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Rocks as poroelastic composites

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ABSTRACT

In Biot's theory of poroelasticity, elastic materials contain connected voids or pores and these pores may be filled with fluids under pressure. The fluid pressure then couples to the mechanical effects of stress or strain applied externally to the solid matrix. Eshelby's formula (for the response of a single ellipsoidal elastic inclusion in an elastic whole space to a strain imposed at infinity) is a very well-known and important result in elasticity. The hardest technical part of Eshelby's work was in computing the elliptic integrals needed to evaluate the fourth-rank tensors for inclusions shaped like spheres, oblate and prolate spheroids, needles and disks. Having a rigorous generalization of Eshelby's results valid for poroelasticity means that the hard part of Eshelby's work can be carried over from elasticity to poroelasticity — and also thermoelasticity — with only trivial modifications. Effective medium theories for poroelastic composites such as rocks can then be formulated easily by analogy to well-established methods used for elastic composites. An identity analogous to Eshelby's classic result has been previously derived by the author for use in these more complex and more realistic problems in rock mechanics analysis. Using these results as the starting point for new methods of estimation, I apply these techniques to the Biot-Willis parameter, which is the technical name for the effective-stress coefficient for total volume strain. The results show that poroelastic parameters can now be estimated as easily as elastic parameters for arbitrary ellipsoidal inclusions using any of the standard effective medium theories.

INTRODUCTION

My subject is the treatment of rocks — and, especially, fluid-saturated and partially saturated rocks — as composite poroelastic media. By this I mean to study and partially answer the question of how the elastic/poroelastic constants of the rock can be estimated from a knowledge of the constituents of the rock, their volume fractions, and possibly the geometry of individual grains and/or pores — when that information is also available. The main new results I obtain concern means of estimating

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the effective-stress coefficient known as the Biot-Willis parameter (Biot and Willis, 1957) and the fluid storage coefficient known as Skempton's coefficient (Skempton, 1954). These two are the key new parameters arising in generalizing from elasticity to poroelasticity, and the ones that are not accounted for in previous theories of elastic/poroelastic composites like rocks.

Having an identity analogous to Eshelby's classic result (Eshelby, 1957) — for the response of a single ellipsoidal elastic inclusion in an elastic whole space to a strain imposed at infinity — available in more complex problems in composites analysis (such as poroelastic or thermoelastic composites) is of great practical interest. In Biot-Gassmann poroelasticity (Biot, 1941; Gassmann, 1951; Biot, 1962), elastic materials contain connected voids or pores and these pores may be filled with fluids under pressure. The fluid pressure then couples to the mechanical effects of an externally applied stress or strain. With a rigorous generalization of Eshelby's formula valid for poroelasticity, the hard part of Eshelby's work (in computing the elliptic integrals needed to evaluate the fourth-rank tensors for inclusions shaped like spheres, oblate and prolate spheroids, needles, disks and/or penny-shaped cracks) can be carried over to these new results with only trivial modifications. Then, effective medium theories for poroelastic composites like rocks can be formulated easily by analogy to well-established theories for elastic composites (Korringa *et al.*, 1979; Berryman, 1980; 1995).

The author (Berryman, 1997) has discovered a simple mathematical trick, applicable to media having isotropic constituents and based on a linear combination of results from two thought experiments, that makes the derivation of such a generalization of Eshelby's formula to poroelasticity an elementary task. In earlier work (Berryman, 1985; 1992), the problem of acoustical scattering by a *spherical* inhomogeneity of one poroelastic material imbedded in another was solved and the results then used to construct various single-scattering-based effective medium theories. The Eshelby generalization now permits incorporation of Eshelby's results for arbitrary ellipsoidal-shaped inclusions into both quasistatic formulations of effective medium theory and/or into scattering formulas. The resulting improved estimates of poroelastic material properties has important applications in geothermal and oil reservoir modeling, nuclear waste disposal, and hydrology, among others.

Generalization of almost all effective medium theories [see Berryman and Berge (1996) for a discussion] now can proceed more easily into the complex realm of poroelastic composites by making use of this generalization of Eshelby's results.

POROELASTICITY AND ESHELBY

The equations of quasistatic poroelasticity, as presented for example by Rice and Cleary (1976), may be written concisely in the form:

$$\varepsilon_{pq} = S_{pqrs} \langle \sigma_{rs} \rangle \,, \tag{1}$$

$$\zeta = (m - m_0)/\rho_0 = \frac{\alpha}{K}\left[\frac{1}{3}\sigma_{qq} + \frac{1}{B}p_f\right]. \tag{2}$$

Commonly understood terms appearing in these equations are the strains ε_{ij}, the solid stress σ_{ij}, the fluid pressure p_f, the elastic compliance tensor S_{ijkl} of the drained porous frame, and the increment of fluid content ζ (which is related to the initial m_0 and current m fluid mass contents, and to the initial density ρ_0 of the fluid). Applying well-known definitions from Biot and Willis (1957), the effective stress (for total volume strain) appearing in (1) is

$$\langle\sigma_{pq}\rangle = \sigma_{pq} + \alpha p_f \delta_{pq}, \tag{3}$$

where the coefficient $\alpha = 1 - K/K_s'$ is the Biot-Willis parameter, K is the bulk modulus of the solid frame (jacketed modulus), and K_s' is the unjacketed solid modulus. The coefficient B is Skempton's pore-pressure buildup coefficient (Skempton, 1954; Green and Wang, 1986; Hart and Wang, 1995), given by

$$\frac{1}{B} = 1 + \frac{\phi_0 K}{\alpha}\left(\frac{1}{K_f} - \frac{1}{K_s''}\right), \tag{4}$$

where ϕ_0 is the initial porosity, K_f is the bulk modulus of the pore fluid, and K_s'' is the unjacketed pore modulus. The equation for the change in porosity ϕ is

$$\phi - \phi_0 = \frac{\alpha}{K}\left[\frac{1}{3}\sigma_{qq} + p_f\right] - \frac{\phi_0}{K_s''}p_f. \tag{5}$$

In other work the present author has often used the alternative notation $K_s = K_s'$ and $K_\phi = K_s''$ for the same two unjacketed bulk moduli.

Starting from these basic equations of poroelasticity I want to formulate methods of computing the effective coefficients in composite poroelastic media when these media are themselves composed of simpler (generally microhomogeneous) poroelastic media. The corresponding problem in elasticity has been studied extensively for at least the last 40 years. It is desirable to try to make the transition from composite elastic media to composite poroelastic media as elegantly as it can possibly be done. One way in which this might be accomplished within effective medium theory is through the use of similar techniques applied to the full poroelastic equations such as was done in Berryman (1992). Another way to reach the same goal is to find new extensions to poroelasticity of some of the classic results like Eshelby (1957) that make the analysis virtually the same as that in the elastic case.

I restrict discussion here to poroelasticity, but the modifications necessary for application to thermoelasticity are not difficult. In my notation, a superscript i refers to the inclusion phase, while superscripts h and $*$ refer to host and composite media, respectively. In this initial analysis, the composite is a very simple one, being an infinite medium of host material with a single ellipsoidal inclusion of the ith phase. The basic result of Eshelby (1957) is then of the form

$$\varepsilon_{pq}^{(i)} = T_{pqrs}\varepsilon_{rs}^{*}, \tag{6}$$

where $\varepsilon^{(i)}$ is the uniform induced strain in the inclusion, ε^* is the uniform applied strain of the composite at infinity, and T is the fourth-rank tensor relating these two strains. The summation convention for repeated indices is assumed in expressions such as (6).

After considering two thought experiments – one when there is no fluid present in the pores and another when a saturating fluid is present and both the confining and pore pressures are chosen so that a uniform expansion of the host medium and inclusion occur (Berryman and Milton, 1991; Berryman and Berge, 1998), I find that the final form of the generalization of Eshelby's formula to poroelasticity is given by

$$\varepsilon_{pq}^{(i)} - e_{pq}(p_f) = T_{pqrs}\left[\varepsilon_{rs}^* - e_{rs}(p_f)\right]. \quad (7)$$

The full analysis shows that, if the pore fluid pressure vanishes (*e.g.*, $p_f = 0$ in the absence of a pore fluid), then the uniform strain e disappears from (7) and it reduces exactly to (6) as it should. In the other limiting case, if the pore pressure has been specified to be a nonzero constant, then the uniform strain e in (7) can be easily computed. So, if the strain at infinity happens to be chosen to be equal to this uniform strain, then from (7) the inclusion strain also takes the value at infinity as it should. Since the equation for $\varepsilon^{(i)}$ is necessarily linear, these two cases are enough to determine the behavior for arbitrary values of ε^* and p_f. (Think of the two thought experiments as independent boundary conditions on the linear equations that then determine the coefficients.) In poroelasticity, the strain e_{pq} can be determined in advance from the applied fluid pressure p_f and the properties of the host and inclusion. In particular, I find that

$$e_{pq} = \left(\frac{\alpha^{(h)} - \alpha^{(i)}}{K^{(h)} - K^{(i)}}\right)\frac{p_f}{3}\delta_{pq}. \quad (8)$$

The formulas presented in the following work form one set of useful applications of this generalization of Eshelby's formula.

EFFECTIVE MEDIUM THEORIES

The analysis to follow requires two main steps for each of the examples to be presented. The first step involves recovering the elastic result for the case when the pore pressure vanishes, *i.e.*, for the drained porous frame. Then, Eqs. (1) and (3) imply, when $p_f = 0$, that

$$\varepsilon_{pq} = S_{pqrs}\sigma_{rs}. \quad (9)$$

Therefore, this step is completely equivalent to the analysis already presented in Berryman and Berge (1996). I will present these results (along with quick derivations for the sake of completeness) because the results are needed to understand the second step of the analysis in each case. The second step is to derive the equivalent effective medium theory expression for $K_s'^*$, or equivalently for the Biot-Willis parameter α^*.

The general result I use for the drained analysis takes the form [see Eq. (19) of Berryman and Berge (1996)]

$$(\mathbf{C}^* - \mathbf{C}^{(r)}) \sum v_i \mathbf{G}^{ri} \varepsilon_r = \sum v_i (\mathbf{C}^{(i)} - \mathbf{C}^{(r)}) \mathbf{G}^{ri} \varepsilon_r, \tag{10}$$

where $\mathbf{C}^*$ is the effective stiffness matrix (inverse of the compliance matrix $\mathbf{S}^*$) to be determined, $\mathbf{C}^{(r)}$ is the stiffness matrix of some convenient elastic reference material, v_i is the volume fraction and $\mathbf{C}^{(i)}$ the stiffness matrix of the ith constituent of the elastic composite, ε_r is the strain in the reference material, and $\mathbf{G}^{ri}$ is the (exact and generally unknown) linear coefficient relating strains in material i to those in material r according to $\varepsilon_i = \mathbf{G}^{ri}\varepsilon_r$.

Coherent potential approximation

The first scheme I consider is sometimes called the Coherent Potential Approximation (CPA) (Gubernatis and Krumhansl, 1975; Berryman, 1992; Berryman and Berge, 1996) or the Self-Consistent Scheme (Korringa *et al.*, 1979; Berryman, 1980).

When there is no pore fluid present (*i.e.*, drained frame conditions), the equations of poroelasticity reduce to those of elasticity (10) for the porous frame material. Within CPA, the idea is to treat all constituents on an equal footing, so no single material serves as host medium for the others. For this reason, the CPA is sometimes known as a symmetrical self-consistent scheme. To find the formulas for the CPA, we take the reference material to be the composite itself, so $r = *$. The formula (10) reduces to

$$\sum v_i (\mathbf{C}^{(i)} - \mathbf{C}^*_{CPA}) \mathbf{T}^{*i} = 0, \tag{11}$$

where I have now approximated the unknown linear coefficient by the Eshelby-Wu tensor $\mathbf{T}^{*i}$ corresponding to inclusions of stiffness $\mathbf{C}^{(i)}$ in host material of stiffness $\mathbf{C}^*_{CPA}$.

To make use of the generalization of Eshelby's formula for poroelasticity in the case when pore fluid and pore pressure are significant factors, I note that each inclusion is effectively imbedded in the composite material *, so it makes sense to consider the formula for inclusion strain

$$\varepsilon^{(i)} = e^{*i}(p_f) + \mathbf{T}^{*i}\left[\varepsilon^* - e^{*i}(p_f)\right], \tag{12}$$

where the strain corresponding to equal expansion or contraction of both materials i and $*$ is given by

$$e^{*i}_{pq} = \left(\frac{\alpha^* - \alpha^{(i)}}{K^* - K^{(i)}}\right)\frac{p_f}{3}\delta_{pq}. \tag{13}$$

If the mixture were composed only of the two materials i and $*$, then the uniform expansion result would apply exactly. In the composite poroelastic material, (12) should be viewed as an estimate of the true strain of the ith constituent. This estimate is conceptually on the same footing as that traditionally used when saying that $\varepsilon^{(i)} = \mathbf{T}^{*i}\varepsilon^*$ is a reasonable approximation of the strain in the ith constituent of an elastic composite, even though there may be many other types of materials present.

To derive a formula within CPA for the Biot-Willis constant α^*, I want to make use of (12). For elasticity, the average confining stress equals the total confining stress, so $\sum v_i\sigma_i = \sigma$. This fact was actually used to derive (10). However, for poroelasticity with finite pore pressure p_f, it is important to distinguish confining stress from the stress in the solid components and so it is no longer true that the average confining stress is equal to the total confining stress, *i.e.*, $\sum v_i\sigma^{(i)} \neq \sigma$. The correct relation for the pertinent stress is more complicated than this. (We could learn some important things about our problem by studying this issue, but the analysis becomes rather more technical than what follows and it seems preferable to avoid this discussion here.) However, it is still necessarily true that the average strain is equal to the total strain, *i.e.*,

$$\sum v_i\varepsilon^{(i)} = \varepsilon^*. \tag{14}$$

Furthermore, this relation is just what is needed to make use of (12). Substituting (12) into (14) and then rearranging terms, I find that

$$\sum v_i(\mathbf{I} - \mathbf{T}^{*i})e^{*i}(p_f) = \sum v_i(\mathbf{I} - \mathbf{T}^{*i})\varepsilon^*, \tag{15}$$

where $\mathbf{I}$ is the identity matrix and I used the fact that $\sum v_i = 1$. Equation (15) is almost what I want, but the right hand side seems to be a problem, because it depends explicitly on ε^*, which is arbitrary. It is known however that $\sum v_i(\mathbf{I} - \mathbf{G}^{*i}) \equiv 0$ (Hill, 1963; Berryman and Berge, 1996), and, since $\mathbf{T}^{*i}$ is my approximation to $\mathbf{G}^{*i}$, it is clear that the right hand side of (15) should be set identically to zero. Thus, after making use of (13) in (15), the CPA for α^* is

$$\sum v_i(1 - P^{*i})\frac{\alpha^*_{CPA} - \alpha^{(i)}}{K^*_{CPA} - K^{(i)}} = 0, \tag{16}$$

where P^{*i} is the coefficient for the compressional modulus, and the corresponding coefficient for the shear modulus is Q^{*i} (see Table 1). Some care should be taken however to check the degree of satisfaction of the subsidiary condition $\sum v_i(\mathbf{I} - \mathbf{T}^{*i}) \simeq 0$ to make sure that it is at least approximately satisfied by the estimate obtained for $\mathbf{C}^*_{CPA}$. It turns out that this condition is satisfied exactly for spherical inclusions. Furthermore, in the case of spheres I have $P^{*i} = (K^* + \frac{4}{3}\mu^*)/(K^{(i)} + \frac{4}{3}\mu^*)$, and it is easy to show that (15) reduces to

$$\sum v_i(\alpha^{(i)} - \alpha^*_{CPA})P^{*i} = 0, \tag{17}$$

which should be compared to

$$\sum v_i(K^{(i)} - K^*_{CPA})P^{*i} = 0, \tag{18}$$

which follows from (11).

TABLE 1. Four examples of coefficients P and Q for spherical and nonspherical scatterers. The superscripts h and i refer to host and inclusion phases, respectively. Special characters are defined by $\beta = \mu[(3K+\mu)/(3K+4\mu)]$, $\gamma = \mu[(3K+\mu)/(3K+7\mu)]$, and $\zeta = (\mu/6)[(9K+8\mu)/(K+2\mu)]$. The expression for spheres, needles, and disks were derived by Wu (1966) and Walpole (1969). The expressions for penny-shaped cracks were derived by Walsh (1969) and assume $K^{(i)}/K^{(h)} << 1$ and $\mu^{(i)}/\mu^{(h)} << 1$. The aspect ratio of the cracks is a.

Inclusion shape	P^{hi}	Q^{hi}
Spheres	$\frac{K^{(h)}+\frac{4}{3}\mu^{(h)}}{K^{(i)}+\frac{4}{3}\mu^{(h)}}$	$\frac{\mu^{(h)}+\zeta^{(h)}}{\mu^{(i)}+\zeta^{(h)}}$
Needles	$\frac{K^{(h)}+\mu^{(h)}+\frac{1}{3}\mu^{(i)}}{K^{(i)}+\mu^{(h)}+\frac{1}{3}\mu^{(i)}}$	$\frac{1}{5}\left(\frac{4\mu^{(h)}}{\mu^{(h)}+\mu^{(i)}}+2\frac{\mu^{(h)}+\gamma^{(h)}}{\mu^{(i)}+\gamma^{(h)}}+\frac{K^{(i)}+\frac{4}{3}\mu^{(h)}}{K^{(i)}+\mu^{(h)}+\frac{1}{3}\mu^{(i)}}\right)$
Disks	$\frac{K^{(h)}+\frac{4}{3}\mu^{(i)}}{K^{(i)}+\frac{4}{3}\mu^{(i)}}$	$\frac{\mu^{(h)}+\zeta^{(i)}}{\mu^{(i)}+\zeta^{(i)}}$
Penny cracks	$\frac{K^{(h)}+\frac{4}{3}\mu^{(i)}}{K^{(i)}+\frac{4}{3}\mu^{(i)}+\pi a\beta^{(h)}}$	$\frac{1}{5}\left(1+\frac{8\mu^{(h)}}{4\mu^{(i)}+\pi a(\mu^{(h)}+2\beta^{(h)})}+2\frac{K^{(i)}+\frac{2}{3}(\mu^{(i)}+\mu^{(h)})}{K^{(i)}+\frac{4}{3}\mu^{(i)}+\pi a\beta^{(h)}}\right)$

To understand the origin of (17), it is helpful to notice that for spheres, needles, and disks (see Table 1) we have

$$\frac{1-P^{*i}}{K^*-K^{(i)}} = -\frac{P^{*i}}{K^*+y^{*i}}, \tag{19}$$

where the value of y^{*i} depends on the shape of the inclusion. Substituting this relation into (15), I find that

$$\sum v_i(\alpha^{(i)}-\alpha^*_{CPA})\frac{P^{*i}}{K^*_{CPA}+y^{*i}} = 0. \tag{20}$$

When y^{*i} is independent of i as it is for spheres, I recover (17) exactly, but for other shapes (17) only follows approximately from (15).

Average t-matrix/Kuster-Toksöz scheme

The second approximation scheme I will consider is sometimes called the Average T-Matrix Approximation (ATA) (Berryman, 1992) and sometimes the Kuster-Toksöz (KT) Scheme (Kuster and Toksöz, 1974).

In the absence of a pore fluid, the poroelastic problem reduces again precisely to the elastic composite problem. Following the analysis of Berryman and Berge (1996), I find that the general result (10) is conveniently written as

$$(\mathbf{C}^*-\mathbf{C}_h)\varepsilon = \sum v_i(\mathbf{C}_i-\mathbf{C}_h)\mathbf{G}^{hi}\varepsilon_h. \tag{21}$$

I obtained this form from (10) by noting that $\varepsilon = \sum v_i \varepsilon_i = \sum v_i \mathbf{G}^{ri} \varepsilon_r$. The Kuster-Toksöz approximation includes the assumptions that $\varepsilon = \mathbf{G}^{h*}\varepsilon_h \simeq \mathbf{T}^{h*}\varepsilon_h$ and that $\mathbf{G}^{ri} \simeq \mathbf{T}^{hi}$. Then, the resulting formula for the approximation is

$$(\mathbf{C}^*_{KT} - \mathbf{C}_h)\mathbf{T}^{h*} = \sum v_i (\mathbf{C}_i - \mathbf{C}_h)\mathbf{T}^{hi}. \tag{22}$$

The further assumption is normally made that the tensor $\mathbf{T}^{h*}$ is always the one for spherical inclusions, while $\mathbf{T}^{hi}$ can be for arbitrary shapes of inclusions.

To derive a formula within ATA/KT for the Biot-Willis constant α^*, I need to make use of the Eshelby generalization again and make appropriate substitutions into the formula (14). The thought experiment for KT is a little more complex than that for CPA, however, so I actually need to do this in two steps. First, note that if I view the composite as a finite sphere and imbed this sphere in a host material (that may be and usually is chosen to be the same as one of the constituent materials), then the appropriate generalized Eshelby formula for the poroelastic case is

$$\varepsilon^{(i)} = e^{hi}(p_f) + \mathbf{T}^{hi}\left(\varepsilon - e^{hi}(p_f)\right), \tag{23}$$

where ε is the applied strain at infinity. Equation (23) can then be averaged to give

$$\sum v_i \varepsilon^{(i)} = \sum v_i \left(\mathbf{I} - \mathbf{T}^{hi}\right) e^{hi}(p_f) + \sum v_i \mathbf{T}^{hi}\varepsilon. \tag{24}$$

But now if I consider that the composite has the effective properties $\mathbf{C}^*_{KT}$ and α^*_{KT} in the composite sphere imbedded in the host material, then I can also write

$$\varepsilon^* = e^{h*}(p_f) + \mathbf{T}^{h*}\left(\varepsilon - e^{h*}(p_f)\right), \tag{25}$$

and, since $\sum v_i \varepsilon^{(i)} = \varepsilon^*$ by construction, (25) should be equated to (24). The final result is

$$\left(\mathbf{I} - \mathbf{T}^{h*}\right) e^{h*}(p_f) = \sum v_i \left(\mathbf{I} - \mathbf{T}^{hi}\right) e^{hi}(p_f) + \dots, \tag{26}$$

where the terms indicated by the ellipsis $\dots$ are of the form $\sum v_i(\mathbf{T}^{hi} - \mathbf{T}^{h*})\varepsilon$ and should vanish for similar reasons to those discussed in the case of a corresponding term in the derivation for CPA, since in this case we have $\mathbf{G}^{h*} = \sum v_i \mathbf{G}^{hi}$ as a rigorous result of the theory. Thus, the KT formula for the Biot-Willis parameter α^* is

$$\left(1 - P^{h*}\right)\frac{\alpha^*_{KT} - \alpha^{(h)}}{K^*_{KT} - K^{(h)}} = \sum v_i \left(1 - P^{hi}\right)\frac{\alpha^{(i)} - \alpha^{(h)}}{K^{(i)} - K^{(h)}}. \tag{27}$$

As in the CPA, I now have a subsidiary condition $\sum v_i(\mathbf{T}^{hi} - \mathbf{T}^{h*}) \simeq 0$ that should be checked for approximate satisfaction by $\mathbf{C}^*_{KT}$. Again, we find this condition is satisfied exactly for spherical inclusions.

For nonspherical inclusions, we can again simplify the result (27) by considering formulas such as

$$\frac{1-P^{h*}}{K^{(h)}-K^{*}} = -\frac{P^{h*}}{K^{(h)}+y^{h*}} \quad \text{and} \quad \frac{1-P^{hi}}{K^{(h)}-K^{(i)}} = -\frac{P^{hi}}{K^{(h)}+y^{hi}}, \tag{28}$$

where the y's again depend on the shape of the inclusion. Substituting into (27) and neglecting the differences in the y's, I find that

$$(\alpha^{*}_{KT}-\alpha^{(h)})P^{h*} = \sum v_i(\alpha^{(i)}-\alpha^{(h)})P^{hi}, \tag{29}$$

which should then be compared to

$$(K^{*}_{KT}-K^{(h)})P^{h*} = \sum v_i(K^{(i)}-K^{(h)})P^{hi}, \tag{30}$$

which follows directly from (22).

Differential effective medium approximation

The third scheme I consider is the Differential Effective Medium (DEM) Approximation (Cleary *et al.*, 1980; Norris, 1985; Avellaneda, 1987). I limit the treatment here to the two-component case, as that is the easiest to explain in a small space. This method is derived by assuming the composite is formed by successively mixing very small (infinitesimal) fractions dy of one inclusion material i in another host material. The host medium changes gradually during this process from material h at $y = 0$ into the desired composite material * at some finite y value. Starting with (10), the resulting formula for the stiffness is the differential equation

$$(1-y)\frac{d}{dy}\mathbf{C}^{*}_{DEM}(y) = \left[\mathbf{C}^{(i)} - \mathbf{C}^{*}_{DEM}(y)\right]\mathbf{T}^{*i}, \tag{31}$$

where the initial value of the stiffness tensor is $\mathbf{C}^{*}_{DEM}(y=0) = \mathbf{C}^{(h)}$. The Eshelby-Wu tensor $\mathbf{T}^{*i}$ is the one corresponding to inclusions of stiffness $\mathbf{C}_i$ imbedded in host material of stiffness $\mathbf{C}^{*}_{DEM}$. The resulting system of coupled equations may be integrated to any desired value of total inclusion volume fraction $y = v_i$ easily using (for example) a Runge-Kutta scheme.

The formula for the Biot-Willis parameter is obtained in this scheme most easily by starting from (27), noting first that the sum on the right is reduced to a single term for the phase that is not the initial host phase, replacing the parameters for the host medium by their values evaluated at concentration y and the $*$ parameters by their values evaluated at concentration $y + dy$. The volume fraction is replaced by $v_i \to dy/(1-y)$ to account for the fact that more than the amount dy of the composite host material must be replaced in order to achieve the new desired volume fraction $y + dy$. Finally, taking the limit as $dy \to 0$ gives the desired formula. For spherical inclusions, the result is

$$(1-y)\frac{d}{dy}\alpha^{*}_{DEM}(y) = \left[\alpha^{(i)} - \alpha^{*}_{DEM}(y)\right]P^{*i}, \tag{32}$$

where $\alpha^*_{DEM}(0) = \alpha^{(h)}$. The corresponding result for the bulk modulus obtained directly from (31) is

$$(1-y)\frac{d}{dy}K^*_{DEM}(y) = \left[K^{(i)} - K^*_{DEM}(y)\right] P^{*i}, \tag{33}$$

where $K_{DEM}(0) = K^{(h)}$. Both results were obtained previously for spherical inclusions (Berryman, 1992), but the present derivation is much more compact. The generalization to nonspherical inclusions is now straightforward.

Since this version of DEM is only valid for two component composites, I can take (32) as the correct generalization formula for nonspherical inclusions with P^{*i} evaluated for arbitrary ellipsoidal inclusions. The motivation for this choice is that, when (32) and (33) are taken together, they guarantee satisfaction of the known exact results for two-component materials, since their ratio gives

$$\frac{\frac{d}{dy}\alpha^*_{DEM}}{\alpha^*_{DEM} - \alpha^{(i)}} = \frac{\frac{d}{dy}K^*_{DEM}}{K^*_{DEM} - K^{(i)}}. \tag{34}$$

Upon integration, (34) gives

$$\frac{\alpha^*_{DEM} - \alpha^{(i)}}{\alpha^{(h)} - \alpha^{(i)}} = \frac{K^*_{DEM} - K^{(i)}}{K^{(h)} - K^{(i)}}, \tag{35}$$

as required (see the discussion in the next section). We could alternatively (and more simply) take (35) as the formula for α^*_{DEM} when K^*_{DEM} has been previously computed.

Mori-Tanaka approximation

The final approximation I consider is the Mori-Tanaka (MT) Scheme of Mori and Tanaka (1973), as described by Weng (1984), Benveniste (1987), and others [see, for example, Berryman and Berge (1996)].

For the drained frame, the Mori-Tanaka approximation is obtained by assuming the composite has a host material with imbedded inclusions and then choosing the host to serve as the reference material, so $r = h$. Making this choice in (10) and then substituting $\mathbf{G}^{hi} \simeq \mathbf{T}^{hi}$, I obtain

$$\sum v_i(\mathbf{C}^{(i)} - \mathbf{C}^*_{MT})\mathbf{T}^{hi} = 0. \tag{36}$$

The Mori-Tanaka result for the bulk modulus with arbitrary ellipsoidal inclusion shapes is

$$\sum v_i(K^{(i)} - K^*_{MT})P^{hi} = 0. \tag{37}$$

Because the Mori-Tanaka scheme can *not* be derived using any analogy to scattering theory (unlike the other three schemes considered so far), there is some ambiguity about how to apply the present method within Mori-Tanaka and different choices of formulas for the Biot-Willis parameter may result. One of the more straightforward approaches can be shown to lead to the formula

$$\sum v_i(\alpha^{(i)} - \alpha^*_{MT})P^{hi} = 0, \tag{38}$$

when the inclusions are all spherical in shape. I stress however that (38) is not the only possible formula that could be obtained or that could be considered to be fully consistent with the Mori-Tanaka scheme.

Note that it is easy to show that both (38) and (32) have the advantage that they reproduce the known exact results (Berryman and Milton, 1991) for two component poroelastic media. This fact provides a useful criterion for choosing among various possibilities that arise when trying to identify the proper generalizations of these theories for the poroelastic case.

CONSISTENCY WITH EXACT RESULTS

One particularly powerful means of checking the validity of any estimation scheme is to compare the results with those of various exact results that may be known for special cases. In the present problem, a result of Berryman and Milton (1991) provides a convenient check on all the formulas derived so far. This result states that for an arbitrary two-component mixture of Gassmann materials the Biot-Willis parameter must satisfy the conditions

$$\frac{\alpha^* - \alpha^{(1)}}{K^* - K^{(1)}} = \frac{\alpha^* - \alpha^{(2)}}{K^* - K^{(2)}} = \frac{\alpha^{(2)} - \alpha^{(1)}}{K^{(2)} - K^{(1)}}. \tag{39}$$

It is not hard to show that all the formulas presented satisfy these constraints as long as the side condition that has been mentioned previously, *i.e.*, $\sum v_i(1 - P^{*i}) = 0$ for CPA or the corresponding side condition $P^{h*} = \sum v_i P^{hi}$ for Kuster-Toksöz. For the other two methods, the formulas were actually designed to guarantee satisfaction of (39) directly. For CPA and Kuster-Toksöz, this satisfaction is especially easy to check in the case of spherical inclusions, but is not limited to that case. Thus, the theories presented here all satisfy this important additional condition that any “good” theory should satisfy, when these constraints are satisfied. Part of the art of constructing good approximations then is contained in our choices of models that satisfy these constraints at least approximately.

EXAMPLES

A series of Tables and Figures will now be presented to illustrate the results obtained using the four methods discussed above. Two pf these methods (CPA and DEM) are known to be realizable (Milton, 1985; Avellaneda, 1987). The other two are not. The particular examples were computed assuming four different shapes for the inclusion, but the method is not restricted to the shapes chosen.

Input parameters are from Table 1 for a clay and Kayenta sandstone mixture. Solid sandstone grains occupy from 100% to 60% of the volume, while porous clay (with fixed porosity of 40%) occupies the remaining 0% to 40%. The overall porosity ϕ therefore ranges from 0% to 16%. CPA treats both components symmetrically, neither being singled out as a host material. The other three models have been computed assuming the strong component (the sand grains) is the host and the weak component (porous clay) is the inclusion. The tables then present results for various choices of inclusion shape: spheres, needles, disks, and penny-shaped cracks.

TABLE 2. Material constants for a clay and a Kayenta sandstone.

material	ϕ	K(GPa)	K_m(GPa)	μ(GPa)	μ_m(GPa)
clay	0.4	0.0625	50.0	0.001	0.002
sand	0.0	37.88	37.88	29.0	29.0

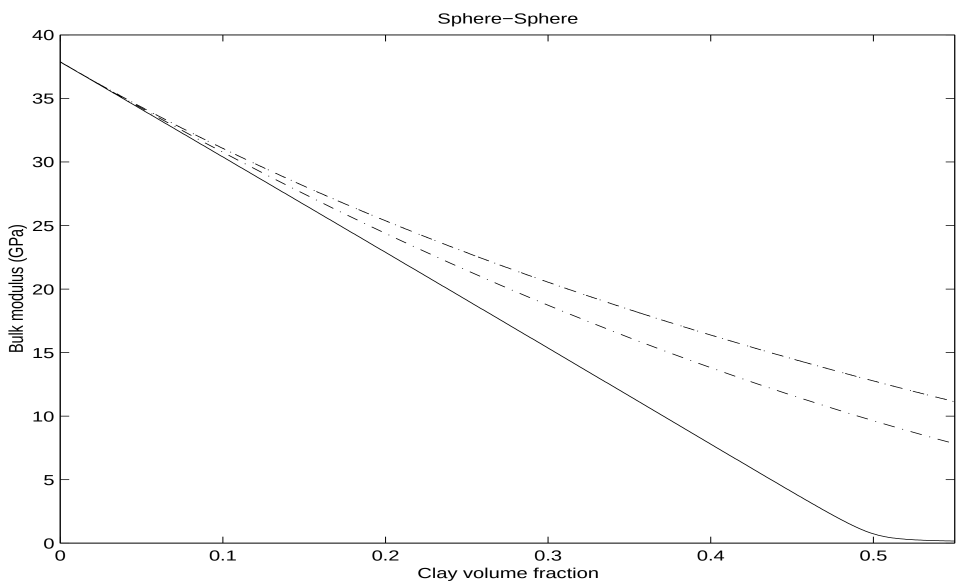

Figure 1: Bulk modulus as a function of clay volume fraction for four models: CPA (solid line), DEM (dot-dash line), Kuster-Toksöz (dotted line), and Mori-Tanaka (dashed line). Both host and inclusion shape factors are assumed to be those for spheres. Note that Kuster-Toksöz and Mori-Tanaka give the same results for this particular case. $\boxed{\text{jim-ssk}}$ [NR]

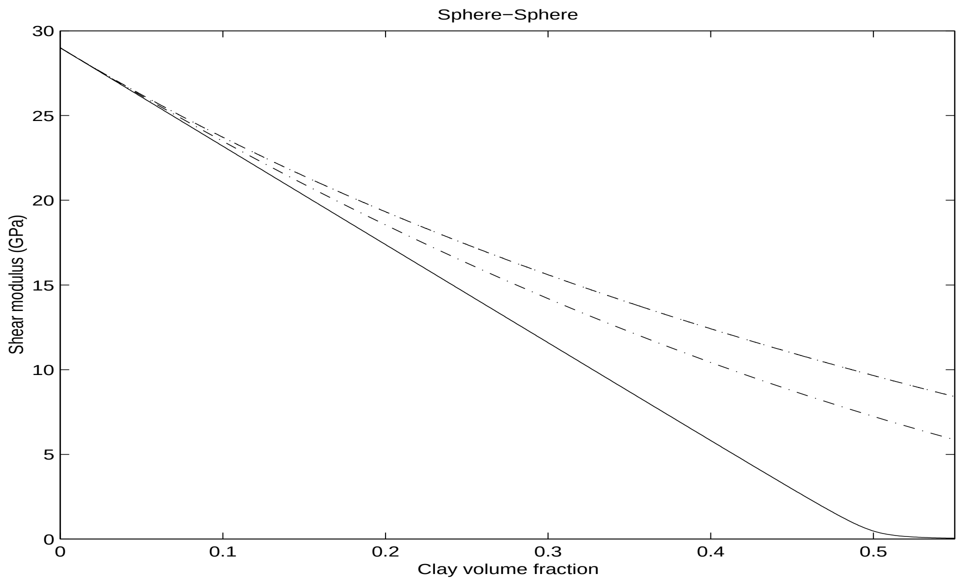

Figure 2: Shear modulus as a function of clay colume fraction for the same four models as in Figure 1. jim-ssm [NR]

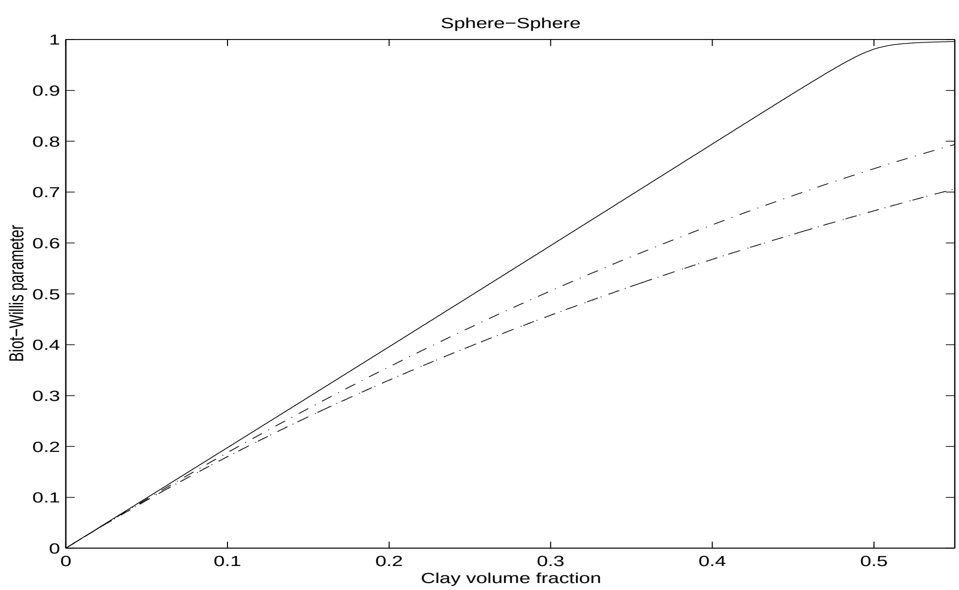

Figure 3: Biot-Willis parameter as a fucntion of clay volume fraction for the same four models as in Figure 1. jim-ssa [NR]

TABLE 3. Computed values of the Biot-Willis parameter α^* using CPA, DEM, Kuster-Toksöz, and Mori-Tanaka, assuming both host and inclusion materials are spherical in shape. The volume fraction of the inclusion is $v_i$and the resulting porosity is ϕ. The three models that distinguish host and inclusion have used sand as host and clay as inclusion for this computation. Note that Kuster-Toksöz and Mori-Tanaka give identical results for this case as expected.

v_i	ϕ	α^*_{CPA}	α^*_{DEM}	α^*_{KT}	α^*_{MT}
0.00	0.00	0.000	0.000	0.000	0.000
0.05	0.02	0.099	0.096	0.094	0.094
0.10	0.04	0.198	0.188	0.180	0.180
0.15	0.06	0.296	0.275	0.258	0.258
0.20	0.08	0.396	0.357	0.330	0.330
0.25	0.10	0.495	0.434	0.397	0.397
0.30	0.12	0.595	0.506	0.458	0.458
0.35	0.14	0.695	0.573	0.515	0.515
0.40	0.16	0.795	0.636	0.568	0.568

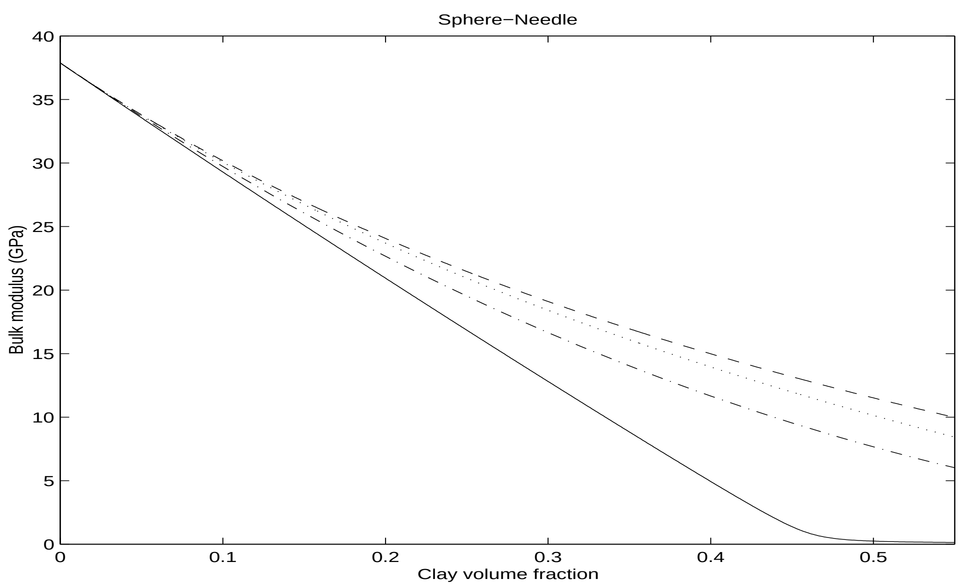

Figure 4: Bulk modulus as a function of clay volume fraction for four models: CPA (solid line), DEM (dot-dash line), Kuster-Toksöz (dotted line), and Mori-Tanaka (dashed line). Host and inclusion shape factors are assumed to be those for spheres and needles, respectively. Note that Kuster-Toksöz and Mori-Tanaka do not give the same results for this case. $\boxed{\text{jim-snk}}$ [NR]

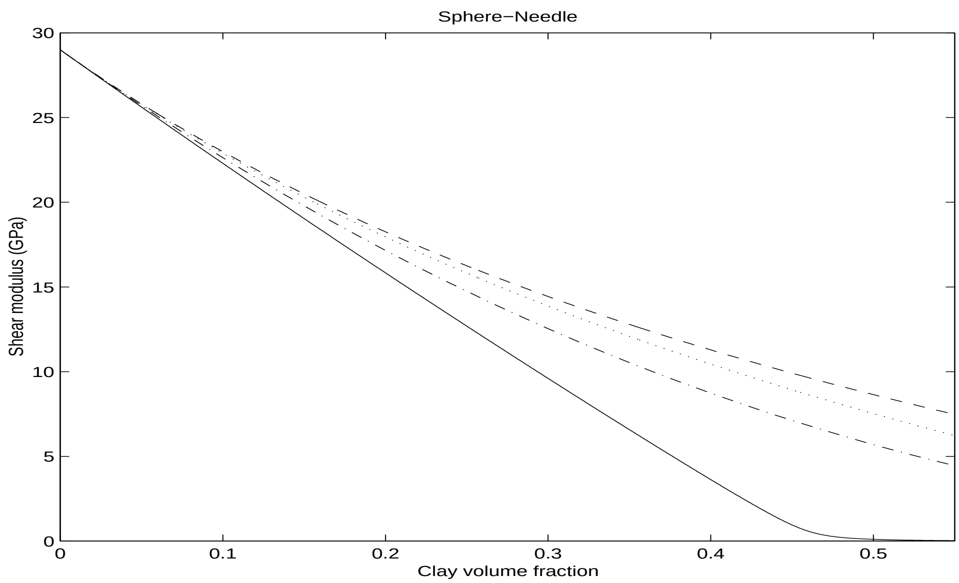

Figure 5: Shear modulus as a function of clay colume fraction for the same four models as in Figure 4. jim-snm [NR]

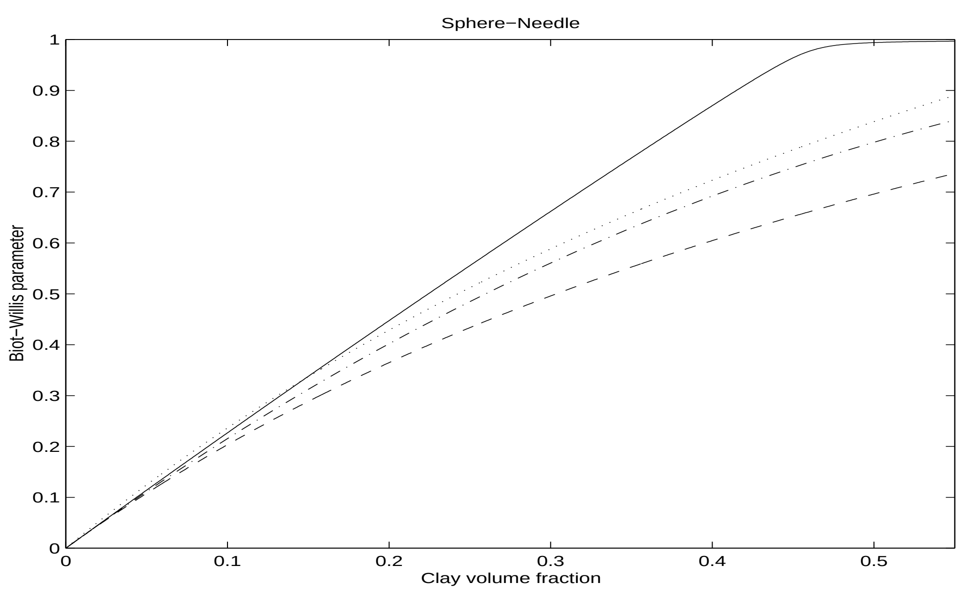

Figure 6: Biot-Willis parameter as a fucntion of clay volume fraction for the same four models as in Figure 4. jim-sna [NR]

TABLE 4. Same as Table 2 except that the host (sand) is assumed to be spherical in shape, while the clay is assumed to be needle-shaped inclusions. Note that Kuster-Toksöz and Mori-Tanaka do not give identical results for this case.

v_i	ϕ	α^*_{CPA}	α^*_{DEM}	α^*_{KT}	α^*_{MT}
0.00	0.00	0.000	0.000	0.000	0.000
0.05	0.02	0.114	0.111	0.124	0.108
0.10	0.04	0.227	0.215	0.236	0.203
0.15	0.06	0.338	0.296	0.337	0.288
0.20	0.08	0.447	0.402	0.429	0.365
0.25	0.10	0.555	0.485	0.512	0.434
0.30	0.12	0.662	0.561	0.588	0.496
0.35	0.14	0.767	0.630	0.659	0.553
0.40	0.16	0.870	0.692	0.723	0.605

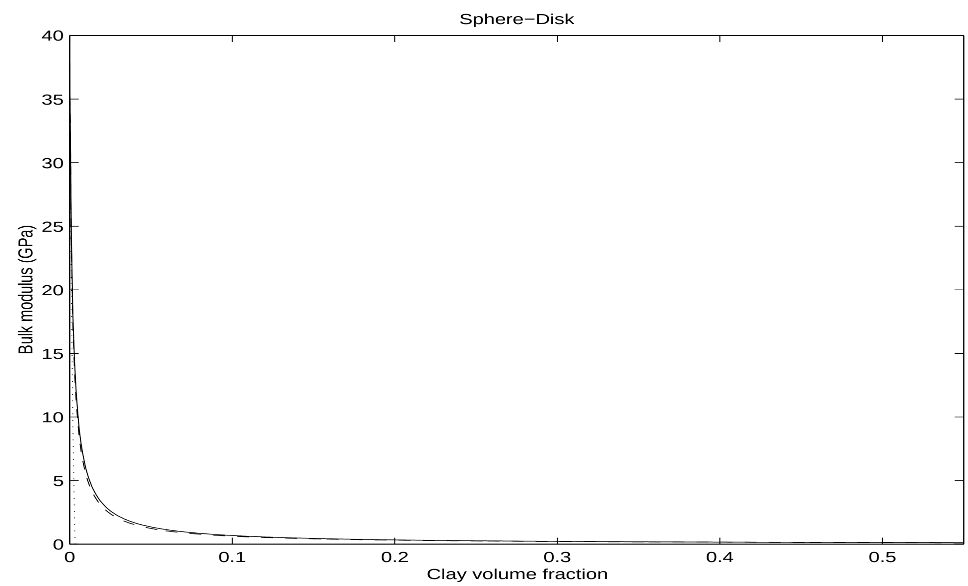

Figure 7: Bulk modulus as a function of clay volume fraction for four models: CPA (solid line), Kuster-Toksöz (dotted line), and Mori-Tanaka (dashed line). Host and inclusion shape factors are assumed to be those for spheres and disks, respectively. DEM is not shown because it predicts that the moduli and Biot-Willis parameter all shift to the values of the clay at very low volume fractions of clay. Note that Kuster-Toksöz and Mori-Tanaka do not give the same results for this case. jim-sdk [NR]

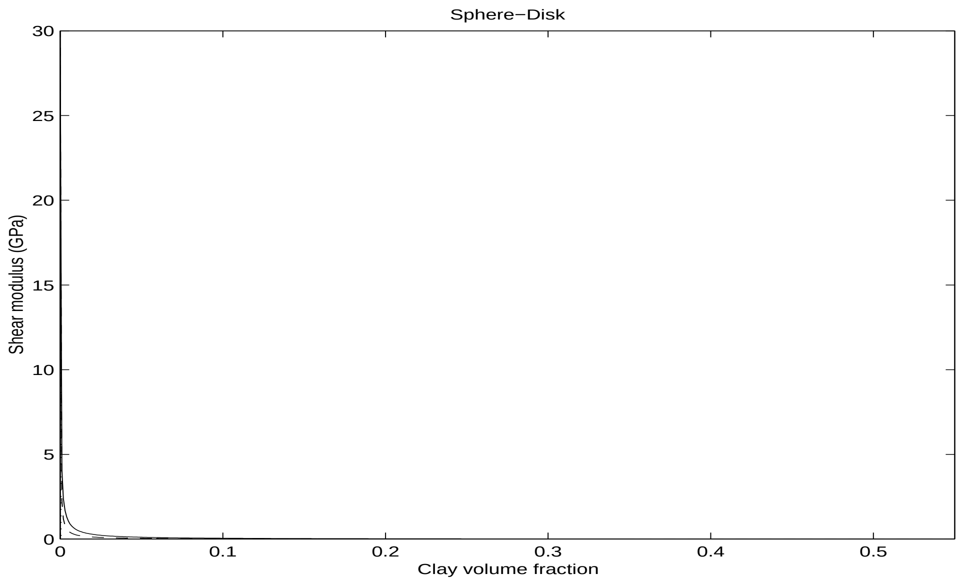

Figure 8: Shear modulus as a function of clay colume fraction for the same three models as in Figure 7. jim-sdm [NR]

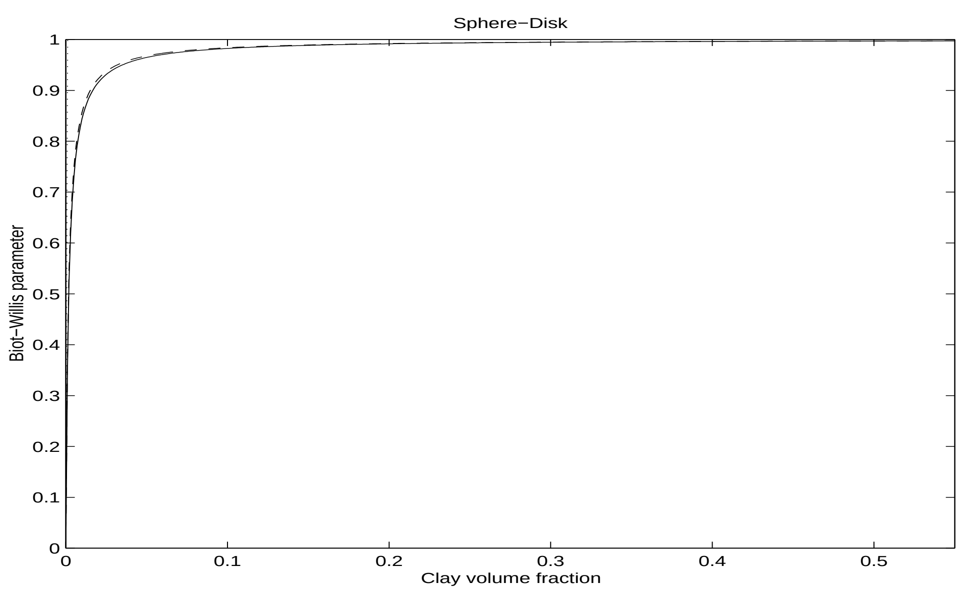

Figure 9: Biot-Willis parameter as a fucntion of clay volume fraction for the same three models as in Figure 7. jim-sda [NR]

TABLE 5. Same as Table 2 except that the host (sand) is assumed to be spherical in shape, while the clay is assumed to be disk-shaped inclusions. Note that Kuster-Toksöz does not give sensible results for this case except at extremely low volume fractions.

v_i	ϕ	α^*_{CPA}	α^*_{DEM}	α^*_{KT}	α^*_{MT}
0.00	0.00	0.000	0.000	0.000	0.000
0.05	0.02	0.965	0.999	—	0.968
0.10	0.04	0.982	0.999	—	0.984
0.15	0.06	0.988	0.999	—	0.989
0.20	0.08	0.991	0.999	—	0.992
0.25	0.10	0.993	0.999	—	0.994
0.30	0.12	0.995	0.999	—	0.995
0.35	0.14	0.995	0.999	—	0.996
0.40	0.16	0.996	0.999	—	0.996

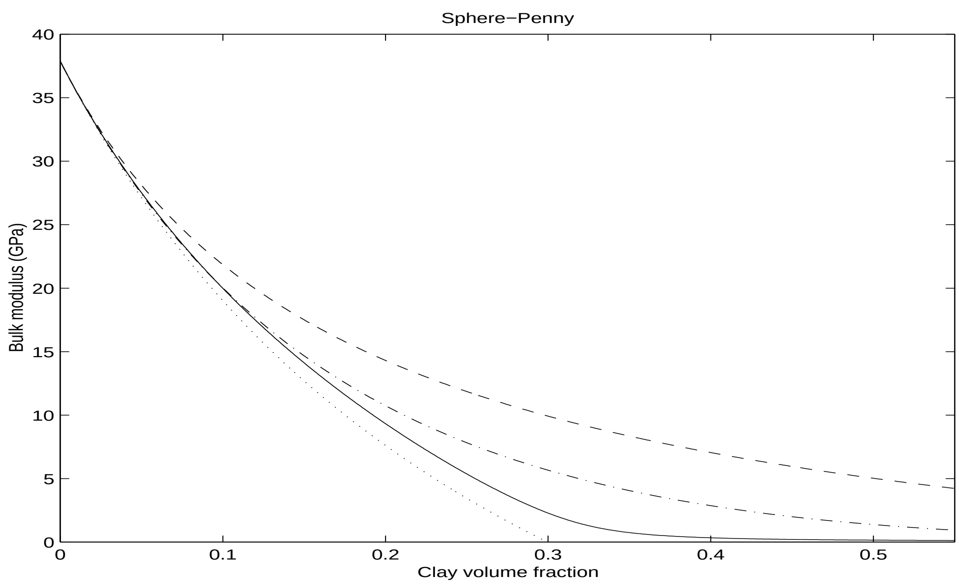

Figure 10: Bulk modulus as a function of clay volume fraction for four models: CPA (solid line), DEM (dot-dash line), Kuster-Toksöz (dotted line), and Mori-Tanaka (dashed line). Host and inclusion shape factors are assumed to be those for spheres and penny cracks, respectively. Note that Kuster-Toksöz and Mori-Tanaka do not give the same results for this case. $\boxed{\text{jim-spk}}$ [NR]

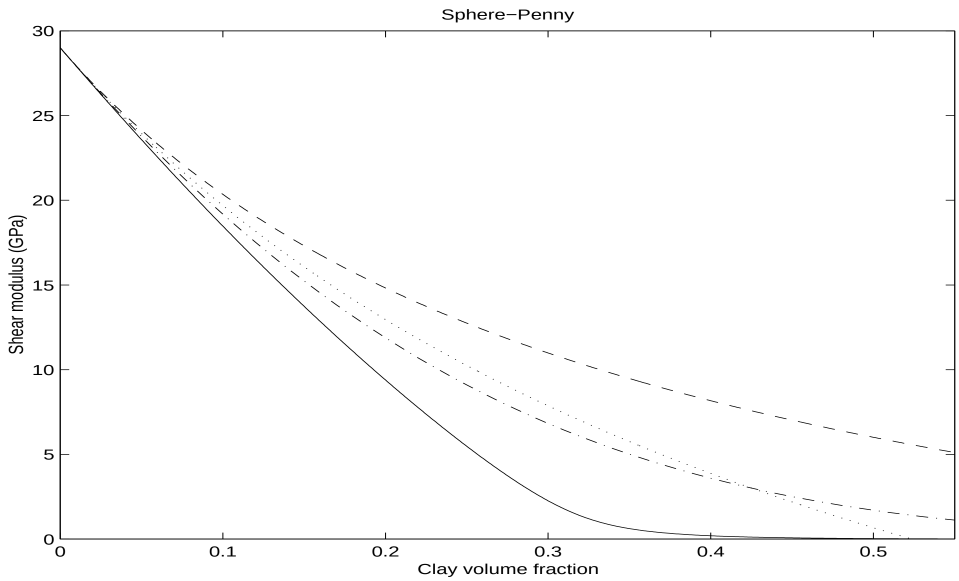

Figure 11: Shear modulus as a function of clay colume fraction for the same four models as in Figure 10. jim-spm [NR]

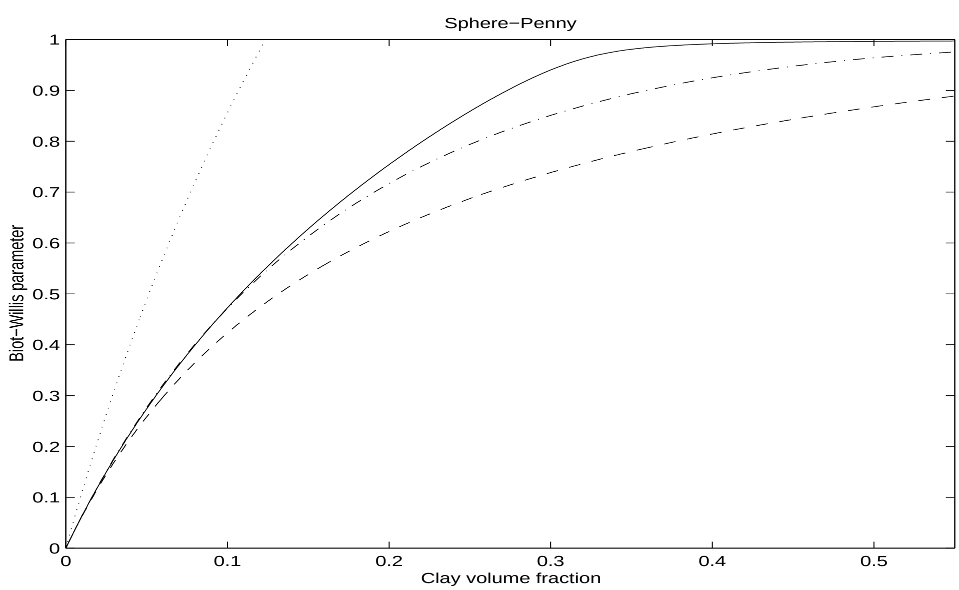

Figure 12: Biot-Willis parameter as a fucntion of clay volume fraction for the same four models as in Figure 10. jim-spa [NR]

TABLE 6. Same as Table 2 except that the host (sand) is assumed to be spherical in shape, while the clay inclusions are assumed to fill penny-shaped cracks. Note that Kuster-Toksöz does not give sensible results for this case, except at very low volume fractions.

v_i	ϕ	α^*_{CPA}	α^*_{DEM}	α^*_{KT}	α^*_{MT}
0.00	0.00	0.000	0.000	0.000	0.000
0.05	0.02	0.274	0.277	0.489	0.258
0.10	0.04	0.473	0.472	0.856	0.423
0.15	0.06	0.628	0.613	—	0.538
0.20	0.08	0.754	0.717	—	0.623
0.25	0.10	0.859	0.794	—	0.687
0.30	0.12	0.940	0.851	—	0.738
0.35	0.14	0.981	0.893	—	0.780
0.40	0.16	0.991	0.925	—	0.814

CONCLUSIONS

I have demonstrated that the generalized Eshelby formula (7) derived earlier (Berryman, 1997) can be successfully used in various well-known effective medium theories to estimate the Biot-Willis parameter when the inclusions are of arbitrary *ellipsoidal* shape. This generalizes other work of the author (Berryman, 1985; 1992) that provided means of computing these same constants but only for the case of *spherical* inclusions. The new formulas are no more difficult to compute that the corresponding formulas for the bulk and shear (empty porous) frame moduli of these materials.

The two most robust theories for the examples shown are clearly the CPA and the Mori-Tanaka theory. Neither of these theories has any problem computing estimates for any of the extreme cases considered. Both DEM and Kuster-Toksöz have problems with the disk or penny-shaped inclusions when the filling material has low shear modulus. This result shows that, although the Kuster-Toksöz and Mori-Tanaka methods are identical for spherical inclusions, they are easily distinguished for non-spherical inclusions and it appears from these examples that Mori-Tanaka may be the preferred explicit scheme of these two.

The work presented is incomplete because it does not yet show how to compute the remaining parameter B (Skempton's coefficient) for a general ellipsoidal inclusion within these various effective medium theories. Nevertheless, the procedure for doing so is a straightforward extension of work published earlier by Berryman and Milton (1991) based on an analysis of (5) and will be presented in a future publication.

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