Chapter 1

From prestack migration to migration to zero-offset

As a condition for further generalization of the migration to zero-offset in variable velocity media, this chapter develops the theory for 2-D and 3-D migration to zero-offset (MZO) in constant velocity media, starting from prestack migration in midpoint-offset coordinates. At the end of the chapter, I arrive at an integral formulation for the MZO operator, analytically derived from the double square root (DSR) prestack migration equation. The integral formulation for MZO is similar in form to the DSR equation, suggesting a generalization to variable velocity media using a phase-shift algorithm. Further chapters treat offset separation and the depth variable $v(z)$, and laterally variable $v(x, z)$ velocity media.

1.1 Introducing the double square root equation

The theory of the double square root (DSR) equation is discussed in detail in the first chapter of Yilmaz's (1979) thesis. Without going into mathematical detail, this section sketches the path of the basic derivation of the DSR migration equation in offset and midpoint coordinates, starting from the wave equation. Readers familiar with the DSR equation can skip directly to the next section.

The scalar wave equation in a 2-D medium of constant density can be written as

$$\frac{\partial^2 p}{\partial z^2} + \frac{\partial^2 p}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}, \tag{1.1}$$

where $p = p(t, x, z)$ is the pressure field, and $v = v(x, z)$ is the earth velocity. The pressure field $p(t, x, z)$ is a finite function and can be therefore expressed as the double Fourier series

$$p(t, x, z) = \sum_{k_x}\sum_{\omega} P(\omega, k_x, z) e^{i(k_x x - \omega t)}. \tag{1.2}$$

Substituting equation (1.2) into equation (1.1), we obtain

$$\sum_{k_x}\sum_{\omega}[\frac{\partial^2 P(\omega,k_x,z)}{\partial z^2} - k_x^2 P(\omega,k_x,z) + \frac{\omega^2}{v^2(x,z)}P(\omega,k_x,z)]e^{i(k_x x-\omega t)} = 0. \qquad (1.3)$$

Equation (1.3) should hold for any values of k_x and ω. This is possible only if each term inside the square brackets is zero. A similar condition states that if a polynomial is zero for any values of x, the coefficients of the polynomial are zero. Therefore we have

$$\frac{\partial^2 P(\omega,k_x,z)}{\partial z^2} = (k_x^2 - \frac{\omega^2}{v^2(x,z)})P(\omega,k_x,z), \qquad (1.4)$$

which is valid for all values of k_x and ω. The problem with this form is that the x-coordinate in the pressure field is Fourier transformed, and there is no direct correspondence between a point (x,z) in the medium, the velocity $v(x,z)$ at that point, and the corresponding value of $p(t,x,z)$ at the same location.

However, if there is no horizontal variation in velocity, we can define a new variable k_z as

$$k_z = \left[\frac{\omega^2}{v^2} - k_x^2\right]^{\frac{1}{2}}, \qquad (1.5)$$

where k_z is constant for two given values of k_x and ω. Equation (1.5) is the well-known dispersion relation. Substituting k_z in equation (1.4), the equation becomes an ordinary differential equation

$$\frac{\partial^2 P}{\partial z^2} = -k_z^2 P, \qquad (1.6)$$

which has the analytic solution

$$P = P_1\, e^{ik_z z} + P_2\, e^{-ik_z z}. \qquad (1.7)$$

To find the solution to equation (1.7), we would need two initial or boundary conditions. We only have the pressure field at $z = 0$ as a boundary condition, but we can still solve the problem if we decide to resolve only the upgoing waves, in other words to use the exploding reflectors principle.

If we know the pressure field (or wavefield) at a certain depth, we can propagate it forward or backward in time. We can also propagate it upward in depth (along the z-axis) or downward. To understand how to determine the direction of propagation, we have to analyze the values and sign of k_z. The function

$$e^{i(k_z z + k_x x - \omega t)}$$

represents a plane wave. Ignoring $k_x x$, which determines the lateral variation, we can introduce a function which we call $phase(z,t)$, defined as

$$phase(z,t) = k_z z - \omega t.$$

The phase is constant along a plane wave, and for a particular plane wave that we are observing, we write

$$k_z z = \omega t + const.$$

The plane wave is moving *downward* when k_z has the same sign as ω because z increases with t, keeping the phase constant. So for the *upward* moving waves we need to reverse the signs of k_z and ω (z decreases while t increases). Thus, in order to have only upgoing waves, we must look at the sign of ω and assign k_z the opposite sign. Therefore equation (1.7) becomes

$$P = \begin{cases} P_1 \, e^{ik_z z} & ; \quad \omega \leq 0 \\ \\ P_2 \, e^{-ik_z z} & ; \quad \omega \geq 0, \end{cases} \tag{1.8}$$

which can be written in a compact form as

$$P = P_0 e^{-i\mathrm{sign}(\omega) k_z z}, \tag{1.9}$$

where

$$\begin{cases} P_0 = P_1 \; for \; \omega \leq 0 \\ P_0 = P_2 \; for \; \omega \geq 0. \end{cases}$$

Setting $z = 0$ in equation (1.9), we identify P_0 as the data recorded at the surface:

$$P_0 = P(\omega, k_x, z = 0).$$

In this form we can use the data recorded at the surface $P(\omega, k_x, z = 0)$ to propagate the wavefield to any depth level, as follows:

$$P(\omega, k_x, z) = P(\omega, k_x, z = 0) e^{-i\mathrm{sign}(\omega) k_z z}. \tag{1.10}$$

The object of zero-offset migration is to estimate $P(t = 0, k_x, z)$ from $P(t, k_x, z = 0)$. This operation can be done in two steps. The downward continuation step in which we find $P(t, k_x, z)$ from the known $P(t, k_x, z = 0)$, and the imaging step in which we extract $P(t = 0, k_x, z)$ from $P(t, k_x, z)$. Knowing the wavefield at any depth z_0, we can find the wavefield at any other depth $z_0 + z$ using equation (1.10). For positive values of z, we must propagate the wavefield back in time (toward $t = 0$) because the wavefield travels upward. If the known wavefield is at depth z_0 and we want to find the wavefield at depth $z_0 - z$, then we propagate the wavefield forward in time. This is the direction used for modeling.

However, for depth-varying velocity $v(z)$, k_z is approximately constant for small-depth intervals (Δz), where we can consider the velocity constant. Therefore, equation (1.10) becomes

$$P(k_x, z_0 + \Delta z, \omega) = P(k_x, z_0, \omega) e^{-i\mathrm{sign}(\omega) k_z \Delta z} \tag{1.11}$$

and can be used to downward or upward extrapolate the wavefield for a small-depth interval.

There are several restrictions on the values of k_z. Equation (1.6) has the solution (1.7) only for real values of k_z, which imposes the condition that

$$\frac{\omega^2}{v^2} - k_x^2 \geq 0.$$

The solution represented in equation (1.10) is for a single Fourier transform component of the wavefield. The general solution in time-space coordinates is obtained by summing all the Fourier coefficients obtained from equation (1.10), as follows:

$$p(t,x,z) = \sum_{k_x}\sum_{\omega} P(\omega, k_x, z_0) e^{-i\mathrm{sign}(\omega) k_z z} e^{i(k_x x - \omega t)}. \tag{1.12}$$

In the case of a seismic experiment with many shots and receivers, we can downward continue the shots and the receivers separately to any depth level. The total phase shift to the same depth level z becomes the phase shift of the shots plus the phase shift of the receivers:

$$k_z(\omega, k_g, k_s)z = -\mathrm{sign}(\omega)\left[\sqrt{\frac{\omega^2}{v^2} - k_g^2} + \sqrt{\frac{\omega^2}{v^2} - k_s^2}\right] z, \tag{1.13}$$

where k_s and k_g are the shot and receiver wavenumbers. This equation assumes that the shots and geophones are on a flat surface at zero depth ($z = 0$). We can change the system of coordinates from shot and receiver to midpoint and offset using the two simple relations:

$$\begin{aligned} y &= \frac{x_g + x_s}{2} \\ h &= \frac{x_g - x_s}{2}, \end{aligned} \tag{1.14}$$

where y and h are the midpoint and offset coordinates, respectively, while x_s and x_g are the shot and geophone surface coordinates. Note that the variable h represents half the total distance between the source and geophone. The total phase shift in the new wavenumber coordinates becomes

$$k_z(\omega, k_y, k_h)z = -\mathrm{sign}(\omega)\left[\sqrt{\frac{\omega^2}{v^2} - \left(\frac{k_y + k_h}{2}\right)^2} + \sqrt{\frac{\omega^2}{v^2} - \left(\frac{k_y - k_h}{2}\right)^2}\right] z, \tag{1.15}$$

where k_y and k_h are the midpoint and offset wavenumbers, and z represents the depth level to which the wavefield is extrapolated. This formulation allows a wavefield organized in midpoint-offset coordinates to be downward-continued to a certain depth level, and it forms the basis for the prestack migration in midpoint and offset coordinates shown in equation (1.16).

1.2 Isolating the zero-offset migration

The basic concept for analytically deriving the MZO from prestack migration is to separate the latter into two processes:

- Migration to zero-offset.
- Zero-offset migration.

Once we extract the zero-offset migration from the prestack migration operator, what remains is assumed to be an operator that transforms the common-offset data into zero-offset data, hence the name of the operator: migration to zero-offset. I define migration to zero-offset as the operation that converts a common-offset section into a zero-offset section. For a constant velocity medium, this operation is equivalent to the sequence of normal moveout (NMO) followed by dip moveout (DMO).

The constant velocity **prestack migration** in offset-midpoint coordinates (Yilmaz, 1979) is formulated as

$$p(t=0,k_y,h=0,z) = \int d\omega \int dk_h \; e^{ik_z(\omega,k_y,k_h)z} p(\omega,k_y,k_h,z=0), \tag{1.16}$$

where $p(\omega,k_y,k_h,z=0)$ is the 3-D Fourier transform of the field $p(t,y,h,z=0)$ recorded at the surface, using Claerbout's (1985) sign convention:

$$p(\omega,k_y,k_h,z=0) = \int dt \; e^{i\omega t} \int dy \; e^{-ik_y y} \int dh \; e^{-ik_h h} p(t,y,h,z=0).$$

The phase $k_z(\omega,k_y,k_h)$ is defined in the dispersion relation as

$$k_z(\omega,k_y,k_h) \equiv -\mathrm{sign}(\omega)\left[\sqrt{\frac{\omega^2}{v^2}-\frac{1}{4}(k_y+k_h)^2}+\sqrt{\frac{\omega^2}{v^2}-\frac{1}{4}(k_y-k_h)^2}\right]. \tag{1.17}$$

The two integrals in ω and k_h in equation (1.16) represent the imaging condition for zero-offset and zero time $(h=0,t=0)$.

In parallel, the constant velocity **zero-offset migration** (Gazdag, 1978) is formulated as

$$p(t=0,k_y,z) = \int d\omega_0 \; e^{ik_z(\omega_0,k_y)z} p(\omega_0,k_y,z=0) \tag{1.18}$$

where $p(\omega_0,k_y,z=0)$ is the 2-D Fourier transform of the field $p(t,y,z=0)$. The phase $k_z(\omega_0,k_y)$ is defined in the dispersion relation as

$$k_z(\omega_0,k_y) \equiv -2\;\mathrm{sign}(\omega_0)\sqrt{\frac{\omega_0^2}{v^2}-\frac{k_y^2}{4}}. \tag{1.19}$$

In order to convert equation (1.16) into a form similar to equation (1.18), I use a change of variables from ω to ω_0 such that after integrating over the variable k_h, equation (1.16) is transformed into

$$p(t=0,k_y,h=0,z) = \int d\omega_0 \; e^{ik_z(\omega_0,k_y)z} p_0(\omega_0,k_y,z=0),$$

where $p_0(\omega_0,k_y,z=0)$ represents the zero-offset data field. The rationale for casting the prestack migration equation in this form is to identify the operations needed to obtain the zero-offset field from the common-offset field. The assumption is that the output of zero-offset migration and prestack-migration is the same image. By dissecting the prestack-migration operator and separating the zero-offset migration, I isolate the migration-to-zero-offset (MZO) operator.

Using Hale's (1983) derivation, I introduce a new variable ω_0 to isolate the zero-offset migration operator. The expression for the new variable ω_0 is found by equating the dispersion relation for prestack migration to the dispersion relation of zero offset migration, as follows:

$$-\frac{2\omega_0}{v}\sqrt{1-\frac{v^2k_y^2}{4\omega_0^2}} = -\frac{\omega}{v}\left[\sqrt{1-\frac{v^2}{4\omega^2}(k_y+k_h)^2}+\sqrt{1-\frac{v^2}{4\omega^2}(k_y-k_h)^2}\ \right]$$

and squaring the two equations twice. The algebra is demonstrated in detail in Hale's thesis (1983), Appendix 3.A, and thus not repeated here. Hale finds the final expression for the variable ω to be

$$\omega \equiv \omega_0\left[1+\frac{v^2k_h^2}{4\omega_0^2-v^2k_y^2}\right]^{\frac{1}{2}}, \tag{1.20}$$

where ω_0 is considered variable and k_y, k_h are constant. Substituting ω into equation (1.17) transforms the downward continuation phase $k_z(\omega, k_y, k_h)$ into

$$k_z \equiv -2\ \mathrm{sign}(\omega_0)\sqrt{\frac{\omega_0^2}{v^2}-\frac{k_y^2}{4}} = -\frac{2\omega}{v}\sqrt{1-\frac{vk_y}{2\omega}}, \tag{1.21}$$

which now has the same form as the phase in equation (1.19). The somewhat lengthy but straightforward algebraic proof appears in Appendix 1.A.

In order to isolate the zero-offset migration operator, after substituting the expression of ω given by equation (1.20) into the prestack migration equation (1.16), the order of integration is changed between ω_0 and k_h. The integration boundaries have to be observed carefully because they are modified after each change of variables and integration order. However, for the sake of simplicity, the following demonstration ignores the integration limits, which are discussed in Appendix 1.D.

Substituting the variable ω into equation (1.16) with its new expression (1.20) as a function of ω_0 and changing the integration order between ω_0 and k_h transforms the prestack migration equation into

$$\begin{aligned} p(t=0,k_y,h=0,z) &= \int dk_h \int d\omega\ e^{ik_z(\omega,k_y,k_h)z}p(\omega,k_y,k_h,z=0) \\ &= \int d\omega_0\ e^{ik_z(\omega_0,k_y)z}\int dk_h\left[\frac{d\omega}{d\omega_0}\right]\tilde{p}(\omega_0,k_y,k_h) \qquad (1.22) \\ &= \int d\omega_0\ e^{ik_z(\omega_0,k_y)z}p_0(\omega_0,k_y). \end{aligned}$$

The new field $\tilde{p}(\omega_0, k_y, k_h)$ represents a remapping (interpolation) from ω to ω_0 of the field $p(\omega, k_y, k_h, z = 0)$. Each value in the new field $\tilde{p}(\omega_0, k_y, k_h)$ with coordinates (ω_0, k_y, k_h) corresponds to the value in the field $p(\omega, k_y, k_h, z = 0)$ with coordinates $(\omega = \omega_0\sqrt{1 + \frac{H^2}{1-Y^2}}, k_y, k_h)$, where for simplicity I define the variables

$$H = \frac{vk_h}{2\omega} \text{ and } Y = \frac{vk_y}{2\omega}.$$

The field $p_0(\omega_0, k_y)$, defined as

$$p_0(\omega_0, k_y) = \int dk_h \left[\frac{d\omega}{d\omega_0}\right] \tilde{p}(\omega_0, k_y, k_h), \tag{1.23}$$

represents the zero-offset field. The Jacobian in equation (1.23) obtained from the change of coordinates from ω to ω_0 is shown in Appendix 1.B to be

$$J = \left[\frac{d\omega}{d\omega_0}\right] = (1 + \frac{H^2}{1 - Y^2})^{-\frac{1}{2}} \left[1 - \frac{H^2 Y^2}{(1 - Y^2)^2}\right]. \tag{1.24}$$

The last equation in (1.22) is, of course, the zero-offset migration equation (1.18), the classic zero-offset downward continuation and imaging described by Gazdag (1978) and Stolt (1978). Equation (1.23) represents a way of obtaining the zero-offset section from prestack data in midpoint-offset coordinates.

So far, the operations needed to obtain the zero-offset stacked section from the prestack field are to

1. Fourier transform the prestack field $p(t, y, h) \rightarrow p(\omega, k_y, k_h)$.

2. Remap (interpolate) the data field from ω into ω_0.

3. Multiply by the Jacobian.

4. Integrate over k_h.

5. Inverse Fourier transform $p_0(\omega_0, k_y) \rightarrow p_0(t_0, y)$.

However, I want to go further and replace the remapping step with an operation that does not require the interpolation of the initial data. The problem to be solved is similar to the one confronted in Stolt migration. After our data is evenly sampled by an FFT, we need to interpolate it for a different variable.

1.3 MZO as phase shift

The interpolated field $\tilde{p}(\omega_0, k_y, k_h)$ in equation (1.23) represents the values of the field $p(\omega, k_y, k_h)$ after remapping from ω to ω_0. It is obtained by first Fourier transforming the initial prestack field along all three (time, midpoint, and offset) axes, $p(t, y, h) \rightarrow p(\omega, k_y, k_h)$, and then interpolating from ω to ω_0. As in Stolt migration (Popovici et al, 1993), we can replace the two steps of

1. Fourier transform with even sampling in ω and
2. interpolation from ω to ω_0,

with the single step of slow Fourier transform with uneven sampling in ω. To do so, we assume that the initial field is already Fourier transformed in the offset and midpoint coordinates: $p(t,y,h) \rightarrow p(t,k_y,k_h)$. We inverse Fourier transform in time equation (1.23) to get

$$\begin{aligned} p_0(t_0,k_y) &= \int d\omega_0 \; e^{-i\omega_0 t_0} \int dk_h \; J \; \tilde{p}(\omega_0,k_y,k_h) \\ &= \int dk_h \int d\omega_0 \; e^{-i\omega_0 t_0} J \; \tilde{p}(\omega_0,k_y,k_h). \end{aligned} \tag{1.25}$$

With this formulation we can reinterpolate back from ω_0 to ω, and drop the original remapping step by changing the integration variable from ω_0 back to ω. The field $\tilde{p}(\omega_0,k_y,k_h)$ thus reverts to the original field $p(\omega,k_y,k_h)$. Appendix 1.C finds the expression of ω_0 function of ω to be

$$\omega_0 = \frac{\omega}{2}\left[\sqrt{(1-Y)^2 - H^2} + \sqrt{(1+Y)^2 - H^2}\right], \tag{1.26}$$

where I used the customary notations

$$H = \frac{v k_h}{2\omega} \text{ and } Y = \frac{v k_y}{2\omega}.$$

Replacing the variable ω_0 with the new expression in ω and simplifying the Jacobian in equation (1.25) yields

$$p_0(t_0,k_y) = \int dk_h \int d\omega \; e^{-i\frac{\omega t_0}{2}\left[\sqrt{(1-Y)^2-H^2}+\sqrt{(1+Y)^2-H^2}\right]} p(\omega,k_y,k_h). \tag{1.27}$$

Equation (1.27) represents a new form for migration to zero-offset. It is analytically derived from the wave equation, and therefore it treats correctly the kinematics of the DMO+NMO operator and is consistent in amplitude with the DSR equation. It is similar in form to the DSR equation, since the complex exponential operator has the sum of two square roots in its phase. However, downward continuation is performed in time in the case of the MZO operator, not in depth as with DSR migration. This difference suggests the use of a time-varying velocity, which could be more convenient since the $v(x,t)$ velocity is information obtained from surface data and requires fewer assumptions about structure.

Another application in which MZO has potential advantages over full prestack migration is velocity estimation. The velocity function assumed for applying MZO influences the alignment in time of the reflections over offset but not their absolute position. In contrast, the choice of the migration velocity influences the absolute position of the reflections (Fowler, 1988).

Because migration to zero-offset can be used to focus the image without knowing the complete velocity structure, anisotropy can be also included in the focusing step (Dellinger and Muir, 1988). Additionally, the focusing analysis can be used to get a better time interval velocity estimation. Equation (1.27) is velocity dependent and the velocity can be used as a parameter for focusing analysis over different offsets, not only for zero-offset. Such an analysis will implicitly handle the anisotropic velocity variations and the depth structure velocity since it is performed in time and not in depth.

The only drawback so far to equation (1.27) is that it performs a Fourier transform and later a summation over the offset variable. The next chapter shows how the offset variable can be separated and thus MZO can be applied to distinct common-offset sections. After applying MZO to separate common-offset sections, I isolate the conventional NMO and DMO processes. Further, Chapter 3 shows how equation (1.27) is applied to variable-velocity media, using a phase-shift algorithm similar to Gazdag migration, and PSPI or split-step migration.

1.4 Appendix 1.A: Transforming the double-square-root phase into single-square-root form

This appendix shows that writing the variable ω function of ω_0 transforms the double-square-root (DSR) phase used in prestack migration to a new form corresponding to the phase used for zero-offset migration. The transformation from ω to ω_0, as defined in equation (1.20), is

$$\omega \equiv \omega_0 \left[1 + \frac{v^2 k_h^2}{4\omega_0^2 - v^2 k_y^2} \right]^{\frac{1}{2}} \equiv \omega_0 \left[1 + \frac{v_h^2}{\omega^2 - v_y^2} \right]^{\frac{1}{2}},$$

where v_h and v_y are defined as $v_h = \dfrac{v k_h}{2}$; $v_y = \dfrac{v k_y}{2}$.

The DSR phase is transformed from

$$k_z(\omega, k_y, k_h) \equiv -\frac{\omega}{v} \left[\sqrt{1 - (Y+H)^2} + \sqrt{1 - (Y-H)^2} \right]$$

to

$$k_z(\omega_0, k_y) \equiv -\frac{2\omega_0}{v} \sqrt{1 - Y^2}.$$

Hale (1983), in Appendix 3.A of his thesis, proves an equivalent assertion with different logic. Comparing the DSR phase with the phase of the zero-offset migration (defined as a single square root), Hale finds the expression of ω_0 in equation (1.20). In this appendix I prove that the new expression for ω_0 indeed transforms the DSR phase into the zero-offset phase.

Using the identity

$$\sqrt{a} + \sqrt{b} \equiv \sqrt{a + b + 2\sqrt{ab}}; \quad \text{for } a \geq 0, b \geq 0,$$

I rewrite the DSR phase as

$$\begin{aligned} k_z &= -\frac{1}{v}\text{sign}(\omega)\left\{\left[\omega^2 - (v_y + v_h)^2\right]^{\frac{1}{2}} + \left[\omega^2 - (v_y - v_h)^2\right]^{\frac{1}{2}}\right\} \\ &= -\frac{\sqrt{2}}{v}\,\text{sign}(\omega)\left[\omega^2 - v_y^2 - v_h^2 + \sqrt{(\omega^2 - v_y^2 - vh^2)^2 - 4v_y^2 v_h^2}\right]^{\frac{1}{2}}. \end{aligned} \tag{1.28}$$

I then examine the expression under the second square root (SSR) in equation (1.28)

$$SSR = (\omega^2 - v_y^2 - v_h^2)^2 - 4v_y^2 v_h^2$$

and substitute for ω the expression in ω_0. The expression under the second square root becomes

$$\begin{aligned} SSR &= (\omega_0^2 + \frac{\omega_0^2 v_h^2}{\omega_0^2 - v_y^2} - v_y^2 - v_h^2)^2 - 4v_y^2 v_h^2 \\ &= \left[\omega_0^2 - v_y^2 + \frac{\omega_0^2 v_h^2 - \omega_0^2 v_h^2 + v_y^2 v_h^2}{\omega_0^2 - v_y^2}\right]^2 - 4v_y^2 v_h^2 \\ &= \left[\frac{(\omega_0^2 - v_y^2)^2 + v_y^2 v_h^2}{\omega_0^2 - v_y^2}\right]^2 - 4v_y^2 v_h^2 \\ &= \frac{1}{(\omega_0^2 - v_y^2)^2}\left[(\omega_0^2 - v_y^2)^4 - 2v_y^2 v_h^2(\omega_0^2 - v_y^2)^2 + v_y^4 v_h^4\right] \\ &= \left[\frac{(\omega_0^2 - v_y^2)^2 - v_y^2 v_h^2}{\omega_0^2 - v_y^2}\right]^2. \end{aligned} \tag{1.29}$$

The DSR in equation (1.28) can be transformed as follows:

$$
\begin{aligned}
k_z &= -\frac{\sqrt{2}}{v}\,\mathrm{sign}(\omega_0)\left[\omega_0^2+\frac{\omega_0^2 v_h^2}{\omega_0^2-v_y^2}-v_y^2-v_h^2+\sqrt{SSR}\right]^{\frac{1}{2}} \\
&= -\frac{\sqrt{2}}{v}\,\mathrm{sign}(\omega_0)\left[\omega_0^2+\frac{\omega_0^2 v_h^2}{\omega_0^2-v_y^2}-v_y^2-v_h^2+\omega_0^2-v_y^2-\frac{v_y^2 v_h^2}{\omega_0^2-v_y^2}\right]^{\frac{1}{2}} \\
&= -\frac{\sqrt{2}}{v}\,\mathrm{sign}(\omega_0)\left[2\omega_0^2-2v_y^2+\frac{\omega_0^2 v_h^2-\omega_0^2 v_h^2+v_y^2 v_h^2-v_y^2 v_h^2}{\omega_0^2-v_y^2}\right]^{\frac{1}{2}} \qquad (1.30) \\
&= -\frac{2}{v}\,\mathrm{sign}(\omega_0)\left[\omega_0^2-v_y^2\right]^{\frac{1}{2}} \\
&= -\frac{2\omega_0}{v}\sqrt{1-Y^2},
\end{aligned}
$$

which is the same equation as (1.19).

1.5 Appendix 1.B: Evaluating the Jacobian of the transformation $\omega \rightarrow \omega_0$

The purpose of this appendix is to evaluate the Jacobian of the transformation from ω to ω_0,

$$
J=\left[\frac{d\omega}{d\omega_0}\right]=(1+\frac{v_h^2}{\omega_0^2-v_y^2})^{-\frac{1}{2}}\left[1-\frac{v_h^2 v_y^2}{(\omega_0^2-v_y^2)^2}\right].
$$

Starting with the transformation of the variable

$$
\omega \equiv \omega_0\left[1+\frac{v^2 k_h^2}{4\omega_0^2-v^2 k_y^2}\right]^{\frac{1}{2}} \equiv \omega_0\left[1+\frac{v_h^2}{\omega_0^2-v_y^2}\right]^{\frac{1}{2}},
$$

and then differentiating gives us

$$
\begin{aligned}
d\omega &= \left\{ \left[1+\frac{v_h^2}{\omega_0^2-v_y^2}\right]^{\frac{1}{2}} - \frac{\omega_0^2 v_h^2}{(\omega_0^2-v_y^2)^2\left[1+\frac{v_h^2}{\omega_0^2-v_y^2}\right]^{\frac{1}{2}}} \right\} d\omega_0 \\
&= (1+\frac{v_h^2}{\omega_0^2-v_y^2})^{-\frac{1}{2}}\left[1+\frac{v_h^2}{\omega_0^2-v_y^2}-\frac{v_h^2\omega_0^2}{(\omega_0^2-v_y^2)^2}\right] d\omega_0 \\
&= (1+\frac{v_h^2}{\omega_0^2-v_y^2})^{-\frac{1}{2}}\left[1+\frac{v_h^2\omega_0^2-v_h^2v_y^2-v_h^2\omega_0^2}{(\omega_0^2-v_y^2)^2}\right] d\omega_0 \\
&= (1+\frac{v_h^2}{\omega_0^2-v_y^2})^{-\frac{1}{2}}\left[1-\frac{v_h^2v_y^2}{(\omega_0^2-v_y^2)^2}\right] d\omega_0,
\end{aligned}
$$

and therefore the Jacobian is

$$
J = \left[\frac{d\omega}{d\omega_0}\right] = (1+\frac{v_h^2}{\omega_0^2-v_y^2})^{-\frac{1}{2}}\left[1-\frac{v_h^2v_y^2}{(\omega_0^2-v_y^2)^2}\right].
$$

1.6 Appendix 1.C: Finding the inverse of the transformation $\omega \to \omega_0$

This appendix demonstrates how to find the inverse of the transformation $\omega \to \omega_0$, or to express ω_0 as a function of ω. Starting with the original transformation of the variable, we have

$$
\omega \equiv \omega_0\left[1+\frac{v^2k_h^2}{4\omega_0^2-v^2k_y^2}\right]^{\frac{1}{2}} \equiv \omega_0\left[1+\frac{v_h^2}{\omega_0^2-v_y^2}\right]^{\frac{1}{2}}.
$$

It is important to note that ω has always the same sign as ω_0. Squaring the equation to obtain

$$
\omega^2 = \omega_0^2 + \frac{\omega_0^2 v_h^2}{\omega_0^2-v_y^2}
$$

and then isolating the terms in ω_0 yields the equation

$$
\omega_0^4 - \omega_0^2(\omega^2+v_y^2-v_h^2) + \omega^2 v_y^2 = 0, \tag{1.31}
$$

which can be solved in ω_0^2. The solutions are

$$
\omega_{0_{1,2}}^2 = \frac{1}{2}(\omega^2+v_y^2-v_h^2 \pm \sqrt{(\omega^2+v_y^2-v_h^2)^2-4\omega^2v_y^2}). \tag{1.32}
$$

The discriminant Δ is

$$
\begin{aligned}
\Delta &= (\omega^2+v_y^2-v_h^2-2\omega v_y)(\omega^2+v_y^2-v_h^2+2\omega v_y) \\
&= (\omega-v_y-v_h)(\omega-v_y+v_h)(\omega+v_y-v_h)(\omega+v_y+v_h).
\end{aligned}
$$

The existence conditions for k_z,

$$| \omega | \quad \geq \quad | v_y | + | v_h |,$$

ensure that Δ is always positive and therefore that ω_0^2 is always real within the ω existence limits. The choice of a positive sign for the discriminant in equation (1.32) is aided by the observation that for $v_h = 0$, the case of a zero-offset data field, the equation becomes an identity as expected. When we choose the positive sign for the discriminant, equation (1.32) becomes

$$\omega_0^2 = \frac{1}{2}(\omega^2 + v_y^2 - v_h^2 + \sqrt{(\omega^2 + v_y^2 - v_h^2)^2 - 4\omega^2 v_y^2}),$$

and applying the observation that ω has the same sign as ω_0, we get

$$\omega_0 = \mathrm{sign}(\omega) \left[\frac{1}{2}(\omega^2 + v_y^2 - v_h^2 + \sqrt{(\omega^2 + v_y^2 - v_h^2)^2 - 4\omega^2 v_y^2}) \right]^{\frac{1}{2}}, \tag{1.33}$$

which can be written in a simpler form using the identity

$$\sqrt{a} + \sqrt{b} \equiv \sqrt{a + b + 2\sqrt{ab}}; \quad \text{for } a \geq 0, b \geq 0.$$

Thus,

$$\begin{aligned}
\omega_0 &= \mathrm{sign}(\omega)\frac{1}{2} \left[2\omega^2 + 2v_y^2 - 2v_h^2 + 2\sqrt{(\omega^2 - 2\omega v_y + v_y^2 - v_h^2)(\omega^2 + 2\omega v_y + v_y^2 - v_h^2)} \right]^{\frac{1}{2}} \\
&= \mathrm{sign}(\omega)\frac{1}{2} \left[\sqrt{(\omega - v_y)^2 - v_h^2} + \sqrt{(\omega + v_y)^2 - v_h^2} \right] \\
&= \mathrm{sign}(\omega)\frac{v}{4} \left[\sqrt{(\frac{2\omega}{v} - k_y)^2 - k_h^2} + \sqrt{(\frac{2\omega}{v} + k_y)^2 - k_h^2} \right] \\
&= \frac{\omega}{2} \left[\sqrt{(1 - Y)^2 - H^2} + \sqrt{(1 + Y)^2 - H^2} \right].
\end{aligned} \tag{1.34}$$

The last part of equation (1.34), in a double-square-root form, is of particular importance in the phase of the MZO operator.

1.7 Appendix 1.D: Tracking the integration limits

In this appendix, I follow the integration boundaries for all the integral transformations from equation (1.16) to equation (1.27). In equation (1.16) the values of the constant k_z, given by equation (1.17), have to be real. This requires that the two conditions

$$| \omega | \quad \geq \quad | v_y + v_h |$$

$$| \omega | \quad \geq \quad | v_y - v_h |$$

be satisfied simultaneously. Considering all four possible sign cases for v_y and v_h represented in Figure 1.1, and the interval of existence for ω displayed in the shaded area, the two requirements can be reduced to the condition

$$|\,\omega\,| \quad \geq \quad |\,v_y\,| + |\,v_h\,|\ . \qquad (1.35)$$

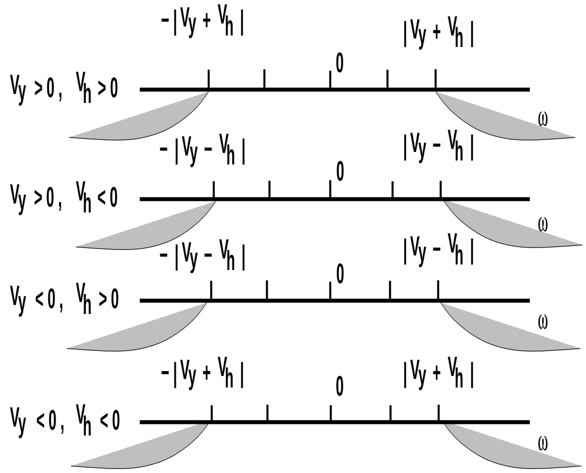

Figure 1.1: Four possible cases of the values of v_y and v_h and the interval of existence of ω. chapter1-DSRbound [NR]

In Figure 1.2, the shaded area represents the region of integration established by equation (1.35) for a constant k_y.

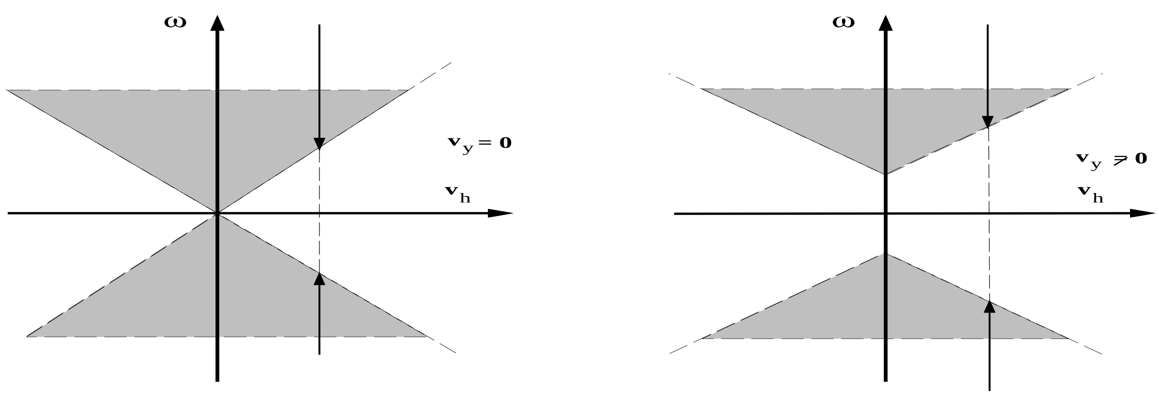

Figure 1.2: Regions of integration. chapter1-khkyomega [NR]

The existence condition for v_h in equation (1.35) requires the integration boundaries in equation (1.16) to be

$$p(t=0,k_y,h=0,z) = \int_{-\infty}^{\infty}\, d\omega \int_{-|\frac{2\omega}{v}|+|k_y|}^{|\frac{2\omega}{v}|-|k_y|}\, dk_h[...].$$

After the change of variable from ω to ω_0 in equation (1.20),

$$\omega \equiv \omega_0 \left[1 + \frac{v_h^2}{\omega_0^2 - v_y^2}\right]^{\frac{1}{2}},$$

we need to determine the new integration boundaries.

In equation (1.22) the new variable ω_0 takes values from $-\infty$ to ∞, but the boundary values for k_h now have to be expressed as a function of the new variable ω_0. Squaring the initial boundary equation (1.35) gives us

$$\omega^2 = v_h^2 + v_y^2 + 2 \mid v_y v_h \mid .$$

Then, by replacing ω with its expression in ω_0 we obtain

$$\omega_0^2 + \frac{v_h^2}{\omega_0^2 - v_y^2} = v_h^2 + v_y^2 + 2 \mid v_y v_h \mid .$$

Multiplying by $\omega_0^2 - v_y^2$ and grouping the terms yields

$$(\omega_0^2 - v_y^2)^2 - 2(\omega_0^2 - v_y^2) \mid v_y v_h \mid + v_h^2 v_y^2 = 0,$$

which is transformed in the condition for k_h as follows:

$$\mid k_h \mid \leq \frac{2}{v} \frac{\omega_0^2 - v_y^2}{\mid v_y \mid}.$$

Therefore, the second line in equation (1.22) has the following integration boundaries:

$$p(t = 0, k_y, h = 0, z) = \int_{-\infty}^{\infty} d\omega_0 \int_{-\frac{2}{v}\frac{\omega_0^2 - v_y^2}{|v_y|}}^{\frac{2}{v}\frac{\omega_0^2 - v_y^2}{|v_y|}} dk_h [...],$$

and subsequently, equation (1.23) has the same integration boundaries in k_h:

$$p_0(\omega_0, k_y) = \int_{-\frac{2}{v}\frac{\omega_0^2 - v_y^2}{|v_y|}}^{\frac{2}{v}\frac{\omega_0^2 - v_y^2}{|v_y|}} dk_h [...].$$

Finally, the change of integration variable from ω_0 in equation (1.25) back to ω in equation (1.27) restores the initial condition for k_h:

$$k_h \in (- \mid \frac{2\omega}{v} \mid + \mid k_y \mid, \mid \frac{2\omega}{v} \mid - \mid k_y \mid).$$

However, since in equation (1.27) the order of integration is reversed, the integration boundaries become

$$p(t_0, k_y) = \int_{-\infty}^{\infty} dk_h \int_{-\frac{v}{2}(|k_y| + |k_h|)}^{\frac{v}{2}(|k_y| + |k_h|)} d\omega [...].$$