10

10-1 CLASSICAL INITIAL-VALUE PROBLEMS IN TIME

It is easy to cover fundamentally with the heat-flow equation in one dimension. The boundary condition, first says that the heat flows, or the temperature gradient, is proportional to the temperature. The second says that the temperature gradient is in proportion to the divergence of heat flow and inversely proportional to the heat capacity of the material.

\[
H = -\sigma \sqrt{\frac{\partial T}{\partial x}} \quad (0.1-1)
\]

\[
\frac{\partial T}{\partial t} = \frac{1}{C} \frac{\partial H}{\partial x} \quad (0.1-2)
\]

\[
\frac{\partial T}{\partial t} = \frac{\partial T}{\partial x} \quad (0.1-3)
\]

The usual procedure is to insert (0.1-1) into (0.1-2) and reflect the derivative of \( T \) with respect to \( x \).

The usual convention in difference equation theory is that temperature \( T_i \) at time \( t \) is written as \( T_{i,j} \), where the subscript denotes time. With the definition \( \Delta t = \sigma / (C \Delta x) \), the derivative of \( T \) can be written

\[
T_{i,j} - T_{i,j-1} = 2\Delta t (T_{i+1,j} - 2T_{i,j} + T_{i-1,j}) \quad (0.1-4)
\]

The temperature \( T_i \) of the spatial position \( k \) for some particular time \( t \), then (0.1-4) may be used to calculate the temperature at \( T_{i,j} \) for some particular time \( t \). The reader should notice that the time derivative is centered at \( T_{i,j} \). This can cause difficulties for the heat-flow equation, as the heat flow is centered at \( T_{i,j} \). The heat-flow difference equation slows the wave down too rapidly. The heat-flow difference equation does the same thing; except that the very short wavelengths will sense the difference in the centering of time and space derivatives.

The result is that the very short wavelengths will not attenuate more and more, thereby making it possible to contain shorter and shorter wavelengths with the grid, simplifying the solution. The situation, called circular, always occurs, therefore, we always hope that the difference in the time difference. But it turns out even worse and creates instability in any \( \sigma / T \). The reason is that the heat-flow difference equation is first-order in time and second-order in space. In this case, the heat loss is the basis, in addition to the heat-flow difference equation, which turns over and over again the heat-flow diffusion.

These problems may all be avoided with the Crank-Nicolson scheme. It will always guarantee stability for any \( \sigma / T \), and can also be applied to the wave equations.
in acoustics, electromagnetics, and elasticity. In the Crank-Nicolson scheme one centers the space difference at $T_{i+1/2}^{n+1}$ in the following way:

$$T_{i+1}^{n+1} - T_i^n = b(T_{i+1}^{n+1} - 2T_i^n + T_{i-1}^n) + b(T_{i+1}^{n+1} - 2T_{i+1}^{n+1} + T_{i-1}^{n+1})$$

(10-1-5)

An apparent problem with the Crank-Nicolson scheme is that the method of getting the $n+1$ time level from the $n$ level is no longer obvious. The $n+1$ terms in (10-1-5) to the left and the $n$ terms to the right, we have

$$-bT_{i+1}^{n+1} + (1 + 2b)T_i^n - bT_{i-1}^n = D_i^n$$

(10-1-6)

The right-hand side $D_i^n$ is a known function of $T^n$. What we have here is a set of simultaneous equations for the $T^n$. Writing this out in full, we see why the set is called a tridiagonal set of equations

$$
\begin{bmatrix}
(1 + 2b) & -b & & & & D_1^n \\
-b & (1 + 2b) & -b & & & D_2^n \\
& -b & (1 + 2b) & -b & & D_3^n \\
& & & \ddots & \ddots & \ddots \\
& & & & -b & -b & D_N^n
\end{bmatrix}
\begin{bmatrix}
T_0^{n+1} \\
T_1^{n+1} \\
T_2^{n+1} \\
\vdots \\
T_{N-1}^{n+1}
\end{bmatrix}
= 
\begin{bmatrix}
D_1 \\
D_2 \\
D_3 \\
\vdots \\
D_N
\end{bmatrix}
$$

(10-1-7)

It turns out that the simultaneous equations in (10-1-7) may be solved extremely easily. As will be shown later there is little more effort involved than in the use of (10-1-4). The scientist who wishes to solve partial differential equations numerically without becoming a computer scientist is well advised to use the Crank-Nicolson scheme. The extra effort required to figure out how to solve (10-1-7) is well rewarded by the ability to use any $\Delta x$ and $\Delta t$ and to forget about stability and the biasing effects of noncentral differences.

Now let us consider steady flow in two spatial dimensions. The heat-flow equation becomes

$$\frac{\partial T}{\partial t} = \frac{\kappa}{C}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

(10-1-8)

A simple, effective means to solve this equation is the splitting method. One uses the different equations at alternate time steps. They are

$$\frac{\partial T}{\partial t} = \frac{2\kappa}{C}\frac{\partial^2 T}{\partial x^2} \quad \text{(all y)}$$

(10-1-9a)

$$\frac{\partial T}{\partial t} = \frac{2\kappa}{C}\frac{\partial^2 T}{\partial y^2} \quad \text{(all x)}$$

(10-1-9b)

Each of these equations (10-1-9a) and (10-1-9b) may be solved by the Crank-Nicolson method.

There are much fancier methods than the splitting method, but their truncation errors (the asymptotic difference between the difference equation and the differential equation) do not go to zero any faster than the truncation error for the splitting method.

Now let us see how to formulate the acoustical problem in a Crank-Nicolson form. Let $u$ and $w$ denote velocities in the $x$ and $z$ directions. Let $P$ denote pressure, $\rho$ denote density, and $K$ denote incompressibility. Acceleration equal to pressure gradient gives

$$\frac{\partial u}{\partial t} = -\frac{\partial P}{\partial x}$$

$$\frac{\partial w}{\partial t} = -\frac{\partial P}{\partial z}$$

and pressure decreasing with the divergence of velocity gives

$$\frac{\partial P}{\partial t} = -K\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)$$

Arranging into a matrix and letting $\partial_i$ denote $\partial / \partial x$, etc., we have

$$\begin{bmatrix}
\partial & P \\
U & W
\end{bmatrix} = 
\begin{bmatrix}
0 & -K\partial_x \quad -K\partial_z \\
-\rho^{-1}\partial_x & 0 \\
-\rho^{-1}\partial_z & 0
\end{bmatrix}
\begin{bmatrix}
P \\
U \\
W
\end{bmatrix}$$

(10-1-10)

The implementation of (10-1-10) by a Crank-Nicolson scheme follows in a direct analogy to the implementation of (10-1-3). The principal difference is that we have vectors and matrices in (10-1-10) but only scalars in (10-1-3). When the splitting method is applied to (10-1-10) we have

$$\begin{bmatrix}
\partial & P \\
U & W
\end{bmatrix} = 
\begin{bmatrix}
0 & -K\partial_x \\
-\rho^{-1}\partial_x & 0 \\
-\rho^{-1}\partial_z & 0
\end{bmatrix}
\begin{bmatrix}
P \\
U \\
W
\end{bmatrix}$$

(10-1-11a)

and

$$\begin{bmatrix}
\partial & P \\
U & W
\end{bmatrix} = 
\begin{bmatrix}
0 & -K\partial_z \\
-\rho^{-1}\partial_z & 0 \\
-\rho^{-1}\partial_x & 0
\end{bmatrix}
\begin{bmatrix}
P \\
U \\
W
\end{bmatrix}$$

(10-1-11b)

at alternate time steps. When formulating boundary conditions for (10-1-11a) it turns out to be convenient to define $P$ and $U$ on alternate squares of a checkerboard. See Fig. 10-1.
A final matter of great practical significance is the fast method of solution to a tridiagonal set of simultaneous equations like (10-1-6) or (10-1-7). A slightly more general set of equations is

\[ A_k T_{k+1} + B_k T_k + C_k T_{k-1} = D_k \]  

(10-1-12)

For the heat-flow equation the elements of \( D_k \) are scalars. In other physical problems we may have to regard \( A, B, \) and \( C \) as 2 x 2 matrices, \( T_k \) as a 2 x 1 vector for each \( k \) at the \( n + 1 \) time level, and \( D_k \) as a 2 x 1 vector function of the field variables known at the \( n \) time level. The method proceeds by writing down another equation \( (E_k, F_k \) yet unknown) with the same solution \( T_k \) as (10-1-12)

\[ T_k = E_k T_{k+1} + F_k \]  

(10-1-13)

Write (10-1-13) with shifted index

\[ T_{k-1} = E_{k-1} T_k + F_{k-1} \]  

(10-1-14)

Insert into (10-1-12)

\[ A_k T_{k+1} + B_k T_k + C_k (E_{k-1} T_k + F_{k-1}) = D_k \]  

(10-1-15)

Rearrange (10-1-15) to resemble (10-1-13)

\[ T_k = -(B_k + C_k E_{k-1})^{-1} A_k T_{k+1} + (B_k + C_k E_{k-1})^{-1} (D_k - C_k F_{k-1}) \]  

(10-1-16)

Comparing (10-1-16) to (10-1-13) we see that they are the same, so that \( E_k \) and \( F_k \) may be developed by the recursions

\[ E_k = -(B_k + C_k E_{k-1})^{-1} A_k \]  

(10-1-17a)

\[ F_k = (B_k + C_k E_{k-1})^{-1} (D_k - C_k F_{k-1}) \]  

(10-1-17b)

Naturally when doing this on a computer for any case where matrices contain zeros, as in (10-1-11), one should use this fact to simplify things.

Now we consider boundary conditions. Suppose \( T_0 \) is prescribed. Then we may satisfy (10-1-13) with \( E_0 = 0, F_0 = T_0 \). Then compute all \( E_k \) and \( F_k \). Then if \( T_N \) is prescribed, we may use (10-1-13) to calculate successively \( T_{N-1}, T_{N-2}, \ldots, T_0 \). Another useful set of boundary conditions is to prescribe the ratios \( r_j = T_j/T_{j+1} \), and \( r_3 = T_0/T_N \). Begin by choosing \( E_0 = r_1, F_0 = 0 \). Compute \( E_k \) and \( F_k \). Then solve the following for \( T_N \). From (10-1-14)

\[ T_{N-1} = E_{N-1} T_N + F_{N-1} \]  

\[ T_N r_2 = E_{N-2} T_N + F_{N-1} \]  

\[ T_N = \left( \frac{1}{r_2} - E_{N-1} \right) F_{N-1} \]

Then compute \( T_{N-1}, T_{N-2}, \ldots \) as before.

As stated earlier, there are many more details associated with numerical solutions to partial differential equations. This chapter has given only the most important tricks for initial-value problems. A program to solve tridiagonal simultaneous equations is given in Fig. 10-2.

**EXERCISES**

1. Consider solving (10-1-8) by a Crank-Nicolson scheme in two dimensions on a 4 x 4 grid. This leads to a 16 x 16 set of simultaneous equations for the unknown \( T_{2x2} \). What is the pattern of zeros in the 16 x 16 matrix? The difficulty in actually solving this set gives impetus to the splitting method.

2. A difference approximation to the heat-flow partial differential equation is

\[ p_{x+1} - p_x = \frac{a \Delta t}{\Delta x^2} \left[ p_{x+1} - 2p_x + p_{x-1} \right] + s \]

utilizing the trial solution \( p = Q e^{(x-b)} \) reduce the equation to a one-dimensional difference equation. Write the reduced equation in terms of \( Z \) transforms. Does this equation correspond to a nondivergent filter for any real values of \( a \)? For any imaginary values of \( a \)? (Use a Fourier expansion for \( x \)).

3. Modify the computer program of Fig. 10-2 so that instead of prescribing zero-slope end conditions, (10-1-7) is solved.

4. Write a computer program to solve equation (10-1-6) with \( b = 0.5 \) and initial conditions \( T(10) \cdot T(20) = 0.0 \) and \( T(21) \cdot T(30) = 1.0 \). Use subroutine TRI.

**10-2 WAVE EXTRAPOLATION IN OPTICS**

In geophysics we generally have measurements along a line on the surface of the earth (x axis) from which we like to make deductions about earth properties below the surface. The first step is often to extrapolate observations at the earth’s surface in a downward direction.

Before looking at numerical methods of extrapolating wave fields in space it will be valuable to review quickly the methods used in optics to extrapolate waves through microscopes and telescopes. An enjoyable, more complete account will be found in Reference 35.

We will take a wave disturbance in two-dimensional cartesian geometry \( p(x, z, t) \) given at \( z_0 \) and show how it is extrapolated down the optic axis. Three common situations arise in the projection of a beam of light down an optic axis: First is the projection of a beam through an aperture or a photographic trans-
pareny. All that is required for a mathematical description is a transmittance function which ranges from 0 to 1 over the aperture or transparency. Taking the optic axis to be the z axis and restricting attention to two-dimensional geometry, the projection through an absorber \( T(x) \) located at \( z_0 + dz/2 \) is

\[
p(t, x, z_0 + dz) = T(x)p(t, x, z_0)
\]  

(10-2-1)

The second common situation is projection through a lens, often approximated as a “thin lens.” Here it is necessary to define a differential delay function \( \tau(x) \) which describes the time delay on propagation through the lens of a ray at \( x \) parallel to the z axis. If the lens is located at \( z_0 + dz/2 \), convolution of the wave field with a delayed impulse is represented as

\[
p(t, x, z_0 + dz) = \int p(t - s, x, z_0) \delta[s - \tau(x)] ds
\]

\[
= p[t - \tau(x), x, z_0]
\]

(10-2-2a)

This time shifting is simply expressed in the frequency domain where the convolution (10-2-2a) becomes a product. Then

\[
P(\omega, x, z_0 + dz) = P(\omega, x, z_0)e^{-i\omega \tau(x)}
\]

(10-2-2b)

The third common situation in optics is the projection of waves across a region of empty space. Surprisingly, this is the most difficult of the three projections. First we recall the wave equation

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) p(t, x, z) = 0
\]

(10-2-3)

Taking the velocity \( v \) to be a constant in time and space, we may use the trial solution

\[
p(t, x, z) = P(\omega, k_x, z)e^{-i\omega t + ik_xx}
\]

which reduces (10-2-3) to the ordinary differential equation

\[
\frac{d^2}{dz^2} P = -\frac{\omega^2}{c^2} P + k_x^2 P
\]

(10-2-4)

This equation has two solutions, \( e^{ik_xz} \) and \( e^{-ik_xz} \), where

\[
k_x = \left( \frac{\omega^2}{c^2} - k_z^2 \right)^{1/2}
\]

(10-2-5)

One of these solutions is a wave down the z axis and the other is a wave going up the axis. Initial conditions (and the no-backscattering approximation at lenses and apertures) enable us to reject one of the solutions, leaving us with

\[
P(\omega, k_x, z) = P(\omega, k_x, z_0)e^{i\omega(zt - z_0)}
\]

\[
= P(\omega, k_x, z_0)e^{i\omega \left[ (z - z_0)/k_z - \sqrt{k_z^2 - k_x^2} \right]}
\]

(10-2-6)

The right-hand side is a product of two functions of \( k_x \). It is also the product of two functions of \( \omega \). This means that with the standard tools of Fourier analysis we could recast (10-2-6) to a convolution in either the time domain or the space domain \( x \) or both. Converting the “filter” transfer function

\[
\exp \left( \frac{\omega^2}{c^2} - k_z^2 \right)^{1/2} (z - z_0)
\]

(10-2-7)

to the space domain will give us an “impulse response” which in this case has the physical meaning of the wave field transmitted through a point aperture. A beam emerging from a point aperture behaves somewhat like a beam from a point source. To recognize the difference, note that the transfer function (10-2-7) has a unit magnitude independent of \( k_z \) but, from Fig. 10-3, the spectral magnitude of a point source is lower near \( k_z = 0 \) and peaks up around \( k_z = \pm \omega/c \). This means that the aperture function does not radiate isotropically like the point source but...
FIGURE 10.4
A snapshot of the wave-equation transfer function. A double Fourier sum of \( \exp(i(\omega t k_x^2 - k_z^2)^{1/2}) \) was done over \( k_x \) and \( \omega \). We see a display of the \( (x, z) \) plane at a fixed \( t \). The result is semi-circular wavefronts with amplitude greatest for waves propagating along the \( z \) axis. Periodicity in \( x \) and \( t \) results from approximating Fourier integrals by sums.

FIGURE 10.5
Seismic profile type displays of the wave-equation transfer function. A double Fourier sum of \( (10-2-7) \) was done over \( k_z \) and \( \omega \). As with a collection of seismograms, we see the \( (x, t) \) plane for a fixed \( z \). The hyperbolic arrival times measure the distance from a point aperture at \( (0, 0) \) to the screen \( (x, z_0) \). Ray theory easily explains the travel time, but the slow amplitude decay along the hyperbola, an obliquity function, is a diffraction phenomenon not easily computed by analytic means, especially far off axis. The obliquity function should not be confused with the hash which arises from attempted representation of a delta function on a grid.

FIGURE 10.6
The real part of the exact transfer function, \( \exp(\sqrt{\omega^2 t^2 - k_z^2 z^2}) \), plotted \( k_z \) vs. \( z \) with \( \omega \) taken as constant. The abrupt change in character of the function occurs at \( \omega^2 t^2 = k_z^2 z^2 \), the transition between propagation and evanescence.

FIGURE 10.7a
Fourier transformation by a lens. A sinusoidal oscillation in the \( x \) domain results from a beam propagating through at some angle. A lens then converts the beam to a point in the \( k_x \) domain. The Fourier transform of a sinusoid is a delta function, the shift of the delta function from the optic axis is in proportion to the rate of oscillation of the sinusoid.
Given $P_0$ and $P_1$, we can determine $P_2$ with the help of (10.2-2) by simply solving it for $P_2$. Note that $P_0$ and $P_1$ are available if we know $P_0$ and $P_1$ at $x = A_0$. Actually, a fundamental difficulty is working out the constant $k_1$, $k_2$, and $k_3$, which are not given in the problem statement. To understand this, let us assume that $k_1$ is independent of $x$ and that $k_2$ becomes Fourier transformed with the dependence on $k_x$. Then (10.3-2) becomes

$$P_{nx} = -\frac{\omega^2}{k_x}P_{nx} + k_x P_{nx}.$$ 

The behavior of $P_{nx}$ will be dramatically affected by the sign of the factor $k_x$.

The terms $P_{nx}$ and $P_{nx}$ can be solved for $P_{nx}$ and $P_{nx}$ separately, so we have two solutions of the form $P_{nx} = e^{k_x x}$. However, the growing exponential solutions will fade as $x$ becomes large. Therefore, we must examine the growing solutions at $x = A_0$ more closely. The equation for the growing exponential solutions is

$$\frac{\partial P_{nx}}{\partial x} = k_x P_{nx},$$

where $k_x$ is the imaginary part of $k_1$. The wave equation is second order in $x$ and hence requires two boundary conditions in $x$. These boundary conditions can be found from the solutions for $P_{nx}$ and $P_{nx}$, and the appropriate boundary conditions can be determined by matching the solutions to those at $x = A_0$. For $x > 0$, an additional boundary condition is required. The boundary condition is that $P_{nx}$ must remain bounded as $x$ becomes large. If this condition is satisfied, $P_{nx}$ will remain bounded as $x$ becomes large.
Saying that \( Q_0 \) is an unknown constant amounts to saying that the wave has unknown amplitude and phase. Next we write

\[
P(x, z) = Q(x, z)e^{i\omega z/\epsilon}
\]

Now, "\( Q(x, z) \) is approximately a constant function of \( x \) and \( z \)" is a rather fuzzy statement which we will proceed to sharpen up. By restricting \( Q(x, z) \) to slowly variable functions we will be restricting \( P(x, z) \) to wave fields which are near to plane waves propagating in the \( z \) direction. In fact, \( P \) might represent plane waves propagating at a small angle from the \( z \) axis, or it might be a small portion of a spherical wave, or it might be the observed backscattered radiation in a seismic reflection survey, or on 90° rotation of the coordinate system it might describe surface waves.

The ratio \( \omega/\epsilon \) occurs often and it is called the spatial frequency of the wave. We define

\[
m = \frac{\omega}{\epsilon(x, z)} \quad (10-3-4)
\]

We also define \( \bar{m} \) as a spatial average of \( m \).

\[
\bar{m} = \frac{\omega}{\epsilon} \quad (10-3-5)
\]

In a material which is homogeneous \( \bar{m} \) will equal \( m \). With this definition we write the wave disturbance as

\[
P(x, z) = Q(x, z)e^{i\omega z/\epsilon} \quad (10-3-6)
\]

Now an additional condition to make \( Q(x, z) \) slowly variable with \( z \) is that \( m(x, z) = \epsilon(x, z) \) be relatively near to \( \bar{m} \). Let us compute some partial derivatives of (10-3-6)

\[
P_x = Q_x e^{i\omega z/\epsilon} \quad (10-3-7a)
\]

\[
P_{xx} = Q_{xx} e^{i\omega z/\epsilon} + \frac{\omega}{\epsilon} Q_x e^{i\omega z/\epsilon} \quad (10-3-7b)
\]

\[
P_z = (Q_z + i\bar{m}Q) e^{i\omega z/\epsilon} \quad (10-3-7c)
\]

\[
P_{zz} = (Q_{zz} + 2i\bar{m}Q_z - \bar{m}^2 Q) e^{i\omega z/\epsilon} \quad (10-3-7d)
\]

Insert (10-3-7b) and (10-3-7d) into (10-3-1) and cancel the exponential, obtaining

\[
Q_{xx} + Q_{zz} + 2i\bar{m}Q_z + (m^2 - \bar{m}^2)Q = 0 \quad (10-3-8)
\]

Now we make the very important step where we assert that for many applications \( Q \) is slowly variable and \( Q_z \) may be neglected in comparison with \( 2i\bar{m}Q_z \). Dropping the \( Q_z \) term will be called the parabolic approximation or the paraxial approximation. This gives us the desired first-order, hence initial-value, equation in \( z \).

\[
Q_{xx} + 2i\bar{m}Q_z = 0 \quad (10-3-9)
\]

In a homogeneous medium, (10-3-9) reduces to

\[
Q_{xx} + 2i\bar{m}Q_z = 0 \quad (10-3-10)
\]

Equation (10-3-10) is really of the same form as the heat-flow equation if \( z \) is associated with time and the heat conductivity is taken to be imaginary. The equation is, in fact, known as the Schrödinger equation. It may be solved numerically by the methods described for the wave equation in Sec. 10-1. Ultimately (10-3-10) will be advocated for quite a number of purposes, so before we proceed let us take a look at what we have lost by dropping \( Q_z \). To facilitate comparison of (10-3-10) to the wave equation, let us convert back from the \( Q \) variable to the \( P \) variable. Rearrange (10-3-6) and form derivatives.

\[
Q = Pe^{-i\bar{m}z} \quad (10-3-11a)
\]

\[
Q_{xx} = P_{xx} e^{-i\bar{m}z} \quad (10-3-11b)
\]

\[
Q_z = (P_z - i\bar{m}P)e^{-i\bar{m}z} \quad (10-3-11c)
\]

Insert (10-3-11b) and (10-3-11c) into (10-3-10) and cancel the exponential, getting the equation which we will call the one-way wave equation.

\[
P_{xx} + 2i\bar{m}(P_z - i\bar{m}P) = 0 \quad (10-3-12)
\]

One technique which may be used to solve any partial differential equation in cartesian coordinates with constant coefficients is to insert the complex exponential \( e^{i(k_x x + k_z z)} \). If \( k_x \) and \( k_z \) turn out to be real, then this trial solution may be interpreted as a plane wave propagating in the \( k = (k_x , k_z) \) direction. Inserting this exponential into both the wave equation (10-3-1) and the one-way wave equation (10-3-12) and canceling the exponential, we get two algebraic equations called dispersion relations. They are

\[
-k_x^2 - k_z^2 + m^2 = 0 \quad (10-3-13)
\]

\[
-k_x^2 - 2i\bar{m}k_z + 2\bar{m}^2 = 0 \quad (10-3-14)
\]

These two equations are graphed in Fig. 10-8, a and b.
FIGURE 10.9
Snapshots of the monochromatic wave-equation transfer function. A Fourier sum over \( k_x \) done over the exact wave-equation transfer function \( \exp(i(1 - k_x^2)u^2/2u^2) \) is displayed (top) in the \((x, z)\) plane for a fixed frequency \( \omega_0 \). Middle is the same for the 15\(^{\circ}\) approximate transfer function \( \exp(i(1 - k_x^2)u^2/2u^2) \) used (bottom). Bottom is the same for the 45\(^{\circ}\) approximation \( \exp\left(\frac{\omega}{v} 4\alpha^2 - \frac{3k_x^2 + 2}{4\alpha^2 - k_x^2} \right) \) of Exercise 2.

The physical picture is of waves passing through small apertures which are periodically spaced along the \( z \) axis.

FIGURE 10.10
Snapshots of the time-dependent wave-equation transfer function and approximations. A double Fourier sum over \( k_x \) and \( \omega \) of the functions of Fig. 10.12 shows the \((x, t)\) plane at a fixed time.

The graph for the wave equation is a circle and illustrates what we already know, namely that the magnitude of the wave number in an arbitrary direction, that is, \((k_x^2 + k_y^2)^{1/2}\) is equal to the constant \( \omega/v \). Such is not the case, however, for the one-way wave equation. Here we have only the approximation \( k_x^2 + k_y^2 \approx \omega^2/c^2 \) for small angles \( \theta \). Figure 10-8a also illustrates geometrically that (10-3-14) is an initial-value problem in \( z \) because Fig. 10-8a gives two values for \( k_x \) corresponding to any \( k_y \), but Fig. 10-8b gives only one value for \( k_x \). Figures 10-9,
FIGURE 10-11
Seismic profile-type displays of the wave-equation transfer function and two approximations to it. Exact, 15° approximate, and 45° approximate forms of the wave-equation transfer function were Fourier summed over \( k_x \) and \( w \). As with a seismic profile, we see a display of the \((x, t)\) plane for a fixed \( z \). The exact solution (top) is a delta function along a hyperbola. The 15° approximation (middle) is a parabola. The approximations die out more rapidly with angle than the exact solution.

FIGURE 10-12
Monochromatic wave-equation transfer functions displayed in the plane of \((k_y, z)\). The real part only is shown. Top is the exact transfer function. Note the abrupt change to evanescent at \( |k_y|w| \approx |\sin 90°| = 1 \). The exponential decay for \( k_x > w^2z \) is perceptible near \( z = 0 \). The 15° approximation (middle) and the 45° approximation (bottom) are all-pass filters and have replaced the evanescent region by an interesting design. In order to eliminate a massive amount of short horizontal wavelength fuzz in the spatial domain on the previous two figures, this evanescent zone was removed with a step function. The implication in a data processing application is that occasionally the approximate transfer functions may well be augmented by a fan-filter. (See Ref. [36].)
10-10, 10-11, and 10-12 show the wave equation transformation function $e^{ik_z x}$ and approximations $e^{ik_z x}$ and Fourier transformations thereof.

What we really want is a one-way wave equation which has the semicircle of Fig. 10-13 for its dispersion relation. The equation for the perfect semicircle is given by

$$k_z = \sqrt{m^2 - k_x^2} \quad (10-3-15)$$

This of course is the basic relation used for extrapolation in optics. By the binomial expansion, (10-3-15) may be written

$$k_z = m \left(1 + \frac{k_x^2}{2m^2} - \frac{k_x^4}{8m^4} + \cdots \right) \quad (10-3-16)$$

This expression converges for all $0 < k_x < m$.

Now for the sudden flash of insight which enables us to write the partial differential equation with this semicircle as its dispersion relation, from (10-3-16) we are inspired to write

$$\partial_z P = im \left(1 + \frac{\partial_{xx}}{2m^2} - \frac{\partial_{xxx}}{8m^4} + \cdots \right) P \quad (10-3-17)$$

Clearly, insertion of the plane wave $\exp(i(k_z x + ik_z z))$ into (10-3-17) immediately gives the desired semicircular dispersion relation (10-3-16). Thus, the greater the angular accuracy desired the more terms of (10-3-17) are required in the calculation. As a shorthand we may choose to write (10-3-17) as

$$\partial_z P = im \sqrt{m^2 + \partial_{xx}} P \quad (10-3-18)$$

It will be of no help to us, but it turns out that (10-3-18) is the relativistic Schrödinger equation.

It is easy to obtain the wave equation from (10-3-18). Just differentiate with respect to $z$

$$\partial_{zz} P = i \sqrt{m^2 + \partial_{xx}} P \quad (10-3-19)$$

Taking $m$ independent of $z$, we may interchange the order of differentiation

$$\partial_z \partial_{zz} P = i(m^2 + \partial_{xx}) \partial_z P$$

inserting (10-3-18)

$$\partial_{zz} P = -(m^2 + \partial_{xx})^{1/2} (m^2 + \partial_{xx})^{1/2} P$$

which is the wave equation.

Figures 10-14, 10-15, and 10-16 show finite-difference solutions to the parabolic approximated wave equation in homogeneous media.

Next, let us turn to the question of using the parabolic approximation in the presence of space variations in material velocity. The exercises go into considerable detail on this matter, but we can easily make some improvements over (10-3-9). The main idea is to approximate a circle by a parabola; the actual radius of the circle does not have anything to do with the approximation. This leads to the suggestion that (10-3-12) or (10-3-14) could be used with $
$ replaced by $m$, as in (10-3-17); hence (10-3-12) would be

$$P_{xx} + 2i(m(x, z))P_x + 2m^2(x, z)P = 0 \quad (10-3-19)$$
FIGURE 10-15
Like Fig. 10-14, but the left-hand boundary is a rigid wall. Waves may be seen reflecting back into the medium from the boundary. The reflected wavefront is indicated by the shorter of the two dashed lines. (From Ref. [5], p. 439.)

FIGURE 10-16
Expanding cylindrical wave. A theoretical solution was put in at the top boundary and extrapolated downward with the equation of Exercise 2. The wavefronts are not quite circular as they would be were it feasible to use (10-3-18). Notice also that the theoretical $r^{-1/2}$ amplitude decay is not exhibited for waves about 60° off the vertical. Such waves attenuate less rapidly because at 60° the phase curve is flatter than a circle. (From Ref. [5], p. 476.)

FIGURE 10-17
Waves impinging on a buried block of low-velocity material. Waves enter at the top of the block and are completely internally reflected from the side of the block. This leaves a shadow on the outside of the block. (From Ref. [5], p. 474.)

With (10-3-19) we no longer need to assume that $\vec{m} \approx \vec{m}$ so we can now deal with a wide range of velocities. Actually, as the exercises will show, the validity of (10-3-19) depends also on the approximation that the logarithmic space gradients of material velocity are small compared with the logarithmic gradients of the waves. In other words, the waves change faster than the material does.

Figures 10-17, 10-18, and 10-19 illustrate the propagation of waves in inhomogeneous materials.

The approximation is evidently best at high frequencies (short wavelengths). This approximation is well known in wave theory. Although it is sometimes called a ray approximation, the reader should not fear that the theory has degenerated to geometrical optics. Actually all the phenomena of physical optics (for example: interference, diffraction, and finite size focus) are still present. In fact we need not go to the physical optics limit at all. Some of the exercises are examples that include the velocity gradients found in lower frequency terms. Whether many or none of these terms is important in practice is a question which is particular to each application.

FIGURE 10-18
A low-velocity block is illuminated from the side. There is partial reflection from the side of the block and interference between waves entering the block through different faces. (From Ref. [5], p. 474.)
The algebraic equation $a + bx + cx^2 = 0$ has two roots. If $b$ is sufficiently large, we may approximate the smallest root with the linear relation $a + bx = 0$. An improved approximation which is still linear in $x$ may be found by substituting $x = -a/b$ back into the quadratic

$$a + bx + c \left( -\frac{a}{b} \right) x = 0$$

$$ab + (b^2 - ac)x = 0$$

Define $k_1 = m - k_0$ and substitute $k_0 = m - k_1$ into $k_0^2 + k_1^2 = m^2$. Find the smallest root for $k_1$. Show that this gives the same partial differential equation as Exercise 2.

Let the velocity $v = v(x)$ be a function of $x$ and define $m = oc(x)$. Define the operator

$$Op = m + \frac{1}{2m} \frac{d}{dx} - \frac{m_0}{2m} \frac{d}{dx}$$

Note that

$$\frac{\partial}{\partial t} P = \text{Op} P$$

$$\frac{\partial}{\partial x} P = \text{i} \frac{\partial}{\partial t} \text{Op} P = \text{-i} \frac{\partial}{\partial t} \text{Op} P = -\text{Op}^2 P$$

$$\left( \frac{\partial}{\partial x} + \text{Op} \right)^2 P = 0 = \text{wave equation} + \text{error}$$

Examine each error term and decide whether it is important (1) at high frequencies (collect terms proportional to nth power of wavelength) and (2) at small or large angles from the z axis.

Review the section on Sylvester’s matrix theorem. How is the square root of a matrix analogous to the square root of an operator?

Deduce the “outgoing wave equation” in cylindrical coordinates.

Deduce the “outgoing wave equation” in spherical coordinates.

Exercise 3 gave a good wide-angle approximation but Exercise 4 works for $m = m(x)$.

To utilize the method of Exercise 3 for $m = m(x)$ it is necessary to note that although $bx - x/b = 0$, it is not true that $(m \partial_x - \partial_x m) P = 0$ unless $m = m(x)$. Salvage the method of Exercise 3 by avoiding the use of commutativity as much as possible.

Consider surface waves propagating on the surface of an imperfect sphere. Deduce an equation, first-order in $\phi$, the longitude coordinate, second-order in $\theta$, the latitude coordinate, for waves beamed roughly along the equator. Assume all quantities are independent of the radial coordinate axis.

Modify the program of Sec. 10-1 in Exercise 4 to compute the solution to (10-3-10). You will need to review the compiler conventions of complex arithmetic. Also, after computing $Q(x, z)$ multiply by $e^{im\theta}$ to give $P(x, z)$. Print only the real part of $P(x, z)$. A physical interpretation of this result is light behind an edge of an opaque screen. Waves diffracted into the shadow zone should have semicircular wavefronts if you have arranged your display to preserve $\Delta x = \Delta x$ on the output.

Let $Z = Z' e^{ix}$ denote a discretization of the $x$ coordinate. Define $A(Z) = \sum a_n Z^n$ by finding $a_n$ such that

$$a_0 + \frac{1}{2z} a_1 \left( Z + \frac{1}{2z} \right) = |k_0|$$

for $|k_0| \Delta x \leq \pi$

Show that either solution to

$$\frac{\partial P(Z)}{\partial z} = \pm A(Z) P(Z)$$

is a solution to Laplace’s differential equation $P_{zz} + P_z = 0$. These solutions may be used for upward and downward continuation.
10-4 EXTRAPOLATION OF TIME-DEPENDENT WAVEFORMS IN SPACE

In Sec. 10-3 we learned how to extrapolate monochromatic waves in space. To extrapolate a time-dependent waveform in space, one could first Fourier transform it into monochromatic waves, then extrapolate them as in the previous section, and finally Fourier transform back into the time domain. Thus, although this section solves, in principle, the same problem as the last section, a direct time-domain method will often be preferable for practical reasons. Although a time-domain study is necessarily more complicated than one in the frequency domain (all time points must be considered together, but each frequency is isolated from the others) there is a great deal more understanding to be gained in the time domain, especially as regards causality. We will discover that wave-extrapolation procedures are like filters (in fact, they are a special kind of multidimensional all-pass filter) and that the feedback parts of these filters must be minimum-phase. There are two independent time-domain derivations.

The first derivation begins by transforming the scalar wave equation

\[ 0 = P_{xx} + P_{zz} - v^{-2}P_{tt} \quad (10-4-1) \]

into a coordinate frame which translates along the z axis at the speed \( \bar{v} \) which we will generally take to equal or exceed \( v \). It does not matter which way energy is propagating in the fixed frame; when it is seen in the moving frame it will remain stationary or fall backward. The coordinate transformation

\[ x' = x \quad (10-4-2a) \]
\[ z' = \bar{v}t - z \quad (10-4-2b) \]
\[ t' = t \quad (10-4-2c) \]

is depicted in Fig. 10-20 for \( \bar{v} = v \). In the primed frame all waves have a velocity component in the plus \( z' \) direction. Knowledge of \( P \) for present and past values at all \( x' \) for fixed \( z' \) should be sufficient to determine \( P \) for present and past values of time at \( (x', z' + \Delta z') \) because anything happens at \( z' + \Delta z' \) something has to happen at \( z' \). Thus, because of the restriction \( \bar{v} \geq v \) we anticipate that the linear operators which we will develop to extrapolate \( P \) in the plus \( z' \) direction should be causal. Let \( P' \) denote the disturbance in the moving frame. We have

\[ P(x, z, t) = P'(x', z', t') \quad (10-4-3) \]

It will be convenient to use subscripts to denote partial derivatives. Obviously,

\[ P_x = P'_x \quad \text{and} \quad P_{xx} = (P'_x)' \quad (10-4-4) \]

Also

\[ P_t = P'_x x'_x + P'_z z'_z + P'_t t'_t = -P'_t \]

so

\[ P_{tt} = P'_{tt} \quad (10-4-5) \]

and

\[ P_{zt} = P'_x x'_x + P'_z z'_z + P'_t t'_t = \bar{v}P'_x + P'_t \]

so

\[ P_{zt} = \bar{v}(P'_x x'_x + P'_z z'_z) + \bar{v}P'_x + P'_t \]

\[ = \bar{v}P'_{xx} + 2\bar{v}P'_x + P'_t \quad (10-4-6) \]

Now we may insert (10-4-4), (10-4-5), and (10-4-6) into (10-4-1) and we obtain

\[ P'_{xx} + \left[ 1 - \left( \frac{\bar{v}}{v} \right)^2 \right] P'_{xx} - 2 \frac{\bar{v}}{v} P'_{x} + \frac{1}{v^2} P''_{tt} = 0 \quad (10-4-7) \]

We will take up the constant velocity case \( \bar{v}(x, z) = \bar{v} \). The case \( v \neq \bar{v} \) is left for the exercises. Our main interest in (10-4-7) is with those waves which propagate with approximately the velocity of the new coordinate frame. In the moving frame such waves are Doppler shifted close to zero frequency. This suggests omitting the \( P''_{tt} \) term from (10-4-7). Thus (10-4-7) becomes

\[ P'_{xx} = \frac{\bar{v}}{2} P''_{xx} \quad (10-4-8) \]

If we Fourier transform out the time coordinate equation (10-4-8) becomes

\[ -i\omega P'_x = \left( \omega / v \right) P''_{xx} \]

which is identical to the monochromatic equation

\[ Q_{xx} + 2imQ_z = 0 \quad (10-4-9) \]

derived in the preceding chapter. Thus, dropping the \( P''_{xx} \) term is the familiar approximation of a circle by a parabola.
Inserting these into the wave equation (10.4-4) we obtain:

\[ P_{\nu} = \frac{\nu}{2} ( P_{x} x + P_{y} y ) = (10.4-4) \]

The last term of (10.4-4) is higher-order small for waves traveling at small angles from the z-axis; this recalls that the solution to the wave equation for waves in a +z direction is an arbitrary function \( f(x, z) \). Thus \( \nu^2 / 2 \nu \) vanishes for a wave along the z-axis. Neglecting \( P_{r} \), we find that (10.4-4) reduces to:

\[ P_{x} x + P_{y} y = (10.4-15) \]

which is the same equation as (10.4-14). Use of \( \epsilon_{\nu} = 0 \) time dependence in either (10.4-8) or (10.4-15) yields the equation (10.3-10) which was developed for extrapolaion of monochromatic waves. Another point of view of this chapter is that we could have obtained the monochromatic equations with \( \delta_{\nu} \) scheme for the solution to (10.3-8) or (10.4-8). Let us refer to the coordinate \( \zeta \) as \( \zeta = \eta + \nu \eta \). Running down the vector will be values of pressure along the x-axis. By removing all the trigonometric relations in the matrix we avoid writing a subscript for the x dependence. Let \( \epsilon_{\nu} \) denote the difference approximation to \( f(1, 2, 1) \) on the diagonal. With all these definitions (10.4-8) or (10.4-15) becomes:

\[ \delta_{\nu} P_{x} - P_{y} = -\delta \epsilon_{\nu} x + P_{y} \]

and the then centered space differencing:

\[ \delta P_{x} = -\delta \epsilon_{\nu} x + P_{y} \]

The first point of view is that \( P_{x} \) is prescribed initially on a grid over \( x \) and \( y \) and then the equation is used for extrapolation in \( t \). The second point of view is that \( P_{x} \) is prescribed initially on a grid over \( x \) and then the equation is used for extrapolation in \( t \). Before developing a numerical method for the solution to (10.4-4) we will derive a new coordinate frame fixed in space relative to the old one. However, let us run at the same speed, but they are initiated in such a way that the plane wave traveling in the +z direction will have the same arrival time measured at all clocks. This is somewhat like a westward-moving plate. The transformation equations are:

\[ x = \frac{x}{x} \]

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A very important question is the one of stability. We will now establish that the recursion (10-4-15) is stable for all positive values of \( \alpha \). If eigenvectors of the eigenvalues of \( \mathbf{T} \) are known for each of the eigenvalues of \( \mathbf{T} \), then the eigenvectors of \( \mathbf{T} \) for any component set or a few different functions of \( x \) are also known. If there is a unique solution, there is a unique solution for the undetermined \( \beta \). Equation (10-4-17) is convoluted with \( e^{-\frac{x}{2}} \).

Thus the eigenvalue is \( -2 \cos k \alpha - 2 \) and it is sufficient to study \( 0<\alpha<\pi/2 \), since any eigenvalue is a function of the arbitrary number \( k \). \( \text{If} \) it is replaced by \( k \), we have a component set for \( (\alpha_{k+1}, \alpha_{k+2}) \). We have shown that for \( \alpha \), there is a unique solution of finding \( \alpha \) and \( \alpha \) for all \( k \). Now, in (10-4-17) bring unknowns to the left.

The important thing for stability is (10-4-14) is that if all \( \alpha_i \) are fixed, then the magnitude of the coefficient of \( \alpha_j \) must exceed that of the 

\[ (1 + \alpha_{j+1}^2) - (1 + \alpha_{j+1}^2) \]

The stability of the recursion (10-4-15) is established by actually computing the stability of all \( \alpha \) and \( \alpha \) for all \( k \). Now, suppose \( \alpha \) were to change, the stability of the recursion would change. By the Z transform [4-33], we mean the stability of the recursion is given by (10-4-19). The filter function for computing \( h(z) \) from \( P_D \) is 

\[ (1 - \alpha_{j+1}^2) - (1 + \alpha_{j+1}^2) \]

We note that for positive \( a \) and for all \( k \) between 0 and 4, the denominator of the right-hand side of the recursion is\( \cos \alpha \), the numerator is \( \cos \alpha \). We have just completed a rather elaborate stability proof. The reader will undoubtedly discover that his own application involves slightly different equation, perhaps \( a = \frac{1}{\cos \alpha} \), or increased angular accuracy. What general advice can we begin with? It is the hope that the rules are the same, of course, obviously, the filter function is also true there. Note that \( \cos \alpha \) for even transformation domains is

\[ \cos \alpha = \frac{a^2}{2} \]

\( (10-4-21) \)
FIGURE 10.23
Possible means of producing a disturbed plane wave. Incident plane wave at bottom is altered by a material inhomogeneity. For example, circulating air cells (center), resulting in the disturbed wave at the top.

Fig. 10.23. Successive frames in Fig. 10.22 depict the subsequent history of the waveform. In optics texts (e.g., Goodman, Reference 35, p. 69) the monochromatic solution is usually obtained at infinity. The most obvious development is that the energy spreads out as one moves to successive frames. The single pulse of the top frame has become an extended oscillatory arrival by the last frame. As time goes on, less and less energy is in the first pulse and more and more is in the oscillatory tail. Another very notable feature is that after some time the first arrivals tend to be aligned again so that disturbances in a wavefront may be said to heal themselves as time goes on. In contrast, the coda (wave tail) develops into a spatially incoherent wave. (This mimics the behavior of most geophysical wave observations.) We may note several other less apparent aspects to Fig. 10.22. Although energy moves back from the first arrival, a point of constant phase in the wave tail (indicated by $X$) moves forward toward the wave onset. Also the dip, or apparent direction of propagation, tends to increase going down a frame. This represents the ray interpretation that late arrivals have taken longer ray paths. Also the $\pi/2$ phase shift of a two-dimensional focus which causes doublets to form may be seen at $A$ in the second frame.

In order to represent a disturbance of infinite extent in $x$ on a finite computer grid, the problem was initialized with a periodic disturbance having zero slope at the side boundaries. Zero-slope boundary conditions are then equivalent to infinite periodic extension in $x$. A value of $v \Delta t/\Delta x^2 = \lambda$ was chosen to give an appropriate variation in progressive frames with each frame in Fig. 10.22 representing five computational iterations. The solution may be rescaled in several ways because of the interdependence of $v \Delta t$, $\Delta x$, and $\Delta z$.

It might be valuable to consider various data enhancement processes in the light of Fig. 10.22. In the process called "beam-steering," observations such as those in Fig. 10.22 would be summed over the $x$ coordinate in an effort to enhance signal and reject noise. Clearly beam-steering will enhance the first arrival while rejecting random noise. It will also tend to cancel signal energy which resides in the oscillatory wave tails. If one is really interested in enhancing signal-to-noise ratio it would hardly seem desirable to use a processing scheme which cancels signal energy. As $\lambda$ or $\Delta z$ is increased the situation becomes increasingly severe, since signal energy moves from the initial pulse toward the oscillatory wave tails. What has often been regarded as "signal-generated-noise" may turn out to be signal in a potentially valuable form. One can indeed expect dramatic results if enhancement techniques are based on entire waveforms rather than only on the initial pulse.

EXERCISES
1. State all the assumptions which must be made to specialize (10-4.7) to

$$n(z) - \delta P' = P_i$$

Derive the analogous equation for the double-prime coordinates.

2. Find a difference scheme for the equation of Exercise 1 which extrapolates from $z'$ to $(z' + \Delta z)$. Show that past time is required if $\delta > 1$ and future time if $\delta < 1$.

3. Let a coordinate transformation be defined by

$$x' = x$$

$$z' = z$$

$$t' - t = \int x'^{-1}(z) dz$$

Put the scalar wave equation into these coordinates.

4. Show that if the transformation velocity $\delta$ in (10-4.9a), (10-4.9b), and (10-4.9c) takes any value less than the $v$ in the wave equation, then stable difference equations will result.

5. Consider the difference equation $(1 + \delta_0/12) \delta_0 P = b \delta_0 P$. For what value of $b$ does it reduce to an explicit scheme? Is the time recurrence stable for that value of $b$?

10.5 BEAM COUPLING

Much of our information about the interior of the earth arises from interfaces within the earth which convert downgoing waves to upgoing waves. In layered media a mathematically strict decomposition of disturbances into downgoing waves $\text{exp}(ik_0z)$ and upgoing waves $\text{exp}(-ik_0z)$ was possible, but at present no such decomposition has been developed for two- or three-dimensional inhomogeneity. What we have is a collection of ad hoc techniques whose rigorous justification depends on the absence of horizontally propagating or evanescent energy. As a practical matter, what we are really interested in is not just the decomposition of waves into downgoing and upgoing parts. We are interested in describing the interactions between more-or-less collimated beams. In holography, these are the incident (or reference) beam and the scattered wave beam. In global seismology, these could be the incident compressional wave beam and the scattered shear wave beam. They need not have any particular orientation to each other or to the vertical.

The wave-extrapolation techniques described earlier can be used to describe beams collimated roughly along the $z$ axis. Now we take up the task of describing the interaction between two such beams. For simplicity, these will initially be taken to be two more-or-less vertically propagating beams, one going down, the
other up, interacting at a planar horizontal interface. The technique developed can then be applied to a great many less restrictive geometries. The accuracy of results in more general geometries is then a practical question whose answer varies from one situation to the next. Accuracy limitations come from many sources, which include

1. Angular dependence of velocity in the collimated beam which arises from Fresnel-like approximations
2. Neglect of evanescent energy
3. Possible inability of two collimated-beam equations to describe all important beams generated at a complicated interface
4. Approximation of elastic compressional waves by the scalar wave equation.

The significance of accuracy limitations must be evaluated in terms of accuracy of experimental work, required accuracy, and accuracy cost of competitive techniques. Such evaluations are completely beyond the scope of our present efforts.

In this section we will describe only the primary reflected seismic energy in reflection seismic exploration. Large-amplitude waves are initiated at the earth’s surface by means of dynamite or other high-energy sources. These waves penetrate into the earth where a small fraction of the energy echoes at weak reflectors and gets sent back to sensitive surface geophones. Occasional situations where a noticeable amount of energy scatters up and down several times (called multiple reflections or just multiples) will be discussed in a later section. For a plane layered medium we can use equation (9.3-13).

\[ \frac{d(U[X])}{dx} = \begin{bmatrix} -iab & Y_x \ Y_z \ 1 \ Y_x \ 1 \ -1 \ D \end{bmatrix} \begin{bmatrix} U \ \ Y_x \ \ Y_z \ \ D \end{bmatrix} \]  

(10-5-1)

Because the practical situation which we are trying to describe satisfies the inequality \( U < D \), we will approximate the lower equation in (10-5-1) by

\[ D_x = \frac{iab}{2} D - \frac{Y_z}{2} D \]  

(10-5-2)

To get a physical understanding of (10-5-2) which is applicable even when \( a, b, \) and \( Y \) are z-variable, note that the solution to (10-5-2) which can be verified by direct substitution, is

\[ D = D_0 Y^{1/2} e^{-i \int_0^z ab \ dz} \]  

(10-5-3)

In other words, \( iab \) controls the phase (or velocity) of the wave and \( Y \) controls amplitude change. Thus, we can interpret the \( Y \) term as providing the physical effect associated with a transmission coefficient. It often happens that the velocity information in \( ab \) is approximately known, but the location of interfaces in the earth given by discontinuities in \( Y \) are totally unknown. This means that we need not abandon our calculation of \( D \) if we are prepared to admit that its amplitude errors by the unknown transmission coefficients.

The basic thrust of Sec. 10-3 was that we can treat nonplanar waves by regarding \( iab \) as the square root of the differential operator \(-i(\omega^2/2 + \phi)\). For a beam collimated downward along the z axis a first approximation to the square root is given by \( i\omega \sqrt{1 + c^2(\partial_{xx}/2\omega^2)} \). With the beam-collimation assumption (\( \partial_{xx} \approx 0 \)) and the unknown admittance gradient taken as zero, the downgoing wave \( D \) can be calculated with the equation

\[ D_x = \frac{i\omega}{2\omega} D + \frac{iv}{2\omega} D_{xx} \]  

(10-5-4)

This would more closely resemble the bulk of our earlier work if we assumed homogeneous velocity \( v = \bar{v} \) and then made the transformation \( D = D'e^{imz} \) where \( m = \omega/\bar{v} \), in which case (10-5-4) would reduce to

\[ D_x = \frac{i\omega}{2\omega} D' + \frac{iv}{2\omega} D'' \]  

(10-5-5)

To solve (10-5-4) or (10-5-5) inside the earth it is only necessary to know values for \( D \) along the surface of the earth (all \( x, z = 0 \)). In a reflection seismic prospecting situation, \( D \) could usually be approximated by a delta function at the shot location. Now let us turn to the calculation of the upgoing wave \( U \). From the top row of (10-5-1) we have

\[ U_x = -iab \ U - \frac{Y_z}{2Y} (U - D) \]  

(10-5-6)

If we care to neglect the transmission coefficient effect on \( U \) while retaining the reflection coefficient interaction of \( U \) and \( D \), this becomes

\[ U_x = -iab \ U + \frac{Y_z}{2Y} D \]  

(10-5-7)

Because reflection coefficient \( c \) is defined as

\[ c = \frac{Y_z - Y_1}{Y_z + Y_1} = -c' \]

we can [for \( Y(z) \) differentiable] write (10-5-7) as

\[ U_x = -iab \ U - c(z) D \]  

(10-5-8)

As with the downgoing waves, we can generalize from plane waves to beams with the square root approximation, obtaining

\[ U_x = -\frac{i\omega}{v} U - \frac{iv}{2\omega} U_{xx} - c'(x, z) D \]  

(10-5-9)

A change of variables to \( U = U'e^{-imz} \) and \( D = D'e^{imz} \) with the homogeneous-velocity, inhomogeneous-admittance assumption converts (10-5-9) to

\[ U_x = -\frac{i\omega}{v} U + \frac{iv}{2\omega} U_{xx} + c'(x, z) D \]  

(10-5-10)
(10-5-10) is turned on and \( U^* \) becomes nonzero from then on upward. This calculation is illustrated in Fig. 10-24.

The calculation can also be done in the time domain. We have the downgoing wave transformation

\[
\begin{align*}
x' &= x \\
z' &= z \\
t' &= t - \frac{z}{\bar{v}}
\end{align*}
\tag{10-5-11a}
\tag{10-5-11b}
\tag{10-5-11c}
\]

and the upgoing wave transformation

\[
\begin{align*}
x' &= x \\
z' &= z \\
t' &= t + \frac{z}{\bar{v}}
\end{align*}
\tag{10-5-12a}
\tag{10-5-12b}
\tag{10-5-12c}
\]

And we have the possibility of expressing \( U \) and \( D \) in either frames (10-5-11) or frames (10-5-12)

\[
\begin{align*}
U(x, z, t) &= U'(x', z', t') = U'(x', z', t') \\
D(x, z, t) &= D'(x', z', t') = D'(x', z', t')
\end{align*}
\tag{10-5-13a}
\tag{10-5-13b}
\]

The chain rule for differentiation gives

\[
\begin{align*}
\partial_x D &= \partial_{x'} D' \\
\partial_z D &= \left(\partial_{z'} + \frac{1}{\bar{v}} \partial_t\right) D' \\
\partial_t D &= \partial_{t'} D' = \partial_{t'} D'
\end{align*}
\tag{10-5-14a}
\tag{10-5-14b}
\tag{10-5-14c}
\]

and

\[
\begin{align*}
\partial_x U &= \partial_{x'} U' \\
\partial_z U &= \left(\partial_{z'} + \frac{1}{\bar{v}} \partial_t\right) U' \\
\partial_t U &= \partial_{t'} U'
\end{align*}
\tag{10-5-15a}
\tag{10-5-15b}
\tag{10-5-15c}
\]

Taking velocity-homogeneous media \( \nu = \bar{v} \), multiplying (10-5-4) and (10-5-9) through by \(-io\), and then identifying \(-io\) with a time derivative, we obtain

\[
\begin{align*}
D_{1t} &= -\frac{1}{\bar{v}} D_{1x} + \frac{\bar{v}}{2} D_{2x} \\
U_{1t} &= \frac{\bar{v}}{\bar{v}} U_{1x} - \frac{\bar{v}}{2} U_{2x} - c(x, z) D_1
\end{align*}
\tag{10-5-16a}
\tag{10-5-16b}
\]

It is important to understand how we can calculate the solution to (10-5-10). First of all, \( D' \) must have been calculated by some other equation before we start on \( U^* \). In the solution of (10-5-10) we will regard \( c(x, z)D' \) as a source term for the generation of \( U^* \). Now there are two important cases. The first one is data synthesis. This is called the forward problem. The other case, called the inverse problem, is where the data sample \( U^* \) is given at the earth’s surface, \( z = 0 \), and the problem is to deduce both \( c(z) \) and \( U'(z) \) as you integrate \( U^* \) downward. The inverse problem is more fully treated in the chapter on seismic data processing. Here we will stick to the forward problem. A boundary condition on \( U^* \) which will enable us to use (10-5-9) to find \( U^* \) everywhere is to prescribe that \( U^* \) vanishes over all \( x \) inside the earth at some depth \( z_0 \) which is suitably great, say beneath all detectable reflectors. Then (10-5-10) is stepped up from \( z_0 \) to \( z_{n-1}, z_{n-2}, \ldots \), etc. \( U^* \) remains zero until we come up to the first illuminated reflector; that is, the deepest place where both \( c(x, z) \) and \( D' \) are nonvanishing. At this point, the source term in
Equations (10-5-16a) and (10-5-16b) are readily converted by means of (10-5-14) and (10-5-15) to

\[ D_{x'y'} = \frac{1}{2} D_{xx'} \]  
(10-5-17a)

\[ U_{x'y'} = -\frac{1}{2} U_{xx'} - c(x', z') D_{x'} \]  
(10-5-17b)

Now (10-5-17a) can be used to compute \( D' \) but (10-5-17b) calls for \( D' \). Subtracting (10-5-12c) from (10-5-11e) we get

\[ \tau = \tau' - \frac{2\tau}{\bar{v}} \]

So, using (10-5-13b) we find (10-5-17b) can be expressed in terms of \( D' \) as

\[ U_{x'y'} = -\frac{1}{2} U_{xx'} - c(x', z') \frac{\partial}{\partial x} D' \]

This time-domain result is the transform of (10-5-10).

**EXERCISES**

1. Show that \( \xi \) and \( \eta \) are the reflection coefficients \( c' \) as seen from above the interface.

2. Consider (9-3-20) that the definition of \( \Delta Y \) includes \( k_z \). This was neglected in the derivation of (10-5-10). Improve (10-5-10) to include the implied \( \partial D' / \partial x \) terms. This improvement allows reflection coefficient to be a function of angle.

**10-6 NUMERICAL VISCOSITY**

Positive numerical viscosity means that the short wavelength deviation of a difference equation from a differential equation is such that the short wavelengths tend to dissipate as the calculation proceeds. The numerical viscosity may also turn out to be negative, causing short wavelengths to amplify rather than attenuate. Whether or not there are good scientific reasons to study numerical viscosity, scientists often get dragged into this study for several reasons: First, even if differential equations do not violate the causality there may be instability due to negative viscosity in the difference equations. Second, the realities of computer economics (especially in a multidimensional problem such that \( P_{xx} = (\alpha(2) P_{xx} \) may require that waveforms be sampled with as few points as practicable. Third, when observational data are to be processed, as when \( P(x, y) \) is extrapolated from \( x_1 \) to \( x_2 \), then the data may be inconsistent with certain assumptions upon which the extrapolating equation is based.

For example, suppose that \( P(x, y) \) has Fourier transform \( P(k_x, \omega) \). Then, since \( k_x^2 + k_y^2 = \omega^2 / v^2 \), freely propagating waves are characterized by \( |k_x| < \omega / \bar{v} \)

so \( P(k_x, \omega) \) should vanish unless \( |k_x| < \omega / \bar{v} \). In the derivation of \( P_{xx} = (\alpha(2) \omega / \bar{v} ^2 \) it was further assumed that the waves have small angles of propagation; hence, the inequality becomes stronger, \( |k_x| < \omega / \bar{v} \). Since observational data will certainly not satisfy these conditions exactly we have two options. First, we can hope to ignore the illegal part of the \( (k_x, \omega) \) space if the data do not have much energy there and if our difference equation does not unacceptably amplify it. Second, we can modify our difference or differential equations so that there is a controlled positive numerical viscosity in the illegal part of the transform space. This kind of operation is sometimes called fan-filtering because of the wedge-shaped region of attenuation in \( (\omega, k_x) \) space.

The operator \( \delta_x \) has the Fourier transform \( -k_x^2 \). The operator \( \delta_x \) amounts to a convolution on the \( x \) axis with the coefficients \( (1, -2, 1) / \Delta x^2 \); thus its Fourier transform is \( \exp(-i k_x \Delta x) = 1 - 2 \exp(ik_x \Delta x) / \Delta x^2 \). We write this as

\[ FT(\delta_x) = -k_x^2 \]  
(10-6-1a)

\[ FT \left( \frac{\delta_x}{\Delta x^2} \right) = -k_x^2 \]  
(10-6-1b)

The approximation \( k_x \) to \( k_x \) is given by

\[ k_x = \frac{2}{\Delta x} \sin k_x \Delta x \]  
(10-6-2)

The error in the approximation \( k_x = k_x \) is tabulated in Fig. 10-25.
The Crank-Nicolson method amounts to another approximation. Here the operator \(\partial / \partial t\) which has the Fourier transform \(-i\omega\) is approximated by the bilinear transformation. The approximation \(\omega\) to \(\omega\) is given by

\[
-i\omega \Delta t = \frac{2(1 - e^{i\omega \Delta t})}{1 + e^{i\omega \Delta t}}
\]

Multiplying top and bottom on the right by \(e^{-i\omega \Delta t/2}\) we get

\[
-i\omega \Delta t = 2 \frac{e^{-i\omega \Delta t/2} - e^{i\omega \Delta t/2}}{e^{-i\omega \Delta t/2} + e^{i\omega \Delta t/2}}
\]

\[
= -2i \tan \frac{\omega \Delta t}{2}
\]

\[
\omega = \frac{2}{\Delta t} \tan \frac{\omega \Delta t}{2}
\]  

(10.6-3)

This approximation is also tabulated in Fig. 10-25.

To see how higher-order difference approximations may be built up, we solve (10.6-2) for \(ik_x\) getting

\[
ik_x = \frac{2}{\Delta x} \arcsinh \left( \frac{i k_x \Delta x^2}{2} \right)
\]

(10.6-4)

Recall the power series for \(\arcsinh\)

\[
\arcsinh u = u - \frac{1}{2} \frac{u^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{u^5}{5} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{u^7}{7} + \cdots
\]

(10.6-5)

The inverse Fourier transform of (10.6-4) using (10.6-5) provides a power series expansion for \(\partial_i\) in terms of powers of \(\delta_x\).

At the present time, reflection seismic data often come close to being undersampled in the horizontal \(x\) coordinate. Hence, it is worthwhile to devise a more accurate approximation than \(\delta_{xx}\) to \(\delta_{xx}\). Squaring (10.6-4) and retaining only the first two terms in the \(\arcsinh\) expansion gives

\[
-k_x^2 \approx \frac{4}{\Delta x^2} \left( u^2 - \frac{u^4}{3} \right)
\]

(10.6-6)

where \(u = ik_x \Delta x/2\). Taking the inverse transform we have

\[
\partial_{xx} \approx \frac{\delta_{xx}(1 - \delta_{xx}/12)}{\Delta x^2}
\]

(10.6-7)

It is most often convenient to use this in the rational form

\[
\partial_{xx} \approx \frac{\delta_{xx}/\Delta x^2}{1 + \delta_{xx}/12}
\]

(10.6-8)

By means of a trick, the rational form can be used without going to higher-order difference operators. Note that (10.6-8) into a differential equation of the type \(P_t = P_{xx}\) leads to

\[
\frac{1 + \delta_{xx}}{12} \delta_{xx} P = \frac{\Delta x}{\Delta t} \delta_{xx} P
\]

(10.6-9)

The new term \(\delta_{xx}\) fits on the old computation star and thus amounts to a just readjustment of coefficients; that is, hardly any increase in computer costs. Reference to Fig. 10-25 shows an astonishing increase in accuracy. On the basis of Fig. 10-25 and the acceptable error for some particular application, say 3 per cent, one determines a minimum acceptable number of points per wavelength, say 10 points per wavelength on \(z\) and \(t\) axes and 31 points per wavelength on the \(x\) axis. Then the useful bandwidth \(-2\pi/10 < \omega < 2\pi/10\) is markedly less than the total bandwidth available (2\(\pi\) is the periodicity interval for transforms of sampled data). In this case, the ratio of useful bandwidth to total bandwidth is 1/5. In order to use more of the available bandwidth it is necessary to put up with more error or to develop more elaborate difference approximations to differential operators. Figure 10-26 depicts the paltry portion of \((\omega, k_x)\) space which is usable.

For examples of the manipulation of numerical viscosity let us take the differential equation \(P_{xx} = v/2P_{xx}\) and modify it to attenuate energy outside the usable bandwidth, say where \(|k_x \Delta x| > \pi/5\). We simply add a term to the right-hand side. That is, we modify

\[
\partial_t P = \frac{v}{2\Delta t} \partial_{xx} P
\]

(10.6-10)

by judicious choice of an additional term

\[
\partial_t P = \frac{v}{2\Delta t} \partial_{xx} P + a \partial_{xx} P
\]

(10.6-11)

To see what numerical value to take for the constant \(a\), we transform the \(x\) coordinate in (10.6-11)

\[
\partial_z P = \left( \frac{v k_x^2}{2\Delta t} - ak_x^2 \right) P
\]

(10.6-12)

Equation (10.6-12) has the solution

\[
P(z) = P(z_0) \exp \left[ \left( \frac{v k_x^2}{2\Delta t} - ak_x^2 \right) (z - z_0) \right]
\]

(10.6-13)

The imaginary part of the exponent merely gives the phase angle, which we will ignore because we are interested only in magnitude. Let \(z - z_0 = d\). Then (10.6-13) becomes

\[
\frac{P(z)}{P(z_0)} = \exp (-ak_x^2d)
\]

(10.6-14)
Thus, the term we added to (10-6-10) to get (10-6-11) has a coefficient which goes to 0 as the squared grid spacing $\Delta x^2$. Inclusion of this term gives the gaussian attenuation function of spatial frequency of (10-6-14). The inclusion of the viscosity term seems to add virtually no cost to a computer program.

Next, let us modify the extrapolation equation so that excessive dips $\sin (\text{dip}) = k_{\nu}/\omega_0$ will be attenuated. This is not exactly numerical viscosity because we will alter the basic differential equation. It is like numerical viscosity in that it is an ad hoc modification intended to correct a certain deficiency. Here we modify the differential equation (10-6-10) to read

$$\frac{\partial_z P}{(1-\omega_0^2 + a_{\nu})} = \frac{v}{2} \frac{\partial_{xx} P}{(1-\omega_0^2 + a_{\nu})} \quad (10-6-16)$$

To see what numerical value to pick for $\omega_0$, we rationalize the denominator

$$\frac{\partial_z P}{(1-\omega_0^2 + a_{\nu})} = \frac{v}{2} \frac{\partial_{xx} P}{(1-\omega_0^2 + a_{\nu})} \quad (10-6-17)$$

Now we may ignore the imaginary part of the right-hand side of (10-6-17) because it contributes only the phase of $P$. Fourier transforming the $x$ coordinate, we have

$$\frac{\partial_z P}{(1-\omega_0^2 + a_{\nu})} = \frac{v}{2} \frac{\partial_{xx} P}{(1-\omega_0^2 + a_{\nu})} \quad (10-6-18)$$

There are two cases. We will pick $\omega_0$ very small so that in the uninteresting case where $\omega < \omega_0$ (10-6-18) reduces to spatial frequency dissipation but in the interesting case $\omega > \omega_0$ (10-6-18) amounts to

$$\frac{\partial_z P}{(1-\omega_0^2 + a_{\nu})} = \frac{v}{2} \frac{\partial_{xx} P}{(1-\omega_0^2 + a_{\nu})} \quad (10-6-19)$$

This is obviously attenuation, which is a gaussian function of dip. It is left for the exercises to find a numerical choice for $\omega_0$.

**EXERCISES**

1. What value of $\omega_0$ in (10-6-16) will attenuate waves propagating from $z_1$ to $z_2$ at a 30° angle from the $z$ axis to $e^{-t}$ times the original amplitude? So that $\omega_0$ may be said to be small, it is necessary to compare it to something with physical dimensions of inverse time. Give examples of a situation where $\omega_0$ is small and a situation where it is not.

2. Show that the parameter $b$ in $P_z = i(2\alpha \omega_0 + b \partial_{xx})P$ may be used to produce a viscosity decay of approximate form $\exp (\frac{-bh^2}{(\omega - \omega_0)}$. This may be useful when $\Delta z$ is taken too large.

3. Consider extrapolation one step in the $z$ direction with the equation $P_z = -\omega_0^2 P$. Insert the bilinear transformation $z = (1 + 2 \gamma - 1)(1 - \gamma) + 2 \gamma - 1$ and deduce that the equation cannot be used since it is a polynomial with a nonminimum-phase divisor results.
4 Show that the equation \( P = \exp(-a\Delta t/2) \exp(ia\Delta t)P \), unlike the equation of Exercise 3, leads to a causal time-domain filter. (Do the extrapolation in \( z \) by the Crank-Nicolson method, i.e., the bilinear transform method.)

5 A given set of data \( P(x, t) \) is believed to satisfy the equation \( P_{tt} = P_{xx} \). It is observed that transformed data \( Q(x, t) \), where \( Q(x, t) = P(x, t)e^{-at} \), fits into a reasonably small numerical range so that \( Q \) may be represented using integer arithmetic. What differential equation does \( Q \) satisfy?

11 SEISMIC DATA PROCESSING WITH THE WAVE EQUATION

The coordinate frames used by theoreticians to describe wave propagation do not include frames in common use by geophysical prospectors to describe observations. Whereas the theoretician generally considers a single source (or shot) location at a time, the experimentalist deals simultaneously with waves which have been generated separately by many shots. Our task in this section is to put the wave equation into some prospectors’ coordinate frames.

11-1 DOWNWARD CONTINUATION OF GATHERS AND SECTIONS

Suboceanic prospecting is generally carried out by a ship which carries a repetitive energy source and which trails a cable that is 2 to 3 kilometers long and packed with sonic receivers. Ideally, the ship’s course is a straight line which we can take to be the \( x \) axis. Ideally, all the seismic waves of interest propagate in a vertical plane through the line of the ship’s course. This plane is called the plane of the seismic section. Despite the fact that it is no great problem to describe waves in