Chapter 1

MZO in 3-D

1.1 Extending the MZO theory to 3-D

Phase-shift methods are fairly easy to generalize to three dimensions, by replacing the scalar wavenumbers with orthogonal vector wavenumbers. In 3-D, the prestack migration equation (Yilmaz, 1979) is

$$p(t=0,\mathbf{k}_y,\mathbf{h}=0,z)=\int d\omega \int d^2k_h\; e^{ik_z(\omega,\mathbf{k}_y,\mathbf{k}_h)z}p(\omega,\mathbf{k}_y,\mathbf{k}_h,z=0),$$

where $p(\omega,\mathbf{k}_y,\mathbf{k}_h,z=0)$ is the 5-D Fourier transform of the field $p(t,\mathbf{y},\mathbf{h},z=0)$ recorded at the surface, and the phase $k_z(\omega,\mathbf{k}_y,\mathbf{k}_h)$ is defined as

$$k_z(\omega,\mathbf{k}_y,\mathbf{k}_h)\equiv -\text{sign}(\omega)\left[\sqrt{\frac{\omega^2}{v^2}-\frac{1}{4}\mid \mathbf{k}_y+\mathbf{k}_h\mid^2}+\sqrt{\frac{\omega^2}{v^2}-\frac{1}{4}\mid \mathbf{k}_y-\mathbf{k}_h\mid^2}\right]. \qquad (1.1)$$

In equation (1.1), $\mid \mathbf{k}_y+\mathbf{k}_h \mid$ is the length of the sum of vectors $\mathbf{k}_h$ and $\mathbf{k}_y$, while $\mid \mathbf{k}_y-\mathbf{k}_h \mid$ is the length of their difference. The 3-D zero offset migration is formulated as

$$p(t=0,\mathbf{k}_y,z)=\int d\omega_0\; e^{ik_z(\omega_0,\mathbf{k}_y)z}p(\omega_0,\mathbf{k}_y,z=0),$$

where $p(\omega,\mathbf{k}_y,z=0)$ is the 3-D Fourier transform of the zero-offset field $p(t,\mathbf{y},z=0)$ recorded at the surface, and the phase $k_z(\omega_0,\mathbf{k}_y)$ is defined as

$$k_z(\omega_0,\mathbf{k}_y)\equiv -\text{sign}(\omega_0)\sqrt{\frac{4\omega_0^2}{v^2}-\mid \mathbf{k}_y\mid^2}. \qquad (1.2)$$

An important step in isolating the MZO operator is the separation of the migration to zero-offset operator from the prestack migration equation. The substitution necessary to convert the double-square root phase into a zero-offset migration operator is found in Appendix 3.A to be:

$$\omega(\omega_0,\mathbf{k}_y,\mathbf{k}_h)=\omega_0\left\{1+\frac{v^2}{4\omega_0^2}\left[k_h^2+\frac{(\mathbf{k}_y\cdot\mathbf{k}_h)}{\frac{4\omega_0^2}{v^2}-k_y^2}\right]\right\}^{\frac{1}{2}}, \qquad (1.3)$$

expression which is consistent with Hale's 1984 formulation.

In order to isolate the zero-offset migration operator the variable ω is replaced by the expression (1.3) in the prestack migration equation. The rest of the steps are similar to the ones in 2-D, where the line integrals in k_h and k_y are replaced by surface integrals. Following the same steps from section 1.2 in Chapter 1, from Appendix 4.A, the expression of the 3-D MZO phase becomes

$$\omega_0^2 = \frac{v^2}{8}\left[\frac{4\omega^2}{v^2} + k_y^2 - k_h^2 + \sqrt{(\frac{4\omega^2}{v^2} + k_y^2 - k_h^2)^2 - 4(\frac{4\omega^2}{v^2}k_y^2 + (\mathbf{k}_y \cdot \mathbf{k}_h)^2 - k_y^2 k_h^2)}\right] \tag{1.4}$$

The final form for the 3-D phase-shift MZO is

$$p(t_0, \mathbf{k}_y) = \int d\omega \int d^2 k_h \; e^{-i\frac{\omega t_0}{\sqrt{2}}\left[1+Y^2-H^2+\sqrt{(1-Y^2-H^2)^2-4(\mathbf{Y}\cdot\mathbf{H})^2}\right]^{\frac{1}{2}}} p(\omega, \mathbf{k}_y, \mathbf{k}_h), \tag{1.5}$$

where I used the customary notations

$$\mathbf{H} = \frac{v\mathbf{k}_h}{2\omega}, \; \mathbf{Y} = \frac{v\mathbf{k}_y}{2\omega}.$$

For the zero-offset case $\mathbf{k}_h = 0$ equation (4.9) becomes an identity. For the flat-layers case $\mathbf{k}_y = 0$, equation (1.5) becomes a Gazdag migration equation with the offset axis as the spatial axis.

1.2 Appendix 4.A: Finding the transformation $\omega \rightarrow \omega_0$ and its inverse

This appendix finds the expression for the variable ω function of ω_0 that transforms the 3-D double-square-root phase used in prestack migration to a new form corresponding to the phase used for 3-D zero-offset migration. The inverse of the transformation $\omega \rightarrow \omega_0$ is the 3-D MZO phase. The following algebraic demonstrations are very boring and only the very determined reader is encouraged to follow them. I included them only to have them handy for reference.

To find the expression for $\omega(\omega_0, \mathbf{k}_y, \mathbf{k}_h)$ and its inverse I start from the double-square-root phase in 3-D and the transformation into a single-square-root form. The 3-D DSR phase is transformed from

$$k_z(\omega, \mathbf{k}_y, \mathbf{k}_h) \equiv -\frac{\omega}{v}\left[\sqrt{1-(Y_1+H_1)^2-(Y_2+H_2)^2} + \sqrt{1-(Y_1-H_1)^2-(Y_2-H_2)^2}\right]$$

to

$$k_z(\omega_0, k_y) \equiv -\frac{2\omega_0}{v}\sqrt{1 - Y_1^2 - Y_2^2}.$$

where I use Yilmaz's notation:

$$H_1 = \frac{vk_{h1}}{2\omega},\ H_2 = \frac{vk_{h2}}{2\omega} \text{ and } Y_1 = \frac{vk_{y1}}{2\omega},\ Y_2 = \frac{vk_{y2}}{2\omega}$$

to denote the components of the normalized offset-midpoint wavenumbers.

To find the transformation of ω that converts (1.1) into (1.2), we equate the two phase expressions and square the equation twice. To reduce the mathematical notations, I introduce again the notations

$$v_{h1} = \frac{vk_{h1}}{2},\ v_{h2} = \frac{vk_{h2}}{2}, \text{ and } v_{y1} = \frac{vk_{y1}}{2},\ v_{y2} = \frac{vk_{y2}}{2}.$$

The first squaring of the equation gives:

$$\omega^2 - 2\omega_0^2 + v_{y1}^2 + v_{y2}^2 - v_{h1}^2 - v_{h2}^2 \qquad =$$

$$\sqrt{\omega^2 - (v_{h1} + vy1)^2 - (v_{h2} + v_{y2})^2}\ \cdot \sqrt{\omega^2 - (v_{h1} - vy1)^2 - (v_{h2} - v_{y2})^2},$$

and after the second squaring we have:

$$\omega^2(\omega_0^2 - v_{y1}^2 - v_{y2}^2) - \omega_0^2(\omega_0^2 - v_{y1}^2 - v_{y2}^2 + v_{h1}^2 + v_{h2}^2) - 2v_{h1}v_{h2}v_{y1}v_{y2} + v_{h1}^2v_{y2}^2 + v_{h2}^2v_{y1}^2 = 0,$$

which allows us to find an expression for ω function of ω_0, that can be used to transform the double square root equation into a single square root equation. The expression for ω becomes

$$\omega^2 = \omega_0^2 + \frac{1}{(\omega_0^2 - v_{y1}^2 - v_{y2}^2)}\left[\omega_0^2(v_{h1}^2 + v_{h2}^2) + 2v_{h1}v_{h2}v_{y1}v_{y2} - v_{h1}^2v_{y2}^2 - v_{h2}^2v_{y1}^2\right]. \quad (1.6)$$

Equation (1.6) can be further transformed by noting that

$$2v_{h1}v_{h2}v_{y1}v_{y2} - v_{h1}^2v_{y2}^2 - v_{h2}^2v_{y1}^2 = -(v_{h1}^2 + v_{h2}^2)(v_{y1}^2 + v_{y2}^2) + (v_{h1}v_{y1} + v_{h2}v_{y2})^2$$

to have

$$\omega^2 = \omega_0^2 + \frac{1}{(\omega_0^2 - v_{y1}^2 - v_{y2}^2)}\left[\omega_0^2(v_{h1}^2 + v_{h2}^2) - (v_{h1}^2 + v_{h2}^2)(v_{y1}^2 + v_{y2}^2) + (v_{h1}v_{y1} + v_{h2}v_{y2})^2\right]. \quad (1.7)$$

Recognizing that k_{h1}, k_{h2} and k_{y1}, k_{y2} are the components of the vectors $\mathbf{k}_h$ and $\mathbf{k}_y$, with respective squared lengths k_h^2 and k_y^2, we have

$$k_{h1}^2 + k_{h2}^2 = k_h^2 \text{ and } k_{y1}^2 + k_{y2}^2 = k_y^2.$$

Moreover, the dot product of the two vectors $\mathbf{k}_h$ and $\mathbf{k}_y$ is

$$\mathbf{k}_h \cdot \mathbf{k}_y = k_{h1}k_{y1} + k_{h2}k_{y2},$$

which can now be introduced in equation (1.7) to obtain a result that is consistent with Hale's (1984) expression 3.19b.

$$\begin{aligned} \omega^2 &= \omega_0^2 + \frac{1}{(\omega_0^2 - \frac{v^2}{4}k_y^2)}\left[\omega_0^2\frac{v^2}{4}k_h^2 - \frac{v^4}{16}k_h^2k_y^2 + \frac{v^4}{16}(\mathbf{k}_y\cdot\mathbf{k}_h)^2\right] \\ &= \omega_0^2\left\{1 + \frac{v^2}{4\omega_0^2}\left[k_h^2 + \frac{(\mathbf{k}_y\cdot\mathbf{k}_h)^2}{\frac{4\omega_0^2}{v^2} - k_y^2}\right]\right\}. \end{aligned} \tag{1.8}$$

The inverse of the transformation (4.8) is an expression of ω_0 function of ω and also the phase of the 3-D MZO:

$$\omega_0^2 = \frac{v^2}{8}\left[\frac{4\omega^2}{v^2} + k_y^2 - k_h^2 + \sqrt{(\frac{4\omega^2}{v^2} + k_y^2 - k_h^2)^2 - 4(\frac{4\omega^2}{v^2}k_y^2 + (\mathbf{k}_y\cdot\mathbf{k}_h)^2 - k_y^2k_h^2)}\right]. \tag{1.9}$$

We can note that equation (4.9) can be transformed back to a double square root expression using the identity

$$\sqrt{a} + \sqrt{b} \equiv \sqrt{a + b + 2\sqrt{ab}}; \quad \text{for } a \geq 0, b \geq 0,$$

only in the form

$$\begin{aligned} \omega_0^2 - \tfrac{v^2k_y^2}{4} &= \frac{v^2}{8}\left[\frac{4\omega^2}{v^2} - k_y^2 - k_h^2 + \sqrt{(\frac{4\omega^2}{v^2} + k_y^2 - k_h^2)^2 - 4(\frac{4\omega^2}{v^2}k_y^2 + (\mathbf{k}_y\cdot\mathbf{k}_h)^2 - k_y^2k_h^2)}\right] \\ &= \frac{v^2}{16}\left[\frac{24\omega^2}{v^2} - 2k_y^2 - 2k_h^2 + 2\sqrt{(\frac{4\omega^2}{v^2} - k_y^2 - k_h^2 - 2(\mathbf{k}_y\cdot\mathbf{k}_h))(\frac{4\omega^2}{v^2} - k_y^2 - k_h^2 + 2(\mathbf{k}_y\cdot\mathbf{k}_h))}\right] \\ &= \frac{v^2}{16}\left[\frac{24\omega^2}{v^2} - 2k_y^2 - 2k_h^2 + 2\sqrt{(\frac{4\omega^2}{v^2} - (\mathbf{k}_y + \mathbf{k}_h)^2)(\frac{4\omega^2}{v^2} - (\mathbf{k}_y - \mathbf{k}_h)^2)}\right] \\ &= \left[\frac{v}{4}(\sqrt{\frac{4\omega^2}{v^2} - (\mathbf{k}_y + \mathbf{k}_h)^2} + \sqrt{\frac{4\omega^2}{v^2} - (\mathbf{k}_y - \mathbf{k}_h)^2})\right]^2 \end{aligned}$$

which is just the initial transformation of a double square root phase to a single square root phase.