\Delta z/\bar{v}}_{k,j}) \\
+ C_{k,j}\,(D^{n+1-2k\Delta z/\bar{v}}_{k,j} - D^{n-2k\Delta z/\bar{v}}_{k,j})\,]
\end{aligned}
\tag{24}
$$

where δ^2_y is the second space differencing operator defined by

$$
\delta^2_y\,U^{n}_{k,j} = \frac{U^{n}_{k,j+1} - 2\,U^{n}_{k,j} + U^{n}_{k,j-1}}{(\Delta y)^2}
$$

In order to simplify the notation and additionally, since we wish to concentrate on the end effects in the (z,d) plane, we will rewrite (24) in terms of a vector U^{n}_{k} having the elements $(U^{n}_{k,1}, U^{n}_{k,2}, \cdots)$ and similar vectors for D^{n}_{k} and C_k . Thus U^{n}_{k} and D^{n}_{k} refer to the up and downgoing waves at a particular time and depth at all midpoints along the profile coordinate. In this notation we may write the second space differencing as

$$\begin{Bmatrix} \delta_y^2 \; U_{k,1}^n \\ \delta_y^2 \; U_{k,2}^n \\ \vdots \\ \delta_y^2 \; U_{k,J}^n \end{Bmatrix} = - \frac{T \, U_k^n}{(\Delta y)^2} \tag{25}$$

where T is a tridiagonal matrix with $(-1, 2 -1)$ on the diagonal. Thus, letting $a = \frac{b \, \Delta z \, \Delta d}{4(\Delta y)^2}$ we have

$$[U_{k+1}^{n+1} - U_{k+1}^{n} - U_{k}^{n+1} + U_{k}^{n}] = a\, T[U_{k+1}^{n+1} + U_{k+1}^{n} + U_{k}^{n+1} + U_{k}^{n}]$$

$$+ \frac{C_{k+1}}{2} (D_{k+1}^{n+1-2k\Delta z/\bar{v}} - D_{k+1}^{n-2k\Delta z/\bar{v}}) \tag{26}$$

$$+ \frac{C_{k}}{2} (D_{k}^{n+1-2k\Delta z/\bar{v}} - D_{k}^{n-2k\Delta z/\bar{v}})$$

Let us now define an upward source term which is the downgoing wave multiplied with the reflection coefficient

$$S_k^n \triangleq C_k \, D_k^{n-2k\Delta z/\bar{v}}$$

Equation (26) in terms of source waves becomes

$$[I-aT](U_{k+1}^{n+1} + U_{k}^{n})-[I+aT](U_{k+1}^{n} + U_{k}^{n+1})- \frac{(S_{k+1}^{n+1} - S_{k+1}^{n} + S_{k}^{n+1} - S_{k}^{n})}{2} = 0 \tag{28}$$

Let us now examine the (z,d) plane of the upcoming waves (see fig. 2). Recall that the y-coordinate has been absorbed in the matrix notation and thus it is important to keep in mind the third dimension. Each cell in figure 2 represents an end view of the wave field along the y-axis. The column of cells at $k = 0$ $(z=0)$ is the upcoming wave field at the surface. The diagonal line of cells represent the first arrival trajectory in (z,d) of a downgoing surface disturbance initiated at t = 0 . Above this diagonal the upcoming waves must vanish since a reflected wave cannot exist prior to the first arrival of a downgoing wave. Thus we are left to dealing with waves in the inner triangular region bounded by the reflection seismogram at the surface and the first arrival trajectory.

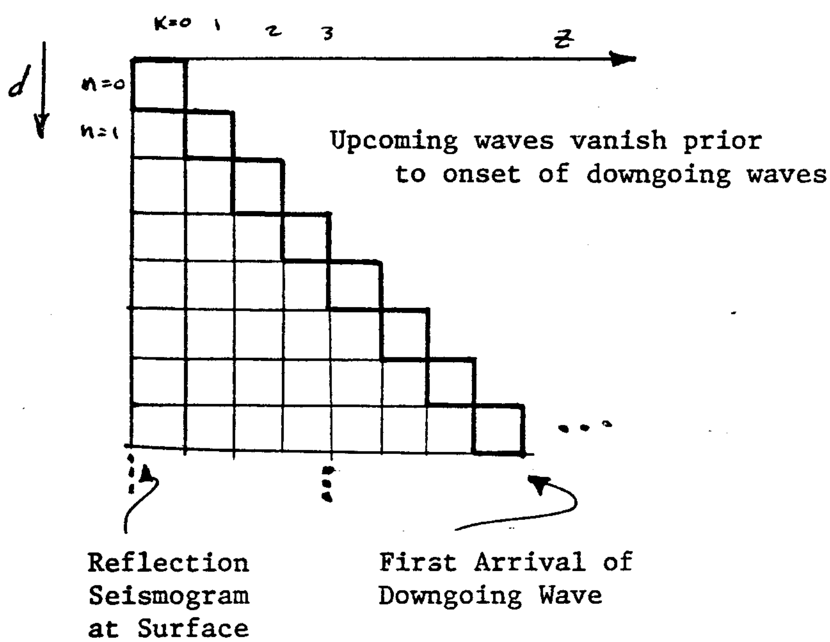

Fig. 2. Upcoming wavefields observed in the (z,d) plane. Normal to this plane is the y (profiling) axis. Eqn. (29) propagates upcoming waves with reflections across the inner triangular region.

The region of interest for the source wave fields is similarly restricted to the same triangle. However, at the surface we will require that the source vanish. Reflections at the free surface will be separately handled by the boundary condition as stated in equation (22c).

Equation (28) may be thought of as a two-dimensional convolution operator acting on four neighboring cells in the (z,d) planes of the U and S waves. If we denote by ⊛ the two-dimensional matrix convolution we may represent equation (28) as

$$\begin{array}{c|c|c|} & k & k+1 \\ \hline n & (I-aT) & -(I+aT) \\ \hline n+1 & -(I+aT) & (I-aT) \\ \hline \end{array} \circledast U - \begin{array}{c|c|c|} & k & k+1 \\ \hline n & -1/2 & -1/2 \\ \hline n+1 & 1/2 & 1/2 \\ \hline \end{array} \circledast S = 0 \qquad (29)$$

where the 2 x 2 operators are laid-down on corresponding cells in the U and S (z,d) plane. Apparently, we have four possible directions (unknowns) to move in. Knowing S and say U_k^n , U_k^{n+1} , and U_{k+1}^n we might try to solve for U_{k+1}^{n+1} via

$$[\ I-aT\ ]\ U_{k+1}^{n+1} = [\ I+aT\ ](U_{k+1}^{n} + U_k^{n+1}) - [\ I-aT\ ] U_k^n + (S_k^{n+1} - S_k^n + S_{k+1}^{n+1} - S_{k+1}^n)$$

However, we find that the matrix I-aT cannot be inverted and U_{k+1}^{n+1} cannot be computed this way. This is also the case for U_k^n unknown. This mathematical predicament is a result of violating causality by attempting to migrate or diffract a wave in an unnatural physical direction. We may think of the problem of trying to migrate a point source diffraction without knowledge of the hyperbolic tales. The two allowable directions, guaranteed stable numerically and causal physically, are ways we move in solving for U_{k+1}^n and U_k^{n+1} .

This is not the case when we have a one-dimensional earth. We then may propagate waves freely in any direction we choose. This fact led to a very simple 1D inversion algorithm, but due to this physical constraint, the 2D inversion is not so easily approached.

In order to best illustrate the steps involved in the forward calculation, let us consider in detail one cycle in the algorithm. We'll need the convolution operator for propagating downgoing waves

$$d' \downarrow \quad \xrightarrow{\quad z \quad}$$

$$\begin{array}{c|c|c|} & k & k+1 \\ \hline n' & -(\ I+aT\ ) & (\ I-aT\ ) \\ \hline n'+1 & (\ I-aT\ ) & -(\ I+aT\ ) \\ \hline \end{array} \circledast D = 0 \tag{30}$$

Note that the allowable directions are, in this case, solving for $D_k^{n'}$ and $D_{k+1}^{n'+1}$. For convenience at this point we'll also define a depth sampling such that $\Delta z = \bar{v}\ \Delta d/2$. Thus, for the presumed frame velocity $\bar{v}$ the 45° diagonal cells represent the theoretical first arrival trajectories.

Let us assume that we know C_k for all space, and further that the first portion of the seismogram R has already been computed. Referring to figure 3 the boundary conditions (22 c,d) give us the values for the cells at the left-hand side of the downgoing grid. With the operator (30) we may continue these wave from surface cells (d_o^o , d_o^1 , d_o^2 , d_o^3) to fill out the grid. The next step is to shift the time axis d' into the d-coordinate system as per $d = d' - 2z/\bar{v}$ or for $\Delta z = \bar{v}\ \Delta d/2$; $n = n' - k$. This downgoing wave referenced in the upcoming system is then cross-multiplied onto the known reflection coefficients generating the upward source terms S_k^n .

These waves, together with the transmitted upcoming waves are brought to the surface by operator (29) and the new row (cell r_4) is developed on the reflection seismogram. The cycle continues by reinserting this wave, after an appropriate 180° phase shift, at the surface back into the downgoing table.

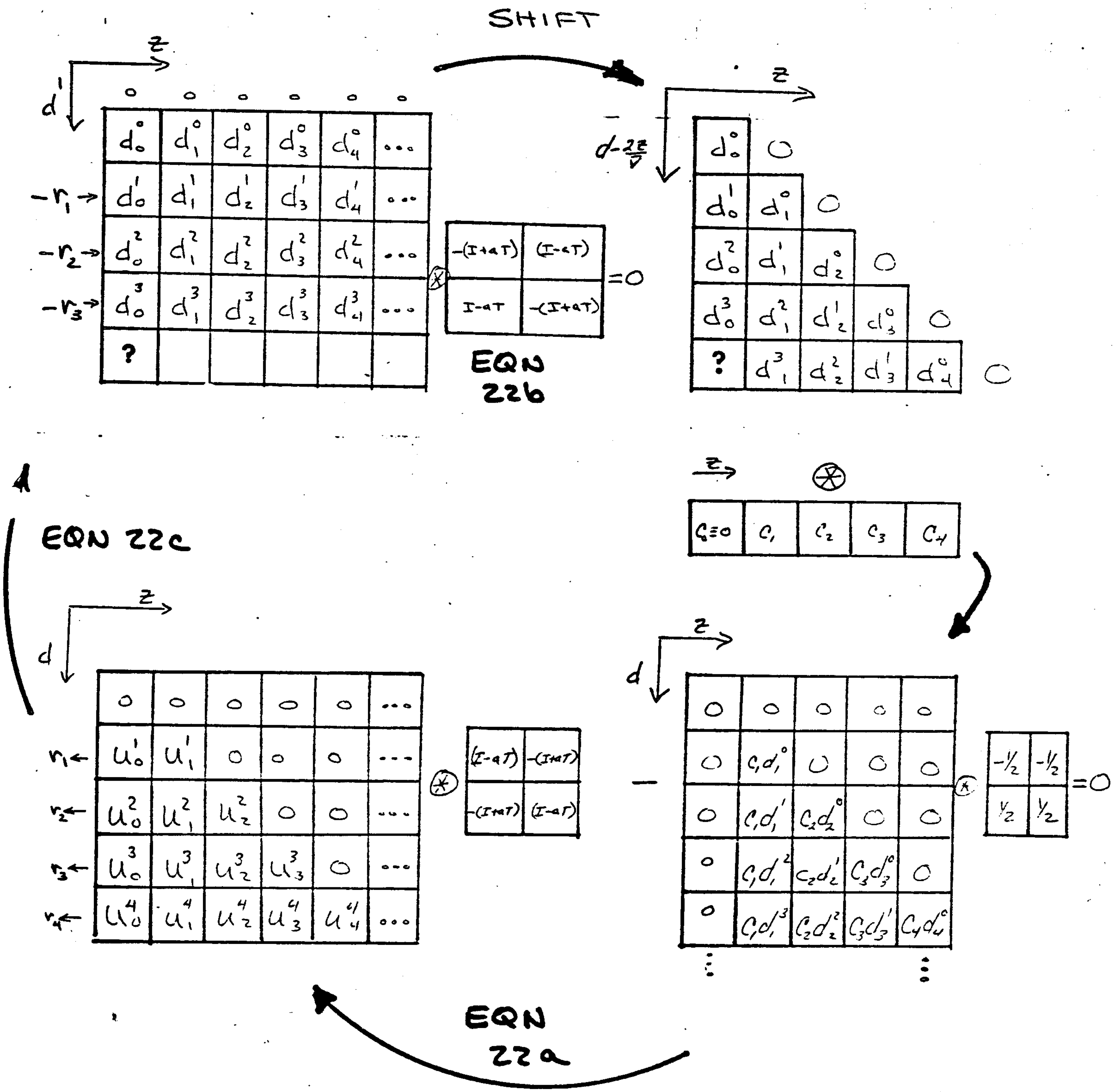

Figure 3.

Two Dimensional Gating Techniques

The number of numerical operations in propagating from the first arrival diagonal up to the surface is obviously proportional to the width or depth of the region spanned. Thus, as we move to later time on the seismogram we find ourselves faced with a growing computational procedure that quickly gets out of hand. If we take a lead from the one-dimensional algorithm, we recognize that some economizing gating may also be done in the two-dimensional algorithm.

First consider short-diffraction-path multiples, i.e. those waves which eventually become trapped in the water layer. These include both returning deep reflections (peglegs) and simple seafloor multiples confined to the water path. If we gate - in the seafloor reflection coefficients, say bounded by N_{1S} N_{2S} , then we will properly model multiples of this class.

On the other extreme we have the long-diffraction-path multiples. These include those waves which reverberate one or more times in the water layer prior to entering the subsurface. Thus, if we gate-in the front portion of the downgoing wave containing these downgoing multiples we approximately model the long-path class. Thus, the long-path gates N_{1L} , N_{2L} encompass as much of the downgoing wave energy as might be considered significant. Figure 4 illustrates the arrangement of these gates in terms of the (z,d) plane diagrams of the D (shifted) and S wavefields.

We may expect the long and short paths to represent distinct multiple processes in the presence of appreciable seafloor or structure topography. A wave transmitted a great distance into the earth, reflected back and trapped by the water layer would be successively stretched by the seafloor topography. However, even for moderate sea depths, this may amount to a small diffraction compared to where the wave gets deformed prior to the long diffraction path. We may expect time coincident arrivals of different paths through the same medium to consist of both large and small diffracted components.

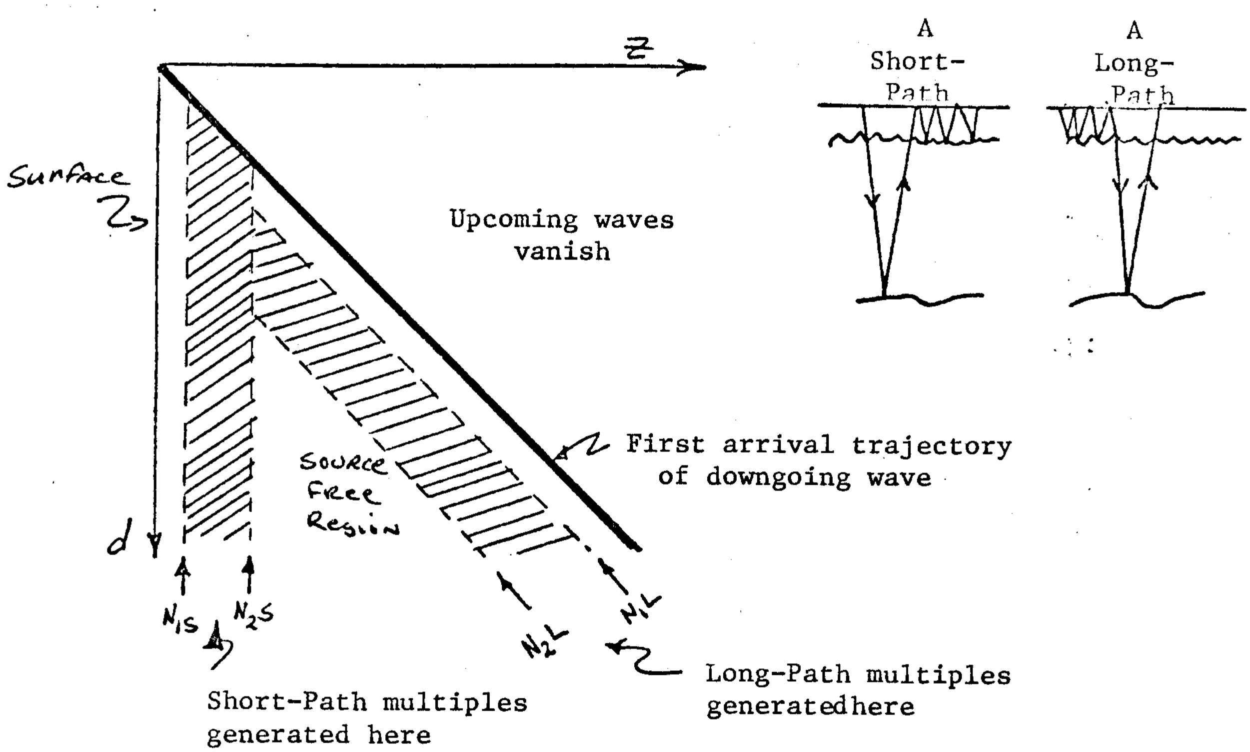

Fig. 4. Gating arrangement for discriminating long and short path multiples. Great economy can be achieved in upward continuing wave fields through source-free regions. For variable seafloor with y the gating may be designed to accomodate such changes.