Two Techniques for Wave Equation Migration

by Jon F. Claerbout

In order to program the migration technique described in the Claerbout-Doherty paper (1972) it is first necessary to master the manipulation of tri-diagonal matrices as described in detail in the Claerbout (1970) paper. Next it is necessary to be able to program the time dependent method as described in some detail in the Claerbout-Johnson (1971) paper. If you are still reading you will know that in simplest form migration deals with the up-coming wave equation

$$U_{tz} = - U_{xx} \tag{1}$$

In this situation we are assuming the velocity to be independent of x and the reflectors to be arranged in two dimensions to satisfy the Fresnel Approximation. Equation (1) could of course be fourier transformed over all three coordinates, but we will transform away only the x-coordinate because we are interested in end-effect interactions on the t and z coordinates. Using $\exp(i k x)$ horizontal dependence equation (1) becomes

$$U_{tz} = k^2 U \tag{2}$$

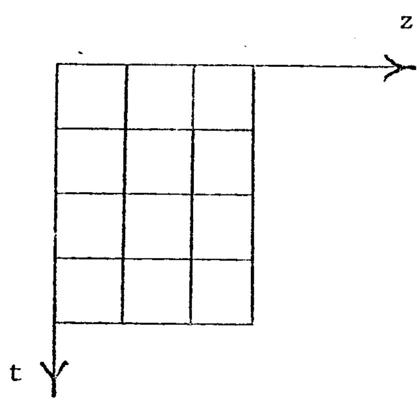
In practice we have always programmed (1) although there seems to be no reason why migration could not be programmed in the spatial frequency domain with (2) exactly as we will now describe it. First we re-express equation (2) as a two dimensional convolution in the $z - t$ plane. Letting $*$ denote two dimensional convolution the equation $k^2 U - U_{tz} = 0$ may be written as

$$\left\{ \frac{k^2}{4} \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array} \; - \frac{1}{\Delta z \; \Delta t} \left[\begin{array}{c|c} 1 & -1 \\ \hline -1 & 1 \end{array} \right] \right\} * U = 0 \qquad (2a)$$

Since k^2 is positive the sum of the two operators always has $|b| \geqslant |s|$ in the form

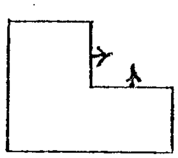
$$\left[\begin{array}{c|c} s & b \\ \hline b & s \end{array} \right] * U = 0 \qquad (2b)$$

Now we consider the task of using (2) to fill in U on a table like



(3)

From equation (2b) we see that given the appropriate three values of U a fourth may be determined by either of the two operations



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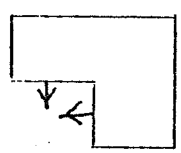
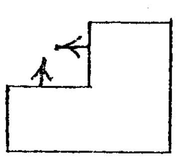


(4a,4b)

Because $|b| \geqslant |s|$ the filling operations implied by



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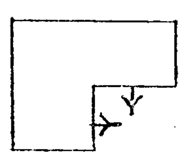
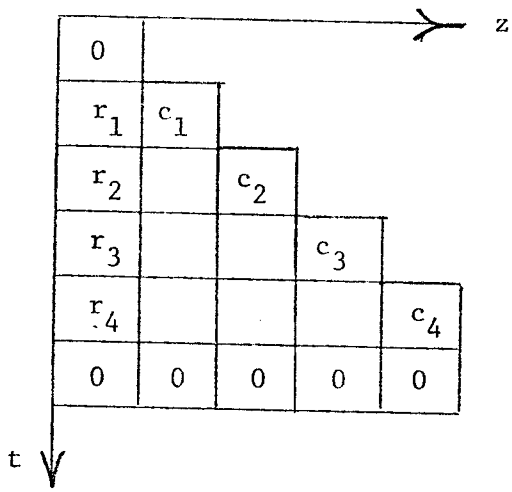


(5)

if continued would quickly lead to divergence as does polynomial division with a non-minimum phase filter. This is the mathematical manifestation of the physical idea of causality. Usually we think of migration and data synthesis on the following grid.



(6)

On this grid $r_0, r_1, \cdots, r_4$ represents the observed surface seismogram and $c_0, c_1, \cdots, c_4$ represents the migrated section. The zeros in the bottom row represent the notion that seismograms vanish at a sufficiently late time. Notice that in filling in the table there is a lot more work (diffraction) in going from r_4 to c_4 than in going from r_2 to c_2. This is because for fixed dip, deep events migrate farther than shallow events. Letting $k^2 = 0$ we are describing up going plane waves. Then for $\Delta z = 1$ and $\Delta t = 1$ we have $b = +1$ and $s = -1$. Starting with surface data $r_0, r_1, \cdots,$ and the bottom row of zeros we can use (4a) to fill in the table (6) and it becomes z →, t ↓

0				
r_1	r_1			
r_2	r_2	r_2		
r_3	r_3	r_3	r_3	
r_4	r_4	r_4	r_4	r_4
0	0	0	0	0

(7)

Inspecting the table (7) we see that numbers remain absolutely constant as we move in z-direction. If we had not taken $k^2 = 0$ but had instead taken k^2 to be small we would see the numbers in the table changing gradually in the z-direction. From the point of view of solving hyperbolic differential equations it is most economical if you can get roughly the same number of points per wavelength (typically 8) on each coordinate axis. This means that in the development till now we have drastically over-sampled the z-axis. The 15 degree limitation of the Fresnel Approximation implies that Δz could always be taken five times or more coarse than $c\, \Delta t$. The subject of optimal selection of grid spacings is somewhat involved. Suffice it to say here that some of the field data migrated in the (1972) paper was done satisfactorily at an $\Delta z / c\Delta t$ ratio as great as 200. Anyway for the purpose of illustrating the point we will now redraw the up-coming wave table with twice as coarse a sampling on the z-axis. Numbers in the two tables below indicate two different possible orderings of use of (4a) for the migration calculation. In either case the resulting migrated section is interpolated (perhaps very crudely) off the diagonal.

z →, t ↓

0			
r_1			
r_2	5		
r_3	4		
r_4	3	8	
r_5	2	7	
r_6	1	6	9
0	0	0	0

z-outer

z →, t ↓

0			
r_1			
r_2	9		
r_3	8		
r_4	6	7	
r_5	4	5	
r_6	1	2	3
0	0	0	0

t-outer

(7a,7b)

To synthesize data from a hypothetical depth section the calculation proceeds in reverse numerical order using (4b). The migrations in the 1972 paper were done by the method of the table on the left, which we call "z-outer" because the outer DO loop is on the z-coordinate and the inner DO loop is on the t-coordinate.

Study of the two optional procedures (7a) and (7b) reveals that the t-outer technique has an extraordinary advantage in the much smaller computer memory requirement.

Finally we will have a more rigorous and general development. Equation (46) in the 1970 paper is

$$p_z^- = -\frac{i}{2m}\left(\varepsilon\, m^2\, p^- + p_{xx}^- + \frac{im}{2}\, \varepsilon_z\, p^+\, e^{i2mz} \right) \tag{8}$$

We assume homogeneous velocity by dropping ε . Recall $m = \omega/\bar{c}$ where $\bar{c}$ is the velocity. Multiply through by $-i\omega$. Use U for p^- and D for p^+ . We get

$$- i\omega U_z = -\frac{\bar{c}}{2} \left(U_{xx} - \frac{\varepsilon_z}{2\bar{c}} (-i\omega D)(e^{i2z\omega/\bar{c}})\right) \quad (9)$$

Inverse Fourier transform.

$$U_{zt} = -\frac{\bar{c}}{2} U_{xx} + \frac{\varepsilon_z}{2\bar{c}} D_t * \delta(t - 2z/\bar{c}) \quad (10)$$

This result is like (1) but the fact that the upcoming waves originate from a coincidence of downgoing waves with a reflector (non-zero ε_z) is explicitly included. Defining reflection coefficient as $c(z) = -\varepsilon_z/2\bar{c}$. We get

$$\partial_{tz} U (x,z,t) = -\frac{\bar{c}}{2} \partial_{xx} U (x,z,t) - c(z)\, \partial_t D (x,z,t-2z/\bar{c}) \quad (11)$$

A confusing aspect of (11) is that time t does not have the same time origin at each depth. It is like describing ballistics in local solar time rather than universal time. The time t seen in (11) represents the arrival or departure time for energy at the earth's surface. Thus a surface downgoing wave at time $t - 2z/\bar{c}$ can interact with a reflector $c(z) = \delta(z)$ at depth z to create an up-coming wave which arrives at the surface at time t. The downgoing wave D can be generated in a table with the diffraction equation. Specializing to down going plane waves this gives a table in which waveforms don't change as the wave goes in the z-direction

→ z

r_0	r_0	r_0	r_0
r_1	r_1	r_1	r_1
r_2	r_2	r_2	r_2
r_3	r_3	r_3	r_3

↓ t = leaving the surface time

(12)

Changing to a surface-arrival-time table we shift successive columns down

→ z

r_0			
r_1	r_0	zeros	
r_2	r_1	r_0	
r_3	r_2	r_1	r_0

↓ t = surface arrival time

(13)

To find the sources for up-coming waves at time t each column of (13) should be multiplied by the reflection coefficient for the appropriate z.

Now assume that the downgoing wave has an impulsive wave form. This excludes the possibility of multiple reflections. Then the table for sources of upcoming waves is

(z →, t ↓)

0	0	0	0
0	c_1	0	0
0	0	c_2	0
0	0	0	c_3

(14)

In tabular form equation (11) at zero dip (i.e. $k^2 = 0$) for $\Delta z = 1$ and $\Delta t = 1$ becomes

$U \cdot (-\partial_{zt}) = \text{source} \; \partial_t$ (15b)

U

0	0	0	0
c_1	$c_1/2$	0	0
c_2	c_2	$c_2/2$	0
c_3	c_3	c_3	$c_3/3$

*

$(-\partial_{zt})$

-1	1
1	-1

=

source

0	0	0	0
0	c_1	0	0
0	0	c_2	0
0	0	0	c_3

*

∂_t

-1/2	-1/2
1/2	1/2

(15b)

The generalization to arbitrary horizontal wavenumbers k is obvious and the generalization to spatial x-dependence is exactly as in the 1970 and 1971 papers.

References

Claerbout, J. F., 1970, Coarse grid calculations of waves in inhomogeneous media with application to delineation of complicated seismic structure, Geophysics, vol. 35, no. 3, pp. 407-418.

Claerbout, J. F., and Johnson, A. G., 1971, Extrapolation of time dependent waveforms along their path of propagation, Geophy. Journal of the Royal Astronomical Society, vol. 26, pp. 285-293.

Claerbout, J. F., and Doherty, S. M., 1972, Downward continuation of moveout corrected seismograms, Geophysics, vol. 37, no. 5, pp. 741-768.

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Oct. 31, 1973

First problem for Post-Doc:

The wave equation in 2-D Cartesian coordinates

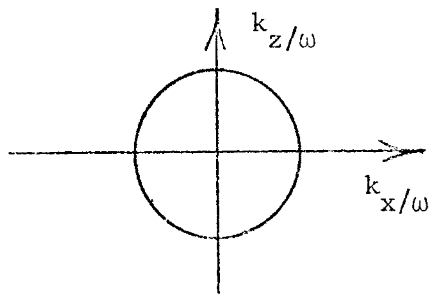
$$P_{xx} + P_{zz} = \frac{1}{c^2} P_{tt}$$

The trial solution is

$$P = Q_0 \exp (-i\omega t + ik_x x + ik_z z)$$

Putting trial solution in wave equation gives the equation of a circle

$$\frac{\omega^2}{c^2} = k_x^2 + k_z^2$$



For geophysical reasons we want an equation with top semi-circle only

$$k_z = + (\omega^2/c^2 - k_x^2)^{1/2}$$

The equation is

$$P_z = \frac{i\omega}{c} (1 + \frac{c^2}{2\omega^2} \partial_{xx} + \frac{c^4}{8\omega^4} \partial_{xxxx} + \cdots) P$$

Presently we solve above by an initial value technique in z, say Crank-Nicolson, and a band matrix in x (usually truncate ∂_{xxxx} and higher).

We believe infinite series approach to square root is much inferior to iterative techniques. We haven't been able to develop a stable iterative technique.

Noah's Method of Deconvolution by Flooding

Jon Claerbout
8 October 1973 : se

The idea here is that it is mainly the presence of the free surface perfect reflector which causes a practical problem with multiples. We wish to replace the seismogram $A = 1 + 2R$ by the seismogram $-C$ in figure 1.

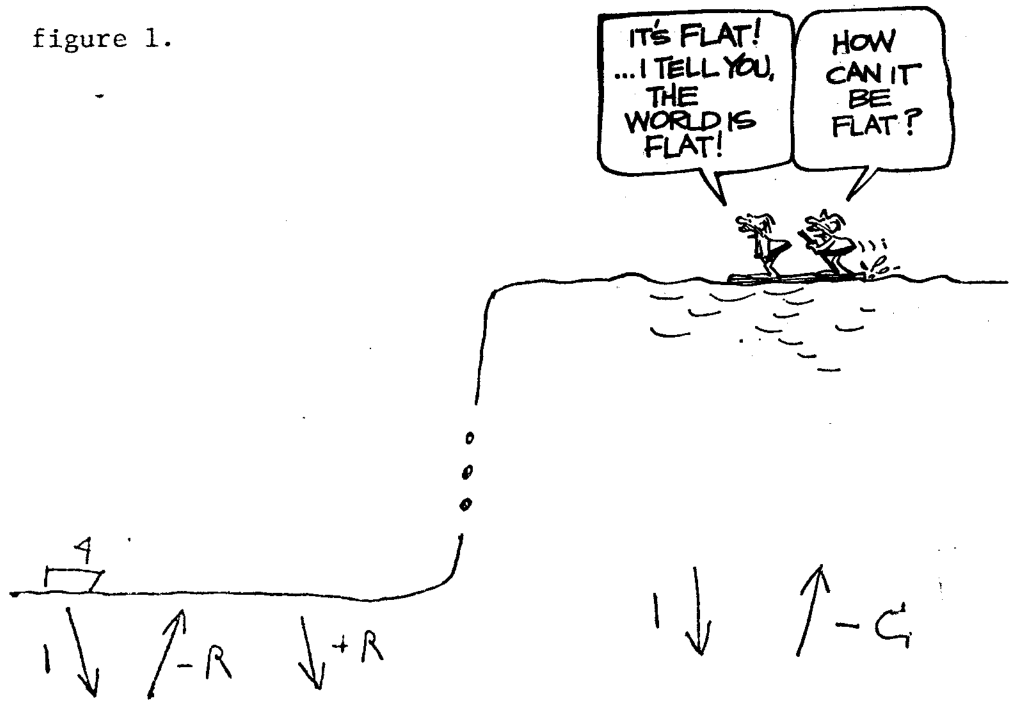


Fig. 1. Our waves on the left, Noah's on the right.

For us the down-going and up-going waves are

$$D = 1 + R$$

$$U = \quad - R$$

and for Noah they are

$$D = 1$$

$$U = - C$$

The admittance A of the earth at ordinary sea level is given by

$$\text{Admittance} = \frac{\text{velocity}}{\text{pressure}} = \frac{D - U}{D + U} = 1 + 2R$$

Noah's seismogram in terms of ours is

$$C = \frac{-U}{D} = \frac{R}{1 + R} = \frac{2R}{2(1+R)} = \frac{A - 1}{A + 1}$$

In terms of z-transforms this is

$$(A(z) + 1)\, C(z) = A(z) - 1$$

Now collect the coefficient of z^t for t greater than zero.

$$c_t + \sum_{k=0}^{t} a_{t-k}\, c_k = a_t$$

If c_t is taken to be unknown but c_{t-1}, c_{t-2}, ... are known then we can get c_t by the recursion

$$(1 + a_0)\, c_t = a_t - \sum_{k=0}^{t-1} a_{t-k}\, c_k \qquad (1)$$

Theoretically $a_0 = 1$ but there is an unknown scale factor, say $2u$ in $a_1, a_2, \ldots a_\infty$ (among other problems). Thus (1) becomes

$$c_t = u\left(a_t - \sum_{k=0}^{t-1} a_{t-k} c_k \right) \qquad (2)$$

Note that c_t immediately feeds back to the calculation of c_{t+1}. Because of the finite extent of the source wavelet it is proposed that (3) will have the good features of (2) but will be safer.

$$c_t = u \left(a_t - \sum_{k=N_1}^{t-1} a_{t-k}\, c_k \right) \tag{3}$$

Figure 2 indicates possible choice of parameters N_1 , N_2 , and N_3 .

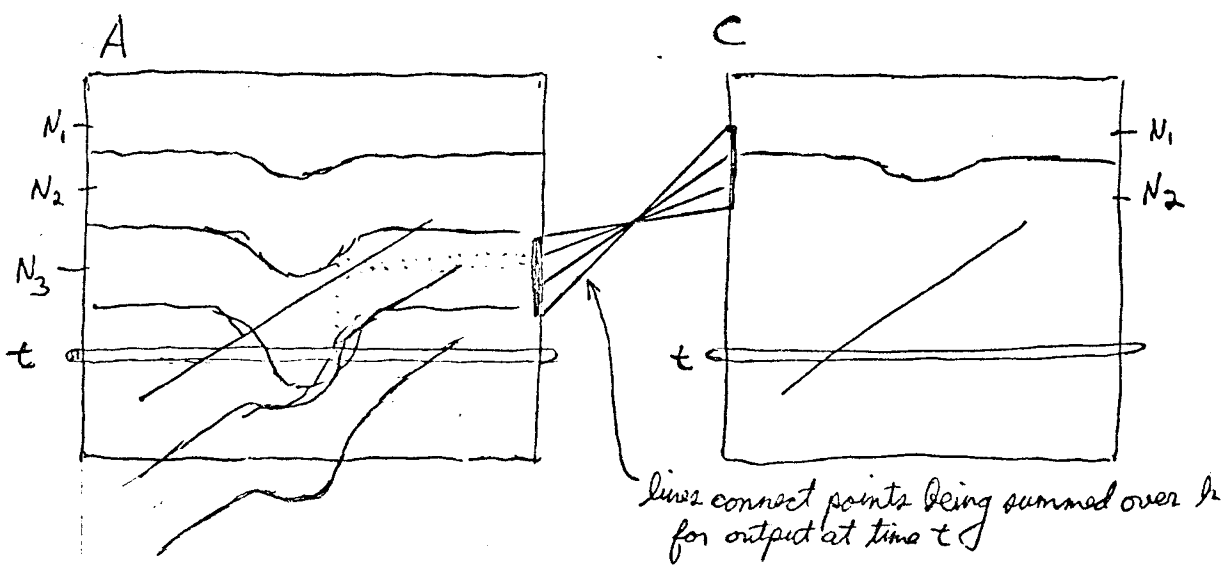


Fig. 2. Our seismogram on the left and Noah's on the right.

Some economy can be achieved if it is desirable to eliminate only sea floor multiples and sea floor peg-legs by limiting the sum in (3) to a maximum index of N_2

$$c_t = u \left(a_t - \sum_{k=N_1}^{N_2} a_{t-k}\, c_k \right) \tag{4}$$

The propagation of the unknown u during the computation goes as indicated in (5)

$$c_t = u \left(a_t - \sum_{k=N_1}^{N_2} a_{t-k}\, c_k \right) \tag{5}$$

Because of this propagation of u and our belief that u can be estimated from the relative strength of the primary and first multiple we get u by the minimization

$$\min_u \sum_{t=N_2}^{N_3} \left(a_t - u \sum_{k=N_1}^{N_2} a_{t-k} a_k \right)^2 \tag{6}$$

To understand (6) define y_t, the convolution of primary on itself by

$$y_t = \sum_{k=N_1}^{N_2} a_{t-k} a_k \tag{7}$$

The minimization (6) is trying to extinguish the multiple by means of the primary convolved on itself. This works perfectly if the gate contains no new structure primaries. Notice that u becomes 1 if the primary has a magnitude equal the reflection coefficient. Finally, let us admit that u is really a waveform, not a scalar. It should be the inverse of the waveform actually transmitted into the earth. Figure 3 shows why u_t is taken to be an anti-causal filter.

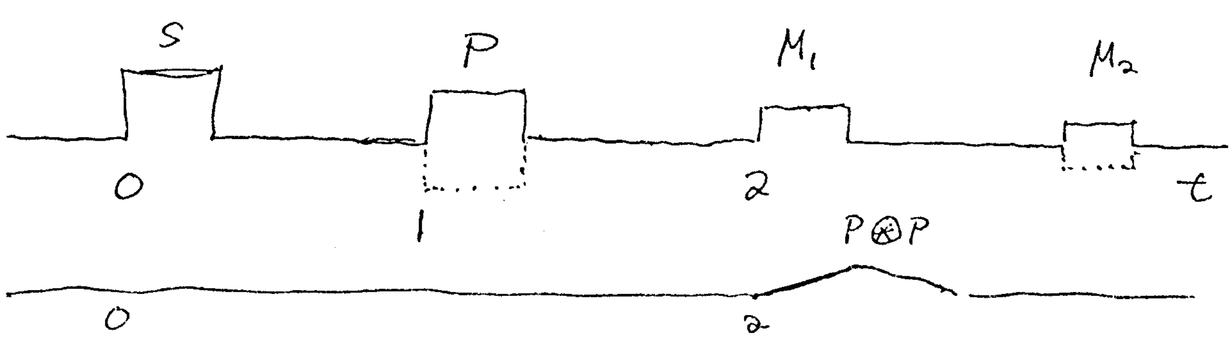


Fig. 3. Since P ⊗ P comes later than M_1 we use an anti-causal filter to push it to an earlier time.

Thus (6) with (7) becomes

$$\min_{u} \sum_{t=N_2}^{N_3} \left(a_t - \sum_{k=0}^{-(N_1 \text{ or less})} u_k \, y_{t-k} \right)^2 \tag{8}$$

Once u_k has been estimated we generalize (4) to a convolution with u_k. Then (4) is run out to large values of time in a completely deterministic fashion. No reflection coefficients are estimated. In generalizing this to diffracting waves a rough guess is that letting z be the outward diffraction operator then (4) becomes

$$c_t = \underbrace{z^{-t}}_{A} \underbrace{\left(a_t - \sum_{k=N_1}^{N_2} \underbrace{z^{k/2} \underbrace{\left(c_k \underbrace{\left(z^{k/2} a_{t-k} \right)}_{B} \right)}_{C}}_{D} \right)}_{E} \tag{9}$$

where the lettered operations have the following interpretation

A: migration of primaries

B: diffraction of downgoing N^{th} multiple

C: reflection at depth k with reflection coefficient c_k

D: diffraction of upcoming wave to give $N + 1^{st}$ multiple

E: cancellation of multiples but not primaries

A more precise statement of (6) with more gating possibility is

$$\min \sum_{t=N_3}^{N_4} \left(a_t - u \sum_{k=N_1}^{\min(t-1,N_2)} a_{t-k} \, a_k \right)^2 \tag{10}$$

Defining the gate $N_4 - N_3$ to be larger than before we include the possibility of estimating u by fitting the N^{th} multiple to the $N + 1^{st}$. Expanding the $N_2 - N_1$ gate includes the fitting of the first multiple self convolved to the third multiple.

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