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An acoustic wave equation for anisotropic media

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ABSTRACT

A wave equation, derived under the acoustic medium assumption for P-waves in transversely isotropic media with a vertical symmetry axis (VTI media), though physically impossible, yields good kinematic approximation to the familiar elastic wave equation for VTI media. The VTI acoustic wave equation is fourth-order and has two sets of complex conjugate solutions. One set of solutions is just perturbations of the familiar acoustic wavefield solutions in isotropic media for incoming and outgoing waves. The second set describes a wave type that propagates at speeds slower than the P-wave for the positive anisotropy parameter, η, and grows exponentially, becoming unstable, for negative values of η. Luckily, most η values corresponding to anisotropies in the subsurface have positive values. Placing the source in an isotropic layer, a common occurrence in marine surveys where the water layer is isotropic, eliminates most of the energy of this additional wave type. Numerical examples prove the usefulness of this acoustic equation in simulating wave propagation in complex models. From this acoustic wave equation, the eikonal and transport equations that describe the ray theoretical aspects of wave propagation are derived. These equations, based on the acoustic assumption (shear wave velocity equals zero), are much simpler than their elastic counterparts, and yet yield exceptionally accurate description of traveltime and geometrical amplitude, or wavefront spreading.

INTRODUCTION

The wave equation is the central ingredient in defining and constraining wave propagation in a given medium. No other constraint, such as the eikonal or raytracing equations, is as conclusive and elaborate (includes all traveltime and amplitude aspects) as the wave equation. Having such an equation in a simple form for transversely isotropic media with a vertical symmetry axis will help us get a better grip on wave propagation in such media.

Wave propagation in the Earth's subsurface is well simulated by solving the elastic

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wave equation. The wavefield, in this case, is described by a three-component displacement vector that includes two types of waves, compressional and shear. Probably, the only drawback of the elastic assumption is that it ignores the inelastic nature of the Earth. Since the wavefield in elastic media is described by a vector, simulating wave propagation using finite-difference techniques requires the dynamic computation of all three components of the wavefield at once, resulting in a process that is computationally expensive. Often, geophysicists have resorted to acoustic media approximations to simulate P-wave propagation. The wavefield in acoustic media is described by a scalar quantity rather than a vector. Kinematically, for P-waves in the far field, the acoustic and elastic wave equations are similar; they both yield the same eikonal equation in isotropic media. The reflection and transmission behavior of waves differs considerably for each of the two media. In elastic media, when encountering an interface, some of the P-wave energy transforms to S-wave energy and vice versa, whereas in acoustic media, all the P-wave energy is conserved.

The importance of the acoustic wave equation in transversely isotropic media with a vertical symmetry axis (VTI media) is not strictly computational. The acoustic wave equation does not yield shear waves, and as a result it can be used for zero-offset modeling of P-waves. No splitting filters (Dellinger and Etgen, 1990) are needed to separate P-waves from S-waves when using this acoustic equation.

In anisotropic media, the acoustic wave equation does not describe a physical phenomenon. This is because acoustic media cannot be anisotropic. If the shear wave velocity equals zero, the medium is rendered isotropic. However, if we ignore the physical aspects of the problem, an acoustic equation for VTI media can be extracted by simply setting the shear wave velocity to zero. Though physically impossible, kinematically the equations resulting from setting the shear wave velocity to zero yield good approximations of the elastic equations.

Numerous attempts through various approximations have been made to simplify the anisotropic equations; some approximations, such as the weak-anisotropy approximation (Thomsen, 1986; Cohen, 1996), elliptical approximations (Helbig, 1983; Dellinger and Muir, 1988), and small dip-angle approximations (Cohen, 1996) do indeed simplify these anisotropic equations considerably. These approximations, nevertheless, have certain limitations that are in some cases unacceptable for use in practice. The approximation considered in this paper (mainly the acoustic assumption of setting the S-wave velocity, V_{S0}, to zero in the new parameter representation) is far more accurate than the weak-anisotropy or small-propagation-angle approximations, as well as yielding simplified equations.

In an earlier paper (1997), I have derived a simple dispersion equation that relates the vertical slowness to the horizontal one in transversely isotropic media. The simplicity of this acoustic equation is a direct result of setting the shear wave velocity to zero. Although the equation results in an approximation, the accuracy is far within the typical accuracies expected in practical geophysical applications. Simply stated, the equation is exact within the confines of seismic error tolerance. This equation served as the starting point for the development of an acoustic wave equation that describes P-wave propagation in VTI media.

In this paper, I first derive the acoustic wave equation for VTI media using the dispersion relation. The acoustic equation is then used to extract the eikonal and transport equations necessary to describe the ray-theoretical aspects of wave propagation in VTI media. Numerical simulations of wave propagation using finite difference techniques demonstrate the accuracy and efficiency of the VTI acoustic wave equation, especially in comparaison with the elastic wave equation.

PARAMETERIZATION IN ANISOTROPIC MEDIA

In this paper, I consider the simplest and probably most practical anisotropic model, that is a transversely isotropic (TI) medium with a vertical symmetry axis. Such a medium is given the same importance in the anisotropic world that $v(z)$ velocity variation has in the inhomogeneous world. Although more complicated anisotropies can exist (such as orthrohombic anisotropy), the large amount of shales (Banik, 1984) present in the subsurface makes the TI model the most influential for P-wave data.

In homogeneous transversely isotropic media with a vertical symmetry axis (VTI media), P- and SV-waves [2] can be described by the vertical velocities V_{P0} and V_{S0} of P- and S-waves, respectively, and the two dimensionless parameters ϵ and δ (Thomsen, 1986), as follows:

$$\epsilon \equiv \frac{c_{11} - c_{33}}{2c_{33}},$$

$$\delta \equiv \frac{(c_{13} + c_{44})^2 - (c_{33} - c_{44})^2}{2c_{33}(c_{33} - c_{44})}.$$

Tsvankin and Thomsen (1994), and more generally Alkhalifah (1997) have demonstrated that P-wave velocity and traveltime are practically independent of V_{S0}, even for strong anisotropy. This finding implies that, for practical purposes, P-wave kinematic signatures can be considered a function of just three parameters: V_{P0}, δ, and ϵ.

Alkhalifah and Tsvankin (1995) further demonstrated that a new representation in terms of just two parameters is sufficient for performing all time-related processing, such as normal moveout correction (including nonhyperbolic moveout correction, if necessary), dip-moveout removal, and prestack and post-stack time migration. These two parameters are the normal-moveout velocity for a horizontal reflector

$$V_{\text{nmo}}(0) = V_{p0}\sqrt{1+2\delta}\,, \tag{1}$$

and the anisotropy coefficient

$$\eta \equiv 0.5\left(\frac{V_h^2}{V_{\text{nmo}}^2(0)} - 1\right) = \frac{\epsilon - \delta}{1+2\delta}\,, \tag{2}$$

[2] I omit the qualifiers in "quasi-P-wave" and "quasi-SV-wave" for brevity

where V_h is the horizontal velocity. Instead of V_{nmo}, I will use v to represent the interval NMO velocity in both isotropic and TI media.

However, if depth is of concern, as in this paper, the vertical P-wave velocity (v_v or V_{P0}) is also needed to characterize the medium. Again, the vertical shear wave velocity has little influence on P-wave propagation in such media, and as a result, it is set to zero with little loss in the accuracy of the VTI equations.

THE VTI ACOUSTIC WAVE EQUATION

I have derived a simple equation that relates the vertical slowness, p_z, to the horizontal one, p_r, in VTI media, based on setting the vertical shear wave velocity to zero (1997). In such media, the slowness surface in the horizontal plane is circular (isotropic), and therefore, p_r can be replaced by $\sqrt{p_x^2+p_y^2}$, where the slowness vector, $\mathbf{p}$, has components in the Cartesian coordinates given by p_x, p_y, and p_z. As a result, the migration dispersion relation in 3-D media is

$$p_z^2 = \frac{v^2}{v_v^2}\left(\frac{1}{v^2} - \frac{p_x^2+p_y^2}{1-2v^2\eta(p_x^2+p_y^2)}\right). \tag{3}$$

Using $\mathbf{k}=\omega\mathbf{p}$, where $\mathbf{k}$ is the wavenumber vector with components in the Cartesian coordinates (k_x, k_y, k_z), and ω is the angular frequency, equation (3) becomes

$$k_z^2 = \frac{v^2}{v_v^2}\left(\frac{\omega^2}{v^2} - \frac{\omega^2(k_x^2+k_y^2)}{\omega^2-2v^2\eta(k_x^2+k_y^2)}\right). \tag{4}$$

Multiplying both sides of equation (4) with the wavefield in the Fourier domain, $F(k_x,k_y,k_z,\omega)$, as well as using inverse Fourier transform on k_z ($k_z \to -i\frac{d}{dz}$) yields

$$\frac{d^2F(k_x,k_y,z,\omega)}{dz^2} = -\frac{v^2}{v_v^2}\left(\frac{\omega^2}{v^2} - \frac{\omega^2(k_x^2+k_y^2)}{\omega^2-2v^2\eta(k_x^2+k_y^2)}\right)F(k_x,k_y,z,\omega). \tag{5}$$

Likewise, using inverse Fourier transform on k_x and k_y ($k_x \to -i\frac{\partial}{\partial x}$, $k_y \to -i\frac{\partial}{\partial y}$), we obtain

$$v_v^2\omega^2\frac{\partial^2 F}{\partial z^2} + 2\eta v^2 v_v^2\left(\frac{\partial^4 F}{\partial x^2\partial z^2}+\frac{\partial^4 F}{\partial y^2\partial z^2}\right) + (1+2\eta)v^2\omega^2\left(\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}\right) + \omega^4 F(x,y,z,\omega) = 0. \tag{6}$$

Finally, applying inverse Fourier transform on ω ($\omega \to i\frac{\partial}{\partial t}$), the acoustic wave equation for VTI media is given by

$$\frac{\partial^4 F}{\partial t^4} - (1+2\eta)v^2\left(\frac{\partial^4 F}{\partial x^2\partial t^2}+\frac{\partial^4 F}{\partial y^2\partial t^2}\right) = v_v^2\frac{\partial^4 F}{\partial z^2\partial t^2} - 2\eta v^2 v_v^2\left(\frac{\partial^4 F}{\partial x^2\partial z^2}+\frac{\partial^4 F}{\partial y^2\partial z^2}\right). \tag{7}$$

This equation is a fourth-order partial differential equation in t. Setting $\eta = 0$ yields the acoustic equation for elliptically anisotropic media with vertical symmetry axis

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 F}{\partial t^2} - v^2\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right) - v_v^2\frac{\partial^2 F}{\partial z^2}\right) = 0. \tag{8}$$

Substituting $P = \frac{\partial^2 F}{\partial t^2}$ gives the familiar second-order wave equation for elliptically anisotropic media with vertical symmetry axis,

$$\frac{\partial^2 P}{\partial t^2} = v^2\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right) + v_v^2\frac{\partial^2 P}{\partial z^2}. \tag{9}$$

For isotropic media, v_v=v and equation (9) reduces to the familiar acoustic wave equation for isotropic media,

$$\frac{\partial^2 P}{\partial t^2} = v^2\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2}\right). \tag{10}$$

Rewriting equation (7) in terms of $P(x, y, z, t)$ instead of $F(x, y, z, t)$, yields

$$\frac{\partial^2 P}{\partial t^2} = (1 + 2\eta)v^2\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right) + v_v^2\frac{\partial^2 P}{\partial z^2} - 2\eta v^2 v_v^2\left(\frac{\partial^4 F}{\partial x^2 \partial z^2} + \frac{\partial^4 F}{\partial y^2 \partial z^2}\right), \tag{11}$$

where

$$F(x, y, z, t) = \int_0^t dt' \int_0^{t'} P(x, y, z, \tau)d\tau.$$

In the numerical implementation, for convenience, I rely on the equation (11).

For comparison, the 2-D elastic wave equation, which is best described in VTI media using the density-normalized elastic coefficients, $A_{ijkl}(= C_{ijkl}/\rho)$, is given by (Aki and Richards, 1980)

$$\frac{\partial^2 u_x}{\partial t^2} = A_{1111}\frac{\partial^2 u_x}{\partial x^2} + A_{1133}\frac{\partial^2 u_z}{\partial x \partial z} + A_{1313}\frac{\partial^2 u_x}{\partial z^2},$$

and

$$\frac{\partial^2 u_z}{\partial t^2} = A_{3333}\frac{\partial^2 u_z}{\partial z^2} + A_{1133}\frac{\partial^2 u_x}{\partial x \partial z} + A_{1313}\frac{\partial^2 u_z}{\partial x^2},$$

where u_x and u_z are the components of the wavefield vector, $\mathbf{u}$, in two dimensions. Solving the elastic wave equation in heterogeneous media requires applying finite-difference computation to two equations (three equations in 3-D media) corresponding to the components of the wavefield. Calculating the wavefield for each component is almost as computationally involved as calculating the acoustic wavefield. This method also incurs the additional expense of input, output, and storage of the wavefield and the corresponding medium parameters in the elastic medium case.

In addition, the solution of the elastic wave equation contains both P- and S-waves, whereas the acoustic equation yields only P-waves. The presence of S-waves in the solution of elastic wave equation makes that equation less desirable when used for modeling P-wave propagation in zero-offset conditions, such as when the exploding reflector assumption is used.

ANALYTICAL SOLUTIONS OF THE VTI WAVE EQUATION

Equations (7) and (11) have four solutions corresponding to the fourth-order nature of these two equations. Two of these four solutions are just perturbations of the isotropic wavefield solutions, and they reduce to these isotropic solutions when η is set to zero; one solution for outgoing waves and the other for incoming waves. The initial condition will resolve whether the wave is outgoing or incoming. Equations (7) and (11) have another set of outgoing and incoming wave solutions. In this section, I solve equation (7) analytically for the case of a homogeneous medium, in hope of providing a better understanding of the two new solutions.

To solve equation (7), we use a plane wave,

$$F(x,z,t) = A(t)e^{i(k_x x + k_z z)},$$

as a trial solution. For simplicity, let us consider the two-dimensional problem, and as a result, drop all terms containing y (or k_y). Substituting the trial solution into the 2-D version of the partial differential equation (7), we obtain the following linear ordinary differential equation in A:

$$\frac{d^4 A}{dt^4} + ((1+2\,\eta)v^2\,{k_x}^2 + {k_z}^2\,{v_v}^2)\frac{d^2 A}{dt^2} + 2\,v^2\,\eta\,{k_x}^2\,{k_z}^2\,{v_v}^2 A = 0. \tag{12}$$

The fact that equation (12) includes only even-order derivates of A implies that we have two sets of complex-conjugate solutions. These solutions are

$$A_1(t) = e^{\pm\sqrt{\frac{a_1}{2}}\,t}, \tag{13}$$

where

$$a_1 = -(1+2\,\eta)v^2\,{k_x}^2 - {k_z}^2\,{v_v}^2 - \sqrt{-8\,v^2\,\eta\,{k_x}^2\,{k_z}^2\,{v_v}^2 + \left(v^2\,{k_x}^2 + 2\,v^2\,\eta\,{k_x}^2 + {k_z}^2\,{v_v}^2\right)^2},$$

and

$$A_2(t) = e^{\pm\sqrt{\frac{a_2}{2}}\,t}, \tag{14}$$

where

$$a_2 = -(1+2\,\eta)v^2\,{k_x}^2 - {k_z}^2\,{v_v}^2 + \sqrt{-8\,v^2\,\eta\,{k_x}^2\,{k_z}^2\,{v_v}^2 + \left(v^2\,{k_x}^2 + 2\,v^2\,\eta\,{k_x}^2 + {k_z}^2\,{v_v}^2\right)^2}.$$

The two sets of solutions differ in the sign in front of the square root in functions a_1 and a_2. Solution (13) reduces to the isotropic medium solution when η=0. Solution (14) represents an additional wave that reduces in the isotropic limit ($\eta \to 0$) to an evanescent wave and, with proper initial conditions, its coefficient reduces to zero. However, this wave will prove to be harmful in the VTI case.

The main concern in equations 13 and 14 is the sign of a_1 and a_2. A negative sign results in an imaginary exponential term that describes wave propagation behavior.

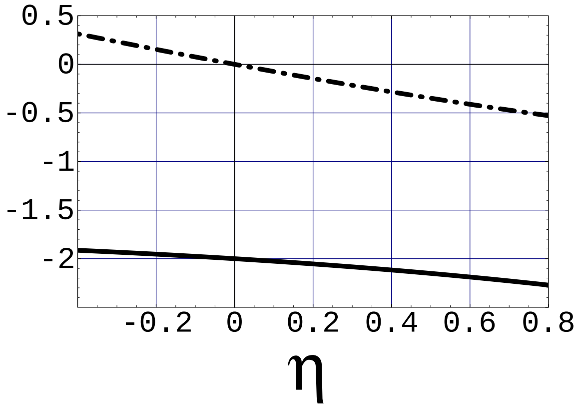

Figure 1: Plots of a_1 (solid curve), and a_2 (dashed curve), from equations (13) and (14), as a function of η. Positive a_1 or a_2 values result in real solutions of the wave equation that either grow or decay exponentially, depending on the sign of the exponent. tariq1-sol2 [NR]

A positive sign results in a real exponential that either decays or grows depending on the sign of the exponent. Considering that we have conjugate solutions, at least one of the solutions will grow exponentially and cause serious instability problems in the numerical implementation regardless of the initial condition. Moreover, numerical noise will insure the instability of the problem when we have a solution that grows exponentially.

Figure 1 shows plots of a_1 (the solid curve), and a_2 (the dashed curve) as a function of η. The value of a_1 is negative independently of the value of η (for practical η values), which corresponds to the P-wave solution. If $\eta = 0$, a_2=0 and the exponential term equals 1. However, for negative η, a_2 is positive and the exponential term grows rapidly with time, a clear area of instability in the problem. Positive η gives negative values of a_2, which results in a wave that is different from the P-wave, propagating in the medium. A numerical example of this phenomenon is given in Figure 2.

Again, two of the solutions [equations (13)] are just perturbations of familiar isotropic medium solutions. Therefore, using the method of perturbation (described in Appendix B), we can obtain these two desired solutions by perturbing the isotropic ones. Doing so should provide accurate results for small η. This procedure is similar to the Born approximation (Cohen and Bleistein, 1979) for small perturbations in velocity.

The unwanted additional wave, which I refer to as an artifact, satisfies the following dispersion relation from a_2:

$$k_z^2 = \frac{\left(\omega^2 - v^2\, k_x{}^2\right) \pm \sqrt{4\, v^2\, \eta\, \omega^2\, k_x{}^2 - 4\, v^4\, \eta\, \left(1+\eta\right)\, k_x{}^4 - \omega^4}}{v_v{}^2}. \tag{15}$$

It has evanescent features (k_z is complex) for $\eta = 0$ (isotropic medium), and therefore exponentially decays as a function of z. Detailed study of this artifact is beyond the scope of this paper. However, putting enough vertical distance between the source and the anisotropic layer will help eliminate this wave.

EIKONAL AND TRANSPORT EQUATIONS

There are two ways to derive the eikonal equation from the above formulas. One way is to substitute $p_x = \frac{\partial t}{\partial x}$, $p_y = \frac{\partial t}{\partial y}$, and $p_z = \frac{\partial t}{\partial y}$ into the dispersion relation [equation (3)] directly. A second method is based on using a ray-theoretical model of the image,

$$F(x,y,z,t) = A(x,y,z)f[t - \tau(x,y,z)],$$

as a trial solution to equation (7). This approach yields the Eikonal equation as well as the transport equation that describes amplitude behavior, $A(x,y,z)$, of wave propagation. Substituting the trial solution into the partial differential equation (7) and considering only the terms with the highest asymptotic order (those containing the fourth-order derivative of F), we arrive at the following eikonal equation:

$$v^2\,(1+2\,\eta)\left(\left(\frac{\partial\tau}{\partial x}\right)^2+\left(\frac{\partial\tau}{\partial y}\right)^2\right)-{v_v}^2\left(\frac{\partial\tau}{\partial z}\right)^2\left(1-2\,v^2\,\eta\left(\left(\frac{\partial\tau}{\partial x}\right)^2+\left(\frac{\partial\tau}{\partial y}\right)^2\right)\right)=1.\quad(16)$$

Again setting $\eta = 0$ and $v_v = v$ gives the isotropic eikonal equation

$$v^2\left(\left(\frac{\partial\tau}{\partial x}\right)^2+\left(\frac{\partial\tau}{\partial y}\right)^2+\left(\frac{\partial\tau}{\partial z}\right)^2\right)=1.\qquad(17)$$

The Eikonal equation includes only the fastest arrival solution, and therefore it includes only P-wave solutions; the slower artifact solution is not given by this eikonal equation.

The next asymptotic order (third-order in derivatives of F) gives us a linear partial differential equation of the amplitude transport, as follows:

$$\left(v^2\,(1+2\,\eta)+2\,v^2\,\eta\,{v_v}^2\left(\frac{\partial\tau}{\partial z}\right)^2\right)\left(2\,\frac{\partial A}{\partial x}\,\frac{\partial\tau}{\partial x}\;+\;2\,\frac{\partial A}{\partial y}\,\frac{\partial\tau}{\partial y}+A\left(\frac{\partial^2\tau}{\partial x^2}+\frac{\partial^2\tau}{\partial y^2}\right)\right)+$$
$${v_v}^2\left(2\,\frac{\partial A}{\partial z}\,\frac{\partial\tau}{\partial z}+A\,\frac{\partial^2\tau}{\partial z^2}\right)-2\,v^2\,\eta\,{v_v}^2\left(A\,\frac{\partial^2\tau}{\partial z^2}\;+\;2\,\frac{\partial\tau}{\partial z}\,\frac{\partial A}{\partial z}\right)\left(\left(\frac{\partial\tau}{\partial x}\right)^2+\left(\frac{\partial\tau}{\partial y}\right)^2\right)-$$
$$8\,v^2\,\eta\,{v_v}^2\,A\,\frac{\partial\tau}{\partial z}\left(\frac{\partial\tau}{\partial x}\,\frac{\partial^2\tau}{\partial x\partial z}\;+\;\frac{\partial\tau}{\partial y}\,\frac{\partial^2\tau}{\partial y\partial z}\right)=0.\qquad(18)$$

The various derivatives of t are computed, as in the case of an isotropic medium, from the solution of the eikonal equation. Despite the apparent complexity of this transport equation, it is linear, and it is also of the first order in derivates of A. Setting $\eta = 0$ and $v_v = v$ in equation (18) gives us the transport equation for isotropic media (Babich and Buldyrev, 1989),

$$2\left(\frac{\partial A}{\partial x}\,\frac{\partial\tau}{\partial x}+\frac{\partial A}{\partial y}\,\frac{\partial\tau}{\partial y}+\frac{\partial A}{\partial z}\,\frac{\partial\tau}{\partial z}\right)+A\left(\frac{\partial^2\tau}{\partial x^2}+\frac{\partial^2\tau}{\partial y^2}+\frac{\partial^2\tau}{\partial z^2}\right)=0.\qquad(19)$$

As expected, computing amplitudes relies on second-order derivatives of traveltime (wavefront curvature). It is also important to keep in mind that equation (18) includes terms corresponding to cross derivates of traveltime (i.e., $\frac{\partial^2\tau}{\partial x\partial z}$ and $\frac{\partial^2\tau}{\partial y\partial z}$) which result from the cross-dependent nature of anisotropy. No $\frac{\partial^2\tau}{\partial x\partial y}$ terms are present in equation (18) because the $x-y$ plane is isotropic.

Another method for calculating traveltimes is ray tracing. Using the method of characteristics, acoustic ray tracing equations for VTI media are derived and shown in Appendix A. Ray tracing is an additional efficient method for calculating traveltime and amplitudes based on the high-frequency approximation. Its main advantage over numerically solving the eikonal equation, is the ability to compute multiarrival traveltimes and amplitudes.

FINITE DIFFERENCE SOLUTIONS OF THE WAVE EQUATION

For simplicity, I use the acoustic wave equation (11), which is second order in t, as opposed to equation (7) [fourth order in derivates of t]. Adding a force function, $f(x, y, z, t)$, to equation (11) yields

$$\frac{\partial^2 P}{\partial t^2} = (1+2\eta)v^2\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right) + v_v^2\frac{\partial^2 P}{\partial z^2} - 2\eta v^2 v_v^2\left(\frac{\partial^4 F}{\partial x^2\partial z^2} + \frac{\partial^4 F}{\partial y^2\partial z^2}\right) + f. \quad (20)$$

In addition, we must solve

$$P = \frac{\partial^2 F}{\partial t^2} \quad (21)$$

for F along with solving equation (20).

Using the second-order finite-difference approach, we can approximate partial derivatives with difference equations, as follows

$$\frac{\partial^2 P}{\partial t^2} \rightarrow \frac{P_{i-1} - 2P_i + P_{i+1}}{\Delta t^2},$$

and

$$\frac{\partial^4 F}{\partial x^2\partial z^2} \rightarrow \frac{4F_{i,j} - 2(F_{i-1,j} + F_{i,j-1} + F_{i+1,j} + F_{i,j+1}) + F_{i-1,j-1} + F_{i+1,j-1} + F_{i+1,j+1} + F_{i-1,j+1}}{\Delta x^2\Delta z^2},$$

where i and j are indexes corresponding to steps in position or time. The later equation is derived from the 2-D convolution of the second-order approximation for each of x and z derivatives, and it is conveniently of a fourth-order accuracy.

Therefore, F in equation (21) is calculated using the following recursive formula:

$$F_{i+1}(x, y, z) = 2F_i(x, y, z) - F_{i-1}(x, y, z) + \Delta t^2 P_i(x, y, z),$$

and P is calculated using

$$P_{i+1}(x,y,z) = 2P_i(x,y,z) - P_{i-1}(x,y,z) + \Delta t^2 \left(\frac{\partial^2 P}{\partial t^2}\right)_i,$$

where $\frac{\partial^2 P}{\partial t^2}$ is the finite-difference approximation of the new Laplacian equation (20).

Higher-order finite-difference approximations (4th order) can be used with equation (7) directly. The set of finite-difference equations (20-21) are subjected to the same constraints and rules used in the isotropic case (such as the CFL condition) to avoid numerical dispersion and instability.

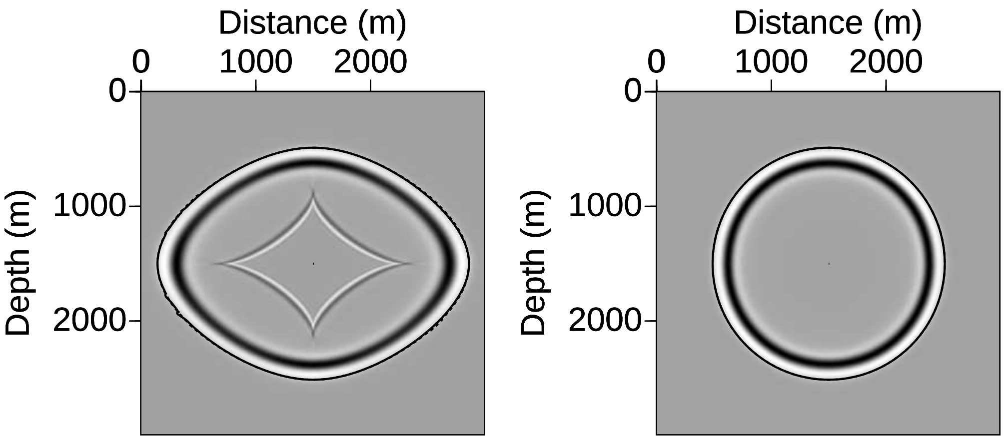

Figure 2: The wavefield at 1 s resulting from a source at center of the section for a VTI medium with η=0.4 (left), and for an isotropic medium (right). In both cases, the velocity (vertical velocity in VTI media) is 1000 m/s. The solid black curves are the solutions of the eikonal equation for both media with the shear wave velocity set to zero, and the dashed curves (not apparent) are the solutions when the shear wave velocity equals half the P-wave velocity. The two curves coincide in the isotropic case and practically coincide in the VTI case. tariq1-fddata [NR]

Figure 2 shows the wavefield at time 1 second caused by an impulse force excited at time 0. On the left side, the medium is homogeneous and VTI with v_v=2 km/s, v=2 km/s, and η=0.4. On the right side, the medium is isotropic with v_v=2 km/s, v=2 km/s, and η=0. Both wavefields are calculated using the second-order finite difference applied to the new acoustic wave equation. In the case of VTI, an additional wave type appears in the section and travels at a speed that is lower than the P-wave velocity. This artifact is the additional solution, mentioned earlier, that behaves like a wave for positive η and exponentially decays or grows for negative η. Such an artifact does not appear in the solution for an isotropic medium. Therefore, we may want to place the source in an isotropic layer and take advantage of the evanescent nature of this wave in isotropic media. The black curves in Figure 2 correspond to solutions of the eikonal equation. The solid curves correspond to the solutions that use the acoustic assumption, in equation (16), in which the shear wave velocity equals zero, while the dashed curves correspond to a shear wave velocity equal to half the P-wave velocity. As expected, in the isotropic case both curves exactly coincide and are therefore indistinguishable. In the VTI case, differences between the two curves exist, but are hardly noticeable. The independence of the eikonal equation on the shear wave velocity in VTI medium is in agreement with the results I obtained in an earlier study (1997).

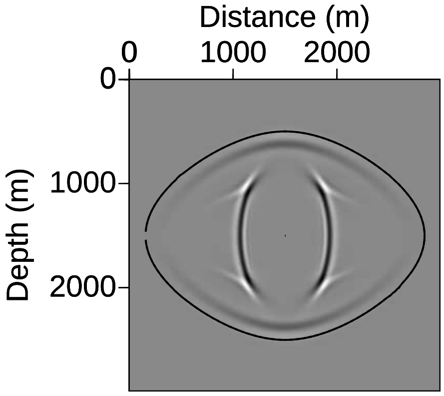

Figure 3: The z-component of the elastic wavefield at 1 s caused by a source at time 0 located at the center. The VTI model is the same as in Figure 2, with v=1000 m/s and η=0.4. The solid curve corresponds to the solution of the acoustic eikonal equation for the same medium. tariq1-fddatatiel [NR]

Figure 3 shows the z-component of the elastic wavefield (computed using the elastic wave equation) for the same model used in Figure 2. The solid curve is the solution of the acoustic eikonal equation. Kinematically, for P-waves, the acoustic and elastic wavefields are similar. Dynamically, they differ considerably; the elastic wavefield includes S-waves (the slower wave), here with triplication, a phenomenon common to S-waves in strongly anisotropic media.

ELIMINATING THE ARTIFACT

To demonstrate some of the features of this additional wave type (the artifact in Figure 2), Figure 4 shows a snapshot taken at 0.13 s of the finite-difference solution of the acoustic VTI wave equation for a two-layered model with the interface at depth 100 m. The first layer is isotropic with velocity equal 1500 m/s, and the second layer is VTI with velocity equal to 2000 m/s and η=0.1. In Figure 4a, the source is placed on the boundary between the isotropic and VTI layers at depth 100 m. As a result, no artifact appears in the isotropic layer, whereas the artifact is clearly apparent in the VTI layer. In Figure 4b, the source is placed in the isotropic layer at depth 95 m from the surface. The resulting additional wave is weaker than in the previous case, where the source was directly interacting with the VTI layer. A greater distance between the source and the VTI layer, as in Figures 4c and 4d, causes the artifact to decay. In fact, when the source is placed at depth 80 m (Figure 4d), only 20 m from the VTI layer, the additional wave practically disappears.

Putting the receivers in the isotropic layer all but assures that no such artifacts appear on synthetic sections. The basic concept is that these waves do not travel in the isotropic layer. Figure 5 shows snapshots of the wavefield for a VTI homogeneous model (top section), a VTI model with a thin isotropic layer of 25 m thickness at top (middle section), and a purely isotropic homogeneous model (bottom section). All three models have a constant velocity of 2000 m/s. As shown earlier, the artifact, illustrated by the diamond shaped wave, appears only when the medium is VTI, as is the case with the top two sections. However, this wave does not travel in the thin isotropic layer in the middle section; it actually reflects at the isotropic-VTI boundary. Therefore, by placing the geophones in this thin isotropic layer, we can avoid recording the artifact. Figure 6 shows common-shot gathers corresponding to the models described in Figure 5. The horizontal line of receivers, in all cases, are placed at depth 10 m from the surface. Therefore, for the VTI model case with the thin isotropic layer (middle section), the artifact, which is apparent for the purely VTI model, disappears. The wave traveltimes, on the other hand, are barely affected by this thin isotropic layer. The purely isotropic model (bottom section) results in slower wave arrival times than those in the VTI models. The wavelet peak frequency of 35 Hz, used in these examples, is typical of surface seismic survey frequencies used nowadays.

A PRACTICAL SUBSURFACE MODEL

Figure 7 shows a velocity model and the corresponding wavefield for a source excited near the surface at lateral distance 1850 m from the origin. The wavefield is calculated using the finite-difference method applied to equation (20) with absorbing boundary conditions. The curves correspond to solutions of the eikonal equation (16) for the VTI model (solid curve) and an equivalent isotropic model (dashed curve). Both models have the same vertical and NMO velocities, and as a result the two wavefront curves coincide at the zero angle from the vertical. The biggest difference between the two wavefronts occurs near horizontal wave propagation, where the influence of the different η values affects the wavefront the most. The corresponding elastic curves (plotted in gray, but indistinguishable) exactly coincide with the acoustic ones. The model was constructed so that the source and receivers are placed in the water layer, which conveniently, is isotropic. As a result, no wave artifacts, similar to the one in Figure 2 appear.

Figure 8 shows common-shot gathers corresponding to the model in Figure 7 computed using geophones placed near the surface. The geophones cover the whole 4-km lateral distance. The top gather in Figure 8 corresponds to the VTI model, and the bottom gather corresponds to the isotropic model. The differences (indicated by arrows) are concentrated at later times, because the largest anisotropies are at depth. Figure 8 also demonstrates the importance of anisotropy in processing; such

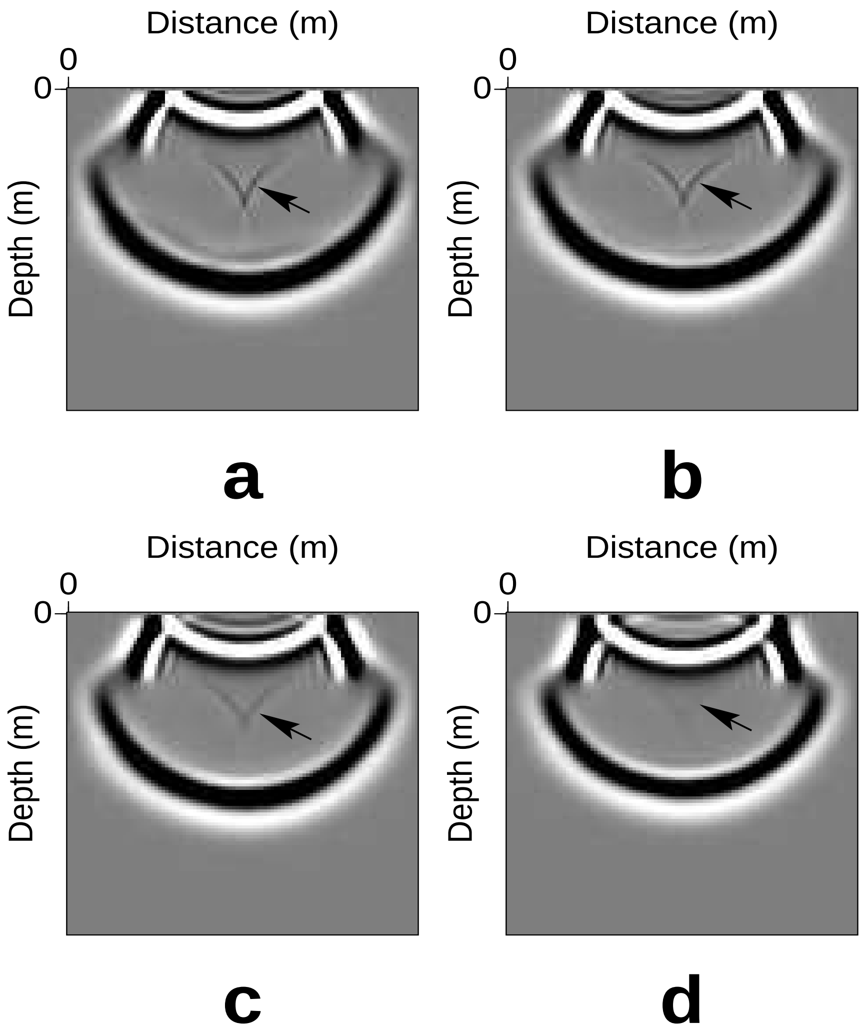

Figure 4: The wavefield at 0.13 s caused by a source at a central location. The model consists of two layers with an interface at depth 100 m. The first layer is isotropic with $v = 1500$ m/s, and the second layer is VTI with v=2000 m/s and $\eta = 0.1$. The source depth varies with (a) the source at depth 100 m (on the interface), (b) the source at depth 95 m, (c) the source at depth 90 m, and (d) the source at depth 80 m. The arrows point to the additional wave as it decays gradually with increasing distance between the source and the VTI layer. tariq1-data-evan [NR]

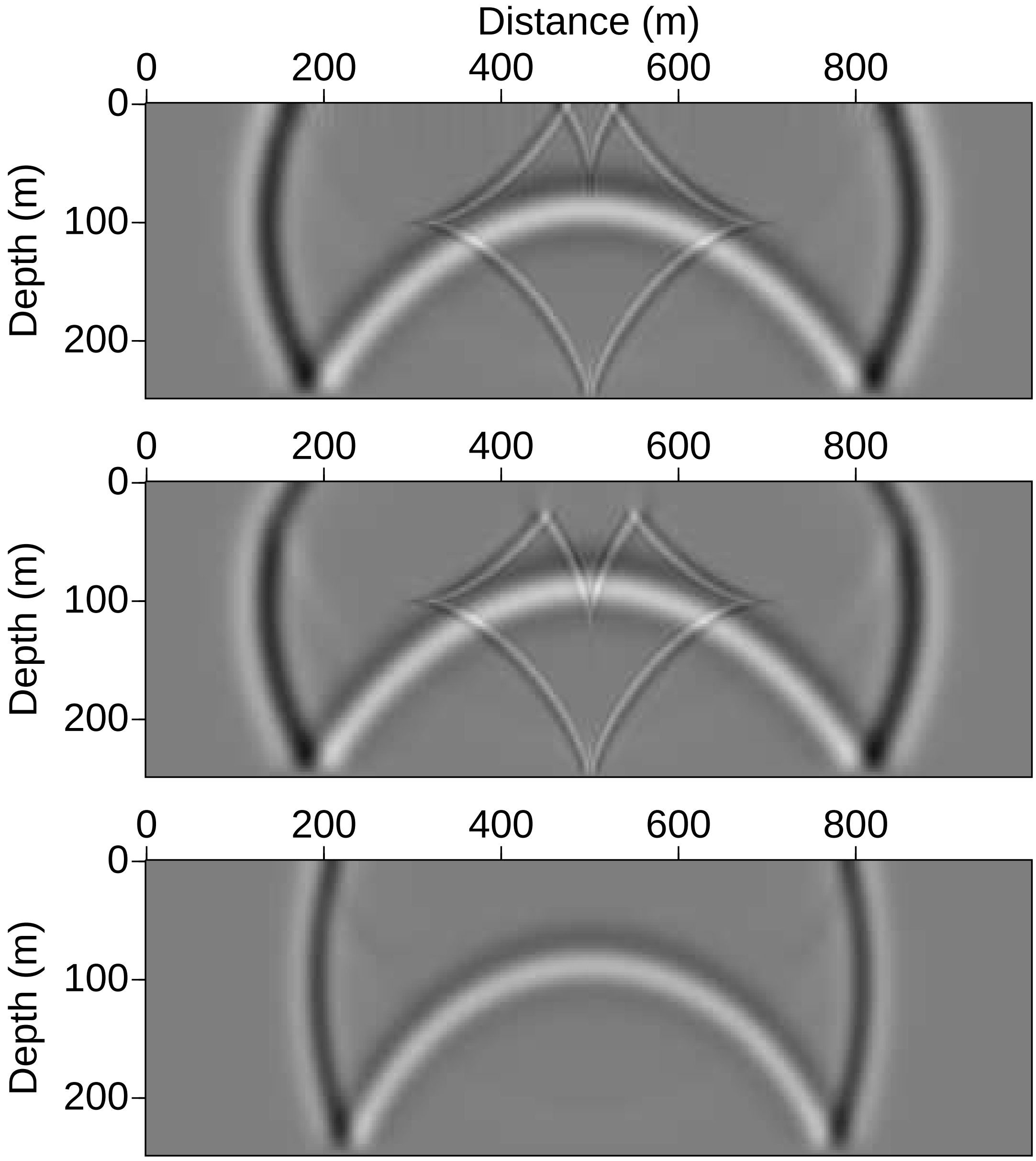

Figure 5: Snapshots of the wavefield at 0.2 s caused by a source at depth 100 m in a central lateral location of 500 m. Top: A snapshot corresponding to a homogeneous VTI medium with η=0.2. Middle: A snapshot for the same medium but with a thin isotropic layer of 25 m thickness at the top. Bottom: A snapshot for an isotropic medium with η=0 throughout. The velocity for all models is the same at 2000 m/s. The peak frequency of the source wavelet is 35 Hz, which is a typical surface seismic survey frequency. tariq1-data-evaneta [NR]

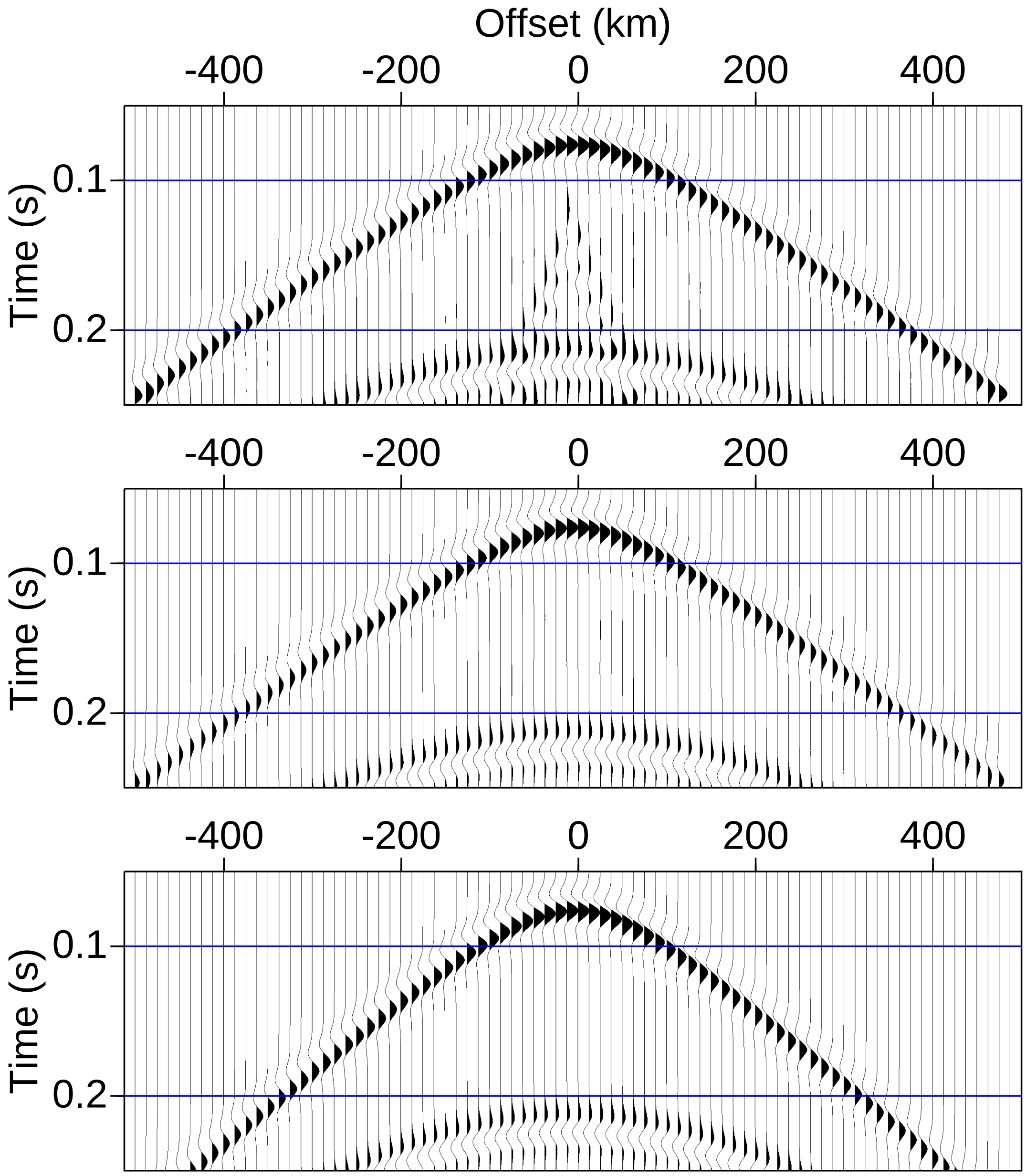

Figure 6: Common-shot gathers for the models described in Figure 5 from top to bottom, respectively. The geophones are placed 10 m under the surface. Top: the gathers are caused by the homogeneous VTI medium with η=0.2. Middle: the gathers are caused by the same homogeneous VTI medium, but with a thin isotropic layer of 25 m thickness at the top. Bottom: the gathers are caused by an isotropic medium. The thin isotropic layer helped remove the artifact, and yet provided accurate traveltimes. tariq1-syn-evan [NR]

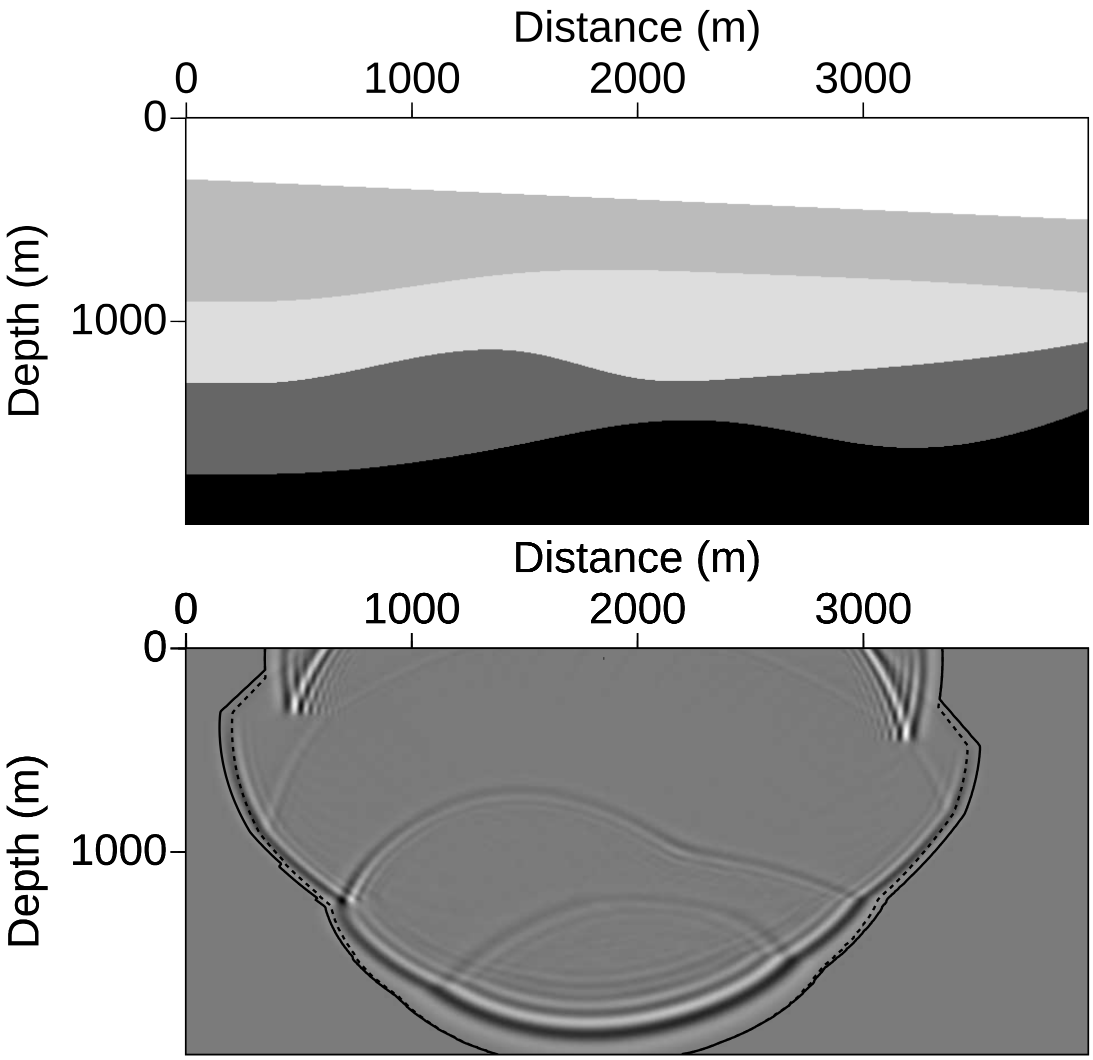

Figure 7: Top: A velocity model consisting of five layers with velocity equal 1500, 1900, 1700, 2400, and 3000 m/s from top to bottom. The corresponding η for the VTI model are 0, 0.1, 0.2, 0.15, and 0.05 from top to bottom. Bottom: The wavefield at 1 s caused by a source at distance 1850 m and depth 50 m for the VTI model. The solid black curve is the solution of the eikonal equation for the VTI model, and the dashed curve is the solution for the corresponding isotropic model. tariq1-fddatamodel [NR]

differences in traveltimes, as well as amplitudes, will considerably hamper isotropic processing when anisotropy similar to that modeled here is ignored.

FINITE DIFFERENCE APPLIED TO THE EIKONAL AND TRANSPORT EQUATIONS

The main feature of applying finite difference to the Eikonal equation, as opposed to the wave equation, is the efficiency gained in calculating traveltimes and computing synthetic sections. A major reason for the cost difference between using the wave equation and Eikonal equation is the difference in the dimensionality of the problem. Using the Eikonal equation we solve for $\tau(x, y, z)$, whereas by using the wave equation we solve for the time-varying wavefield, $W(x, y, z, t)$. Add to that, the fine grid necessary to solve the wave equation to avoid dispersion. However the eikonal equation is a high frequency asymptotic approximation that would only provide traveltime information for the fastest arrival (Vidale, 1990). In complex media, the fastest arrival is not necessary the most energetic, and as a result, traveltime solutions obtained by applying the finite difference scheme on the eikonal equation might result in less sufficient solutions.

CONCLUSIONS

Though physically impossible, the acoustic wave equation for P-waves in transversely isotropic media with a vertical symmetry axis (VTI media) that I have derived yields good kinematic approximations to the familiar elastic wave equation for VTI media. The fourth-order nature of this acoustic equation results in two sets of complex conjugate solutions. One set of solutions are just perturbations of the familiar acoustic wavefield solutions in isotropic media for incoming and outgoing waves. The second set describes a wave type that propagates at speeds slower than the P-wave for the positive anisotropy parameter, η, and grows exponentially, becoming unstable, for negative values of η. Most η values corresponding to anisotropies in the subsurface are likely to have positive values. Placing the source or receivers in an isotropic layer, a common occurrence in marine surveys where the water layer is isotropic, will eliminate most of the energy of this additional wave type. Numerical examples, provided in this paper, prove the usefulness of this acoustic equation in simulating wave propagation in VTI media.

A ray theoretical (high-frequency) approximation is used to derive the eikonal and transport equations that describe the traveltime and amplitude behavior, respectively. These equations are also simpler than those we have grown accustomed to in anisotropic media.

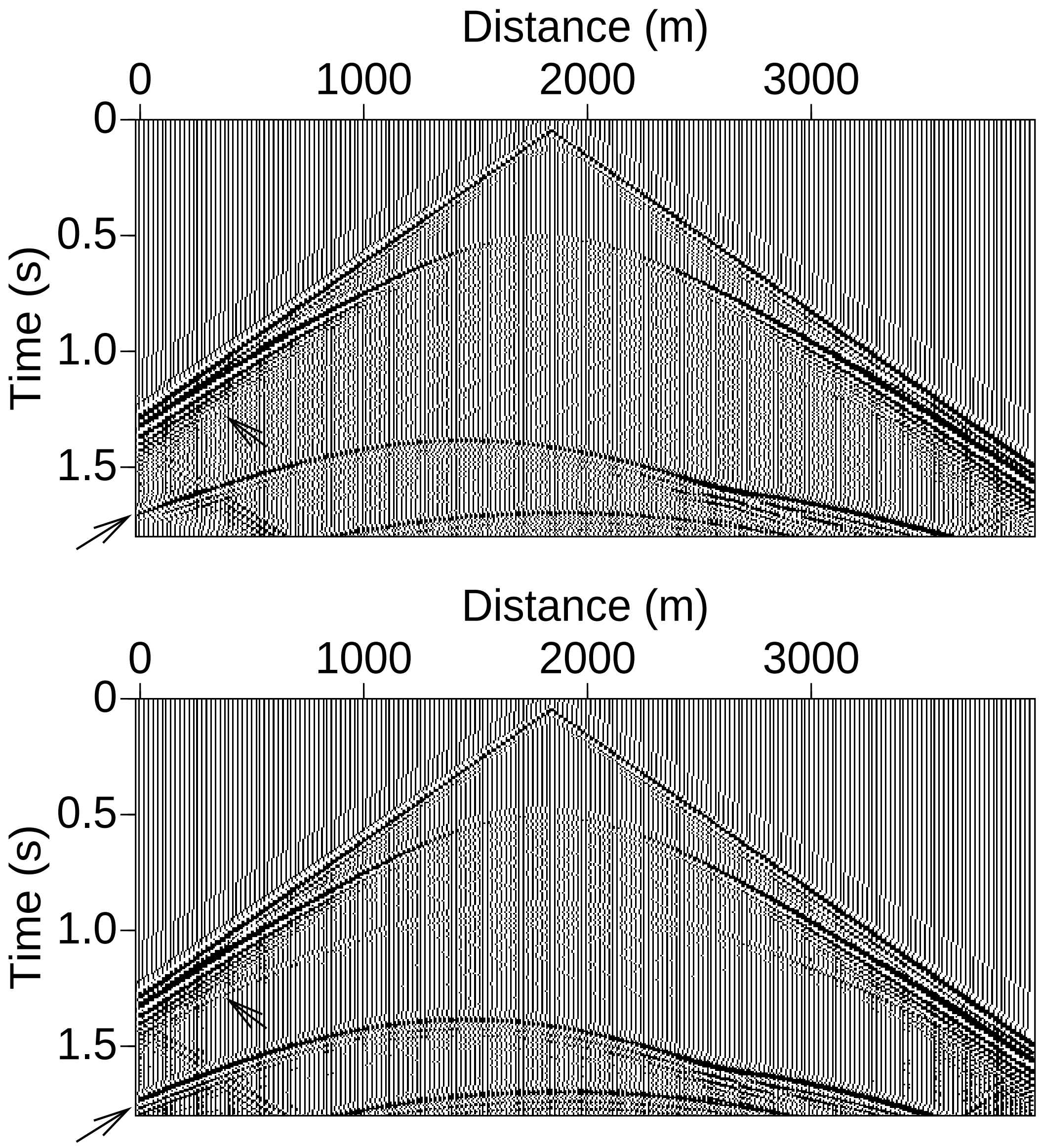

Figure 8: Common-shot gathers for the model in Figure 7 using geophones that span the 4-km distance, buried at depth 50 m from the surface. On top is a shot gather corresponding to the VTI model; on the bottom, a shot gather corresponding to the isotropic model. Both synthetic gathers were calculated using the acoustic wave equation (20). The arrows point to some of the differences in energy arrival times between the two media. tariq1-synmodelf [NR]

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REFERENCES

Aki, K., and Richards, P. G., 1980, Quantitative seismology: Theory and methods: W. H. Freeman and Co., New York.

Alkhalifah, T., and Tsvankin, I., 1995, Velocity analysis for transversely isotropic media: Geophysics, **60**, 1550–1566.

Alkhalifah, T., 1997, Acoustic approximations for seismic processing in transversely isotropic media: accepted for Geophysics.

Babich, V. M., and Buldyrev, V. S., 1989, Asymptotic methods in short-wavelength diffraction theory: Springer Ser. Wave Phen., Vol. 4, Springer, Berlin, Heidelberg.

Banik, N. C., 1984, Velocity anisotropy of shales and depth estimation in the north sea basin: Geophysics, **49**, no. 9, 1411–1419.

Červený, V., and Hron, F., 1980, The ray-series method and dynamic ray tracing for three-dimensional inhomogenous media: Bull. Seism. Soc. Am., **70**, 47–77.

Cohen, J. K., and Bleistein, N., 1979, Velocity inversion procedure for acoustic waves: Geophysics, **44**, no. 6, 1077–1087.

Cohen, J., 1996, Analytic study of the effective parameters for determination of the nmo velocity function in transversely isotropic media: Colorado School of Mines, CWP-191.

Dellinger, J., and Etgen, J., 1990, Wavefield separation in 2-D anisotropic media: Geophysics, **55**, 914–919.

Dellinger, J., and Muir, F., 1988, Imaging reflections in elliptically anisotropic media: Geophysics, **53**, 1616–1618.

Helbig, K., 1983, Elliptical anisotropy - its significance and meaning: Geophysics, **48**, no. 7, 825–832.

Thomsen, L., 1986, Weak elastic anisotropy: Geophysics, **51**, no. 10, 1954–1966.

Tsvankin, I., and Thomsen, L., 1994, Nonhyperbolic reflection moveout in anisotropic media: Geophysics, **59**, no. 8, 1290–1304.

Vidale, J. E., 1990, Finite-difference calculation of traveltimes in three dimensions: Geophysics, **55**, no. 5, 521–526.

APPENDIX A

RAY TRACING EQUATIONS

Using the method of characteristics, we can derive a system of ordinary differential equations that define the ray trajectories. To do so, we need to transform equation (16) to the following form:

$$F\left(x, y, z, \frac{\partial t}{\partial x}, \frac{\partial t}{\partial y}, \frac{\partial t}{\partial z}\right) = 0, \tag{K-1}$$

or

$$F\left(x, y, z, p_x, p_y, p_z\right) = 0, \tag{K-2}$$

where $p_x = \frac{\partial t}{\partial x}$, $p_y = \frac{\partial t}{\partial y}$, and $p_z = \frac{\partial t}{\partial z}$. According to the classic rules of mathematical physics, the solutions of this kinematic equation satisfy

$$\begin{aligned} \frac{dx}{ds} &= \frac{1}{2}\frac{\partial F}{\partial p_x} \quad, \quad \frac{dy}{ds} = \frac{1}{2}\frac{\partial F}{p_y}, \quad \frac{dz}{ds} = \frac{1}{2}\frac{\partial F}{\partial p_z}, \\ \frac{dp_x}{ds} &= -\frac{1}{2}\frac{\partial F}{\partial x} \quad, \quad \frac{dp_y}{ds} = -\frac{1}{2}\frac{\partial F}{\partial y}, \quad \frac{dp_z}{ds} = -\frac{1}{2}\frac{\partial F}{\partial z}, \end{aligned} \tag{K-3}$$

where s is a running parameter along the ray, and it is related to traveltime as follows:

$$\frac{dt}{ds} = p_x\frac{\partial F}{\partial p_x} + p_y\frac{\partial F}{\partial p_y} + p_z\frac{\partial F}{\partial p_z}.$$

As a result,

$$\begin{aligned} \frac{dx}{dt} &= \frac{dx}{ds}\bigg/\frac{dt}{ds} \quad, \quad \frac{dy}{dt} = \frac{dy}{ds}\bigg/\frac{dt}{ds}, \quad \frac{dz}{dt} = \frac{dz}{ds}\bigg/\frac{dt}{ds}, \\ \frac{dp_x}{dt} &= \frac{dp_x}{ds}\bigg/\frac{dt}{ds} \quad, \quad \frac{dp_y}{dt} = \frac{dp_y}{ds}\bigg/\frac{dt}{ds}, \quad \frac{dp_z}{dt} = \frac{dp_z}{ds}\bigg/\frac{dt}{ds}. \end{aligned} \tag{K-4}$$

Using equation (16) to calculate the various derivates of F, we obtain the following system of ordinary differential equations:

$$\frac{dx}{ds} = v^2\, p_x\, \left(1 + 2\,\eta\, \left(1 - {p_z}^2\, {v_v}^2\right)\right), \tag{K-5}$$

$$\frac{dy}{ds} = v^2\, p_y\, \left(1 + 2\,\eta\, \left(1 - {p_z}^2\, {v_v}^2\right)\right), \tag{K-6}$$

$$\frac{dz}{ds} = \left(1 - 2\,v^2\,\eta\, p_r^2\right)\, p_z\, {v_v}^2, \tag{K-7}$$

$$\begin{aligned} \frac{dp_x}{ds} - = &- v\,\left(1 + 2\,\eta\right)\, p_r^2\, v_x - 2\,v^2\, p_r^2\,\eta_x + 2\,v\, p_r^2\, {p_z}^2\, {v_v}^2\,\left(2\,\eta\, v_x + v\,\eta_x\right) \\ &-2\,\left(1 - 2\,v^2\,\eta\, p_r^2\right)\, {p_z}^2\, v_v\,(v_v)_x, \end{aligned} \tag{K-8}$$

$$\frac{dp_y}{ds} - = -v\ (1+2\,\eta)\ p_r^2\, v_y - 2\,v^2\, p_r^2\, \eta_y + 2\,v\, p_r^2\, {p_z}^2\, {v_v}^2\ (2\,\eta\, v_y + v\, \eta_y) \\ -2\left(1 - 2\,v^2\,\eta\, p_r^2\right)\, {p_z}^2\, v_v\, (v_v)_y, \qquad \text{(K-9)}$$

$$\frac{dp_z}{ds} - = -v\ (1+2\,\eta)\ p_r^2\, v_z - 2\,v^2\, p_r^2\, \eta_z + 2\,v\, p_r^2\, {p_z}^2\, {v_v}^2\ (2\,\eta\, v_z + v\, \eta_z) \\ -2\left(1 - 2\,v^2\,\eta\, p_r^2\right)\, {p_z}^2\, v_v\, (v_v)_z, \qquad \text{(K-10)}$$

and

$$\frac{dt}{ds} = \left(v^2\ (1+2\,\eta)\ \left({p_x}^2 + {p_y}^2\right) + \left(1 - 4\,v^2\,\eta\, {p_x}^2 - 4\,v^2\,\eta\, {p_y}^2\right)\, {p_z}^2\, {v_v}^2\right),$$

where $v_x = \frac{\partial v}{\partial x}$, $v_y = \frac{\partial v}{\partial y}$, and $v_z = \frac{\partial v}{\partial z}$, and the same holds for η and (v_v). Recall that $p_r^2 = {p_x}^2 + {p_y}^2$.

Figure K-1 shows sixteen rays originating from a source on the surface at the position $x = 0$ for inhomogeneous models all of which have v_v=v=1.5+0.5 z+0.2 x km/s and $\eta = 0.1 + 0.05z + 0.05x$. These models differ in their shear wave velocities with rays given in the black curves corresponding to the ratio, r, of the vertical S-wave to P-wave velocities of 0.5, and rays given in the gray curves to r=0. The sixteen rays have ray parameters ranging from zero to the maximum value of $1/V_h$ (V_h is the horizontal velocity), with a fixed ray-parameter spacing of $1/(15V_h)$. These rays terminate at the same time of 8 s, and the wavefronts (given by the dashed curves) are plotted at about 1.6-s intervals. The wavefronts corresponding to the different models are practically coincident. This implies that traveltimes extracted from ray tracing are independent of the shear wave velocity, and the acoustic ray equations derived above are almost as accurate as the conventional anisotropic ray tracing equations (Červený and Hron, 1980). The distribution of the rays also indicates that the geometrical spreading features of the two models are practically identical.

APPENDIX B

THE PERTURBATION METHOD

The general procedure of the perturbation method is to identify a small parameter (i.e., η), such that when this parameter is set to zero the problem (for example, the differential equation) becomes soluble. Thematically, the approach decomposes a tough problem into an infinite number of relatively easy ones. Hence, perturbation method is most useful when the few first steps reveal the important features of the solution and the remaining ones give small corrections. Another feature of the perturbation method is that it does not add additional solutions to the solutions already obtained from the unperturbed medium (i.e., background medium). The method simply perturbes those solutions obtained for the background medium (in our case isotropic) to accommodate the perturbation in the medium.

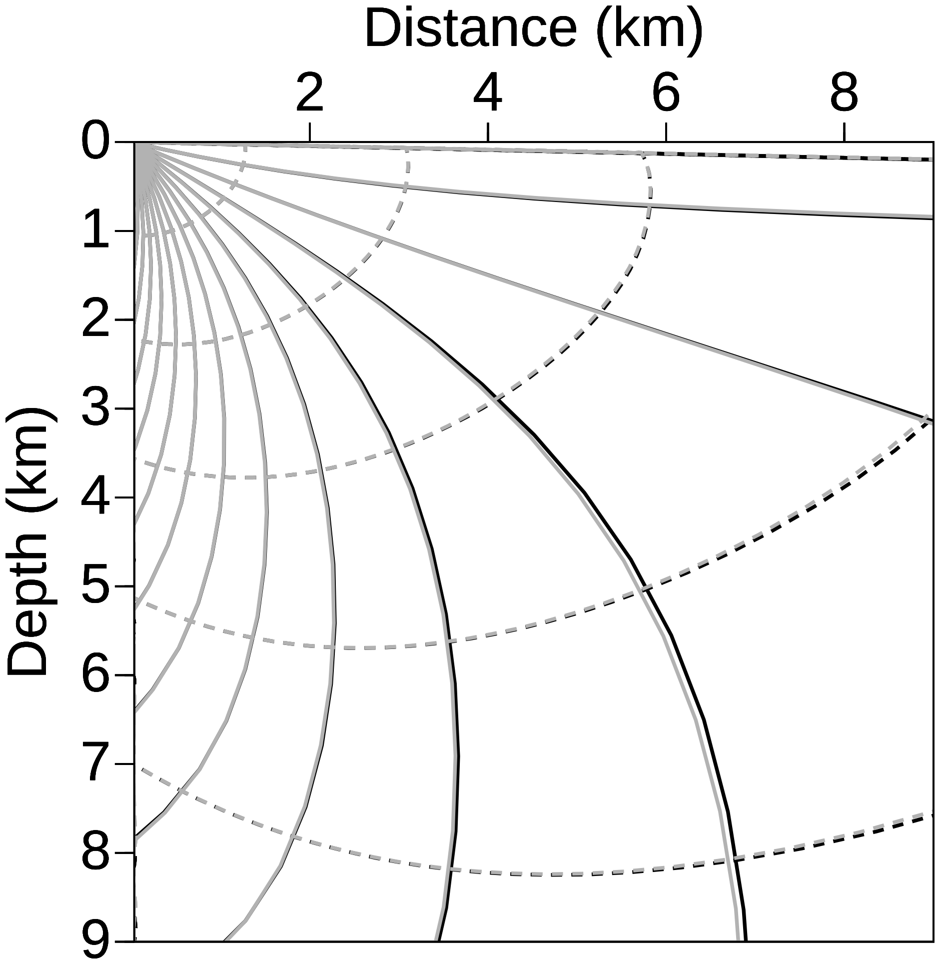

Figure K-1: Raypaths (solid curves) and corresponding wavefronts (dashed curves) for an inhomogeneous VTI model with v_v=v=1.5+0.5 z+0.2 x km/s and $\eta = 0.1 + 0.05z+0.05x$. The black curves correspond to a ratio of the vertical S-wave to P-wave velocity of one half, and gray curves correspond to a zero vertical S-wave velocity. In this case, the curves practically overlap; they are only barely distinguishable. $\boxed{\text{tariq1-raysxz}}$ [NR]

From equation (7), one parameter that can clearly be small is η. Setting η=0 yields the acoustic wave equation for elliptically anisotropic media. For isotropic (or elliptically anisotropic) acoustic media, we have two complex-conjugate solutions; one corresponds to outgoing waves and the other to incoming waves. Perturbation from a background medium of η=0 will result in only two solutions as well.

According to the perturbation theory (Bender and Orszag, 1978), the solution of equation (7) can be represented in a power-series expansion in terms of η. Since in practical applications, this power series is truncated, we use only the first four terms. Therefore, to find the perturbed solution, we write

$$F(x,z,t) = A_0(x,z,t) + A_1(x,z,t)\eta + A_2(x,z,t)\eta^2 + A_3(x,z,t)\eta^3,$$

as a trial solution of equation (7) for the two-dimensional problem. In addition, we consider

$$P(x,z,t) = \frac{\partial^2 F}{\partial t^2} = B_0(x,z,t) + B_1(x,z,t)\eta + B_2(x,z,t)\eta^2 + B_3(x,z,t)\eta^3,$$

as a trial solution of equation (11). Because η is a variable, we can set the coefficients of each power of η separately to equal zero. This will provide a recursive method of solving for the coefficients, A_i and B_i. For example, B_0 is obtained directly from setting η=0, which corresponds to the solution of the elliptically anisotropic problem. Although, η can vary with position, its variation is taken to be much smaller than those associated with the wavefield (which is true asymptotically, where $\omega \to \infty$).

Substituting the trial solution into the partial differential equation (11), and considering only the term of zero-power in η yields

$$\frac{\partial^2 B_0}{\partial t^2} = v_v^2 \frac{\partial^2 B_0}{\partial z^2} + v^2 \frac{\partial^2 B_0}{\partial x^2}. \tag{B-1}$$

If a force function was present in equation (11), the force function would transport as is to this zero-order equation. Equation (B-1) is just the acoustic wave equation for elliptically anisotropic media, and can be solved using the finite-difference approach.

The coefficient of first power in η yields

$$\frac{\partial^2 B_1}{\partial t^2} = v_v^2 \frac{\partial^2 B_1}{\partial z^2} + v^2 \frac{\partial^2 B_1}{\partial x^2} + f_0, \tag{B-2}$$

where

$$f_0 = 2v^2 \left(\frac{\partial^2 B_0}{\partial z^2} - v_v^2 \frac{\partial^4 A_0}{\partial t^2 \partial z^2} \right), \tag{B-3}$$

and

$$B_0 = \frac{\partial^2 A_0}{\partial t^2}.$$

Equation (B-3) is calculated using the solution B_0 of equation (B-1). Note that equations (B-1) and (B-2) are the same differential equations with different forcing functions. In other words, they share the same Green's function, that of the elliptically anisotropic background medium.

The next power in η gives

$$\frac{\partial^2 B_2}{\partial t^2} = v_v^2 \frac{\partial^2 B_2}{\partial z^2} + v^2 \frac{\partial^2 B_2}{\partial x^2} + f_1, \tag{B-4}$$

where

$$f_1 = 2v^2 \left(\frac{\partial^2 B_1}{\partial z^2} - v_v^2 \frac{\partial^4 A_1}{\partial x^2 \partial z^2} \right), \tag{B-5}$$

and

$$B_1 = \frac{\partial^2 A_1}{\partial t^2},$$

which is the same as the previous power η term. Actually, all terms that follow have the same form. A Green's function is calculated for the elliptically anisotropic background problem can be used to recursively estimate all B_i from the different forcing functions, and subsequently to estimate the wavefield.

APPENDIX C

YET ANOTHER VTI WAVE EQUATION

For η=0, the VTI acoustic wave equation reduces to the simple elliptically anisotropic wave equation (9), which is second order in t, and therefore has only two solutions. Assuming that η is small, let us insert, when possible, the 2-D elliptical wave equation

$$\frac{\partial^2 F}{\partial t^2} = v^2 \frac{\partial^2 F}{\partial x^2} + v_v^2 \frac{\partial^2 F}{\partial z^2}. \tag{C-1}$$

in place of one $\frac{\partial^2 F}{\partial t^2}$ in each term of the 2-D version of equation (9),

$$\frac{\partial^4 F}{\partial t^4} = (1+2\eta) v^2 \frac{\partial^4 F}{\partial x^2 \partial t^2} + v_v^2 \frac{\partial^4 F}{\partial z^2 \partial t^2} - 2\eta v^2 v_v^2 \frac{\partial^4 F}{\partial x^2 \partial z^2}. \tag{C-2}$$

The resultant formula after some manipulation is

$$\frac{\partial^2}{\partial t^2} \left(v^2 \frac{\partial^2 F}{\partial x^2} + v_v^2 \frac{\partial^2 F}{\partial z^2} \right) = (1+2\eta) v^4 \frac{\partial^4 F}{\partial x^4} + v_v^4 \frac{\partial^4 F}{\partial z^4} + 2v^2 v_v^2 \frac{\partial^4 F}{\partial x^2 \partial z^2}. \tag{C-3}$$

This equation is second order in time, and therefore has two solutions. To arrive at this form we had to ignore the difference between P and F, which is valid since our main objective is to treat the kinematics of wave propagation.

To understand this equation kinematically, we Fourier transform equation (C-3) in x, z, and t, and obtain the following dispersion relation:

$$v^2\,\omega^2\,{k_x}^2 - v^4\,\left(1 + 2\,\eta\right)\,{k_x}^4 + \omega^2\,{k_z}^2\,{v_v}^2 - 2\,v^2\,{k_x}^2\,{k_z}^2\,{v_v}^2 - {k_z}^4\,{v_v}^4 = 0, \tag{C-4}$$

with k_z satisfying

$$k_z^2 = \frac{\omega^2 - 2\, v^2\, {k_x}^2 \pm \sqrt{\omega^4 - 8\, v^4\, \eta\, {k_x}^4}}{2\, {v_v}^2}. \tag{C-5}$$

Figure C-1 compares k_z given by this dispersion relation with that given by equation (4), which is the exact eikonal equation for acoustic VTI media. The difference is small for $\eta = 0.2$, which implies that equation (C-3) can serve as a valid substitute, kinematically, for equation (11).

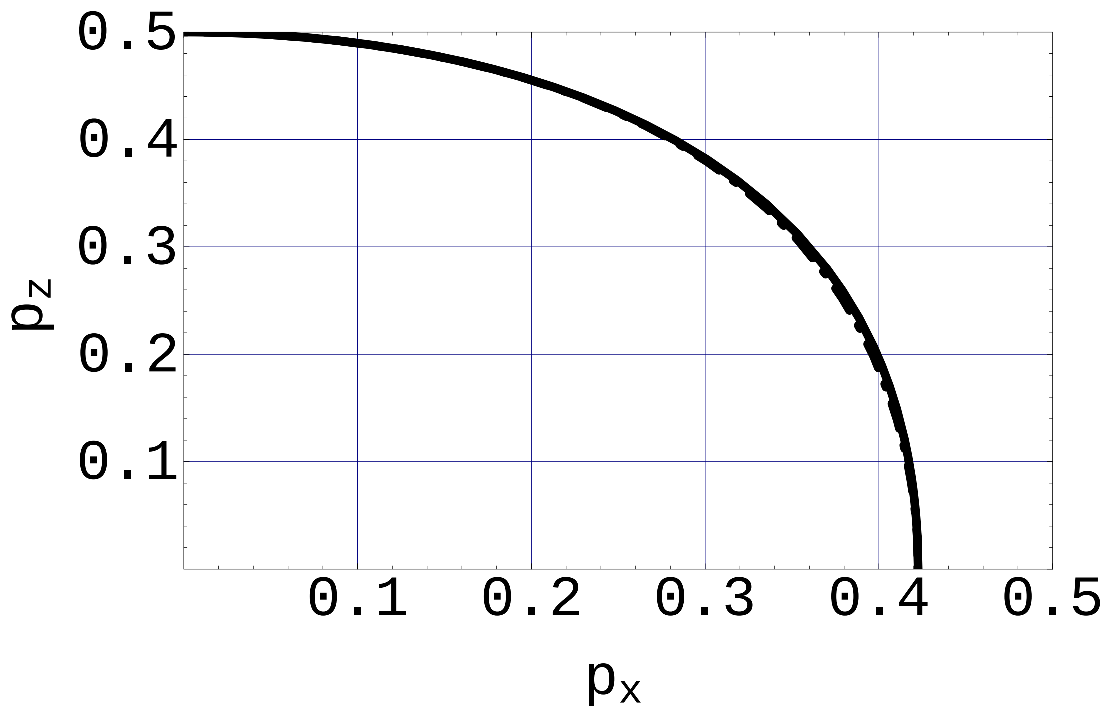

Figure C-1: The vertical component of slowness as a function of the horizontal component in a VTI medium with $\eta = 0.2$ and v =2 km/s. The solid curve corresponds to using approximation (C-5), while the dashed curve corresponds to the exact acoustic eikonal equation for VTI media. tariq1-pxpz2 [NR]