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Explicit schemes for estimating elastic properties of multiphase composites

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ABSTRACT

Explicit schemes of Mori-Tanaka and Kuster-Toksöz for estimating elastic properties of multiphase composites are compared and contrasted. Both methods are known to have a limited range of validity. Reexamination of the theories and comparisons to experiment lead to the conclusion that these explicit schemes should not be used to estimate properties of systems in which the host material occupies less than 50% of the total volume. A unified approach to derivation of these explicit methods, as well as some implicit methods, is also presented.

INTRODUCTION

Methods for computing estimates of elastic constants for composites are either implicit or explicit. Implicit methods are generally computed either by iteration [for example, self-consistent methods such as those described by Gubernatis and Krumhansl (1975) or Berryman (1980)] or by integration [for example, differential effective medium methods such as those discussed by Cleary *et al.* (1980) or Norris (1985)]. On the other hand, explicit methods are easier to evaluate because they supply *formulas* for estimates of constants, not requiring any further computation except direct substitution.

Two of the more common implicit methods — the self-consistent (SC) method and the differential effective medium (DEM) method mentioned in the preceeding paragraph — are known to be realizable (Milton, 1985; Norris, 1985; Avellaneda, 1987), *i.e.*, they compute constants that correspond to values realized for specific microgeometries (Berryman and Berge, 1993). Realizability of an estimator implies that the estimator will *always* satisfy rigorous bounds on the constants [such as Hashin-Shtrikman (1963) and Walpole (1969) bounds]. On the contrary, the two most common explicit methods and the ones I intend to study here — Mori-Tanaka (1973) and

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Kuster-Toksöz (1974) — are known not to be realizable, and are in fact both known to violate some of the rigorous bounds (Norris, 1989; Ferrari, 1991; Berryman, 1980) in limiting cases with extreme microgeometries (such as high concentrations of thin disk inclusions).

Nevertheless there are circumstances in which explicit schemes are useful. In one situation frequently encountered in geophysical/rock mechanics studies, it is desirable to invert the formulas for constants to try to estimate other physical properties of the composite/rock, such as the crack aspect ratio distribution (Toksöz *et al.*, 1976; Cheng and Toksöz, 1979) or the porosity (Berge *et al.*, 1992), from measurements of seismic velocities or elastic constants. Such an inversion procedure is practically intractable with an implicit scheme but quite feasible with any explicit estimation scheme. The main limitations on the use of explicit schemes for inversion are the nonuniqueness of the results obtained (a common difficulty in geophysical inversion problems) and the inherent inaccuracies of the approximate estimators themselves. My present purpose is not to study the inverse problem, but rather to analyze two of the most commonly used explicit schemes: Mori-Tanaka (MT) and Kuster-Toksöz (KT). I do this with a view to determining relations between them and also which of them might be more accurate and therefore more useful for such inversion studies. Range of validity of the two methods will also be discussed.

To achieve these goals, I first introduce a unified approach for obtaining these approximation schemes and use this method to present different derivations of the SC, MT, and KT estimates. For example, I show how to obtain the KT approximation within a quasistatic analysis, whereas this approximation has generally been derived using a long-wavelength approximation within scattering theory (Kuster-Toksöz, 1974; Berryman, 1980).

BACKGROUND AND GENERAL ANALYSIS

The constituents of the composites under consideration are linear elastic, isotropic materials whose average stresses σ_i and average strains e_i are related by the constitutive relations

$$\sigma_i = \mathbf{L}_i e_i \quad \text{and} \quad e_i = \mathbf{M}_i \sigma_i. \tag{1}$$

The subscript i refers to the i-th constituent, of which I assume there are N. The components of the fourth ranked stiffness tensor $\mathbf{L}_i$ are defined by

$$(\mathbf{L}_i)_{mnpq} = \lambda_i \delta_{mn} \delta_{pq} + \mu_i (\delta_{mp} \delta_{nq} + \delta_{np} \delta_{mq}), \tag{2}$$

where λ_i and μ_i are the Lamé parameters of the i-th constituent. The indices m, n, p, q take the values 1,2,3, corresponding to Cartesian axes $x_1 = x$, $x_2 = y$, and $x_3 = z$. The bulk modulus $K_i = \lambda_i + \frac{2}{3}\mu_i$, while the shear modulus is μ_i. Following Hill (1963) [also see Gubernatis and Krumhansl (1975)], I treat $\mathbf{L}_i$ as a six-by-six matrix, with

σ_i and e_i being treated as six component vectors. Then, the compliance tensor $\mathbf{M}_i$, when also viewed as a six-by-six matrix, satisfies

$$\mathbf{M}_i\mathbf{L}_i = \mathbf{I} = \mathbf{L}_i\mathbf{M}_i, \tag{3}$$

so $\mathbf{M}_i$ is the matrix inverse of $\mathbf{L}_i$.

I assume for simplicity that the overall behavior of the composite is also linear elastic and isotropic, and that the effective constitutive laws are given by

$$\sigma = \mathbf{L}^* e \quad \text{and} \quad e = \mathbf{M}^*\sigma, \tag{4}$$

with

$$\mathbf{M}^*\mathbf{L}^* = \mathbf{I} = \mathbf{L}^*\mathbf{M}^*. \tag{5}$$

My problem is to find ways of relating the effective tensors $\mathbf{L}^*$ and $\mathbf{M}^*$ to the properties of the components contained in the constituents' tensors $\mathbf{L}_i$ and $\mathbf{M}_i$.

General results

The volume fraction of the i-th constituent is v_i and I assume that the constituents of the composite are space filling so that $\sum v_i = 1$. Then, the average strain and average stress are given by the volume averages

$$e = \sum v_i e_i \quad \text{and} \quad \sigma = \sum v_i \sigma_i. \tag{6}$$

It is an elementary exercise to show that (6), (4), and (1) imply

$$\sum v_i(\mathbf{L}_i - \mathbf{L}^*)e_i = 0 \tag{7}$$

and

$$\sum v_i(\mathbf{M}_i - \mathbf{M}^*)\sigma_i = 0. \tag{8}$$

Following Hill (1963), then recall that, since e_i must be a linear function of e and since σ_i must be a linear function of σ, I can write

$$e_i = \mathbf{A}^{*i} e \quad \text{and} \quad \sigma_i = \mathbf{B}^{*i}\sigma, \tag{9}$$

where $\mathbf{A}^{*i}$ and $\mathbf{B}^{*i}$ are unknown dimensionless fourth ranked tensors depending on the properties and microgeometry of the composite and its constituents. From (6), these tensors must also satisfy the normalization conditions

$$\sum v_i\mathbf{A}^{*i} = \mathbf{I} = \sum v_i\mathbf{B}^{*i}. \tag{10}$$

From (9), (4), and (1) follow the consistency conditions

$$\mathbf{B}^{*i} = \mathbf{L}_i\mathbf{A}^{*i}\mathbf{M}^*. \tag{11}$$

Substituting (9) into (7) and (8), I find (since e and σ may be considered arbitrary) that

$$\sum v_i(\mathbf{L}_i - \mathbf{L}^*)\mathbf{A}^{*i} = 0 \tag{12}$$

and

$$\sum v_i(\mathbf{M}_i - \mathbf{M}^*)\mathbf{B}^{*i} = 0. \tag{13}$$

There has been no approximation made in arriving at these equations, but they are of only limited usefulness since the tensors $\mathbf{A}^{*i}$ and $\mathbf{B}^{*i}$ are unknown. If I could find approximations to $\mathbf{A}$ and $\mathbf{B}$ based on their definitions (9), then such approximations would lead immediately to approximations for $\mathbf{L}^*$ and $\mathbf{M}^*$ using (12) and (13).

Wu tensors and single inclusion problems

Eshelby (1957) calculated the strain field in an elastic ellipsoidal inclusion imbedded in an elastic host material subjected to a displacement with uniform strain field at infinity. The important general result of Eshelby is that for all ellipsoidal inclusions the resulting strain in the inclusion is homogeneous. Host and inclusion strains are therefore related algebraically. For arbitrary spheroidal inclusions, Wu (1966) subsequently found the components of the tensor $\mathbf{T}^{hi}$ relating the strain in the host e_h to the strain in the inclusion e_i according to

$$e_i = \mathbf{T}^{hi}e_h. \tag{14}$$

If the composite of interest is isotropic, no particular orientation of the spheroidal inclusions is preferred. Then, it is not actually $\mathbf{T}$ that is of most interest but rather its isotropic average. This average is obtained by considering

$$P^{hi} \equiv \frac{1}{3}(\mathbf{T}^{hi})_{mmpp} \quad \text{and} \quad Q^{hi} \equiv \frac{1}{5}\left[(\mathbf{T}^{hi})_{mnmn} - \frac{1}{3}(\mathbf{T}^{hi})_{mmpp}\right], \tag{15}$$

where the summation convention is implied for repeated subscripts. I treat $\mathbf{T}$ from this point on in the paper as an isotropic tensor of the form

$$(\mathbf{T})_{mnpq} = \frac{1}{3}(P - Q)\delta_{mn}\delta_{pq} + \frac{1}{2}Q(\delta_{mp}\delta_{nq} + \delta_{mq}\delta_{np}). \tag{16}$$

Specific results on P and Q for inclusions shaped like spheres, needles, disks, and penny-shaped cracks are given in Table 1. Formulas for more general prolate and oblate spheroidal inclusions were presented by Berryman (1980), based on the work of Wu (1966) and Kuster and Toksöz (1974).

TABLE 1. Four examples of coefficients P and Q for spherical and nonspherical scatterers. The superscripts h and i refer to host and inclusion phases, respectively. Special characters are defined by $\beta = \mu[(3K+\mu)/(3K+4\mu)]$, $\gamma = \mu[(3K+\mu)/(3K+7\mu)]$, and $\zeta = (\mu/6)[(9K+8\mu)/(K+2\mu)]$. The expressions for spheres, needles, and disks were derived by Wu (1966) and Walpole (1969). The expressions for penny-shaped cracks were derived by Walsh (1969) and assume $K_i/K_h << 1$ and $\mu_i/\mu_h << 1$. The aspect ratio of the cracks is α.

Inclusion shape	P^{hi}	Q^{hi}
Spheres	$\frac{K_h+\frac{4}{3}\mu_h}{K_i+\frac{4}{3}\mu_h}$	$\frac{\mu_h+\zeta_h}{\mu_i+\zeta_h}$
Needles	$\frac{K_h+\mu_h+\frac{1}{3}\mu_i}{K_i+\mu_h+\frac{1}{3}\mu_i}$	$\frac{1}{5}\left(\frac{4\mu_h}{\mu_h+\mu_i}+2\frac{\mu_h+\gamma_h}{\mu_i+\gamma_h}+\frac{K_i+\frac{4}{3}\mu_h}{K_i+\mu_h+\frac{1}{3}\mu_i}\right)$
Disks	$\frac{K_h+\frac{4}{3}\mu_i}{K_i+\frac{4}{3}\mu_i}$	$\frac{\mu_h+\zeta_i}{\mu_i+\zeta_i}$
Penny cracks	$\frac{K_h+\frac{4}{3}\mu_i}{K_i+\frac{4}{3}\mu_i+\pi\alpha\beta_h}$	$\frac{1}{5}\left(1+\frac{8\mu_h}{4\mu_i+\pi\alpha(\mu_h+2\beta_h)}+2\frac{K_i+\frac{2}{3}(\mu_i+\mu_h)}{K_i+\frac{4}{3}\mu_i+\pi\alpha\beta_h}\right)$

General equations using a reference material

Now consider a reference material r (with stiffness and compliance tensors $\mathbf{L}_r$ and $\mathbf{M}_r$) which may or may not be one of the actual constituents of the composite under study. If there is one material in the composite that acts as host to the others, the reference material may be chosen to be the host material h. However, in general, I want to leave the choice of reference material open for later consideration.

Define tensors relating (possibly fictitious) strains e_r and stresses σ_r to the actual average strains and stresses in the composite by the linear relations

$$e_i = \mathbf{G}^{ri} e_r \quad \text{and} \quad \sigma_i = \mathbf{H}^{ri}\sigma_r. \tag{17}$$

The consistency relation between the new tensors is

$$\mathbf{H}^{ri} = \mathbf{L}_i \mathbf{G}^{ri} \mathbf{M}_r. \tag{18}$$

Note that, in the case where the reference material is chosen to be the composite material itself (*i.e.*, $r \to *$), the new tensors satisfy $\mathbf{G}^{*i} \equiv \mathbf{A}^{*i}$ and $\mathbf{H}^{*i} \equiv \mathbf{B}^{*i}$. These new tensors are therefore simply formal generalizations of $\mathbf{A}$ and $\mathbf{B}$ suitable for the wide variety of circumstances that may arise in my modeling efforts.

Again substituting (17) into (7) and (8) yields the equations

$$(\mathbf{L}^* - \mathbf{L}_r)\sum v_i \mathbf{G}^{ri} e_r = \sum v_i (\mathbf{L}_i - \mathbf{L}_r)\mathbf{G}^{ri} e_r \tag{19}$$

and

$$(\mathbf{M}^* - \mathbf{M}_r)\sum v_i \mathbf{H}^{ri} \sigma_r = \sum v_i (\mathbf{M}_i - \mathbf{M}_r)\mathbf{H}^{ri} \sigma_r. \tag{20}$$

These equations are not in simplest form (there are redundant terms proportional to either $\mathbf{L}_r$ or $\mathbf{M}_r$ on both side of the equations) for reasons that will become apparent. Nevertheless, these two equations are exact (*i.e.*, no approximations have been made) and mutually consistent. Consistency may be checked by deleting the redundant terms from (20), multiplying the result by $\mathbf{L}^*$, and then including the redundant terms found in (19). The result is exactly (19).

It is sometimes claimed that certain theories are superior to others because the compliance tensor $\mathbf{M}^*$ can be found using an equation analogous to the one used to find $\mathbf{L}^*$. However, since all the theories considered in this paper may be derived from the pair of equations (19) and (20) which are themselves consistent in this sense, I have found that all these theories possess this desirable property. Thus, this criterion appears to be too weak to force choices among the theories.

The advantage of the introduction of the reference material will become apparent as I derive various approximate expressions from (19) and (20) using Eshelby's and Wu's results for single ellipsoidal inclusions amd isotropic averages of ellipsoidal inclusions. These equations also enable me to present a unified approach to derivation of the various approximation schemes considered.

APPROXIMATION SCHEMES

Starting with (19) and (20), I derive various approximation schemes by making use of Wu's tensor $\mathbf{T}$ to replace the unknown tensor $\mathbf{G}$. The art of approximation scheme design enters through clever choices of reference materials.

Self-consistent scheme

The first approximation scheme considered is an implicit method called the self-consistent (SC) scheme (Gubernatis and Krumhansl, 1975; Berryman, 1980). This scheme should be distinguished from another scheme by Wu (1966) that is also called a self-consistent scheme, but generally differs from the present one. For completeness, Wu's scheme is also derived in the Appendix.

My approximation may be obtained by supposing the reference material is a medium with the same stiffnesses and compliances as the composite of interest. Thus, $r \to *$ in (19) and (20), so the left hand side of each equation vanishes. To obtain a useful approximation, suppose that $\mathbf{G}^{*i}$ is well approximated by Wu's tensor $\mathbf{T}^{*i}$. Then, considering e_r and σ_r arbitrary, the self-consistent approximation is given by

$$\sum v_i(\mathbf{L}_i - \mathbf{L}^*_{SC})\mathbf{T}^{*i} = 0. \tag{21}$$

The corresponding equation for the compliance tensor is

$$\sum v_i(\mathbf{M}_i - \mathbf{M}^*_{SC})\mathbf{W}^{*i} = 0, \tag{22}$$

where $\mathbf{W}^{*i} \equiv \mathbf{L}_i\mathbf{T}^{*i}\mathbf{M}^*_{SC}$. Both of these equations have the same general form as the exact equations (12) and (13), but nevertheless the results obtained are necessarily approximate since the exact factors $\mathbf{A}^{*i}$ and $\mathbf{B}^{*i}$ are only approximated by the factors $\mathbf{T}^{*i}$ and $\mathbf{W}^{*i}$, respectively.

I find easily that, by multiplying (22) on both the left and right by $\mathbf{L}^*_{SC}$, it reduces to (21) and is therefore consistent with it in the sense that it can be derived from it. This shows that it is sufficient to solve either equation and then use the uniqueness of the matrix inverse to find the solution of the other. I will continue to point out the existence of this property for the other theories as they are derived, but will not dwell on the proofs, since these proofs are completely analogous to the one just given in all cases.

Mori-Tanaka scheme

The first explicit scheme considered is that due to Mori and Tanaka (1973) as described by Weng (1984) and Benveniste (1987). To obtain this approximation from (19) and (20), assume that the composite has a host material with inclusions and choose the host to serve as reference material, so $r \to h$. Then, approximating $\mathbf{G}^{hi} \simeq \mathbf{T}^{hi}$, I find

$$\sum v_i(\mathbf{L}_i - \mathbf{L}^*_{MT})\mathbf{T}^{hi} = 0, \tag{23}$$

in simplest form. The corresponding equation for the compliance is

$$\sum v_i(\mathbf{M}_i - \mathbf{M}^*_{MT})\mathbf{W}^{hi} = 0, \tag{24}$$

where $\mathbf{W}^{hi} \equiv \mathbf{L}_i\mathbf{T}^{hi}\mathbf{M}_h$. Again by multiplying (24) on the right by $\mathbf{L}_h$ and on the left by $\mathbf{L}^*_{MT}$ to check consistency, I find that (23) is recovered, as shown previously by Weng (1984) and Benveniste (1987). The equation for $\mathbf{L}^*_{MT}$ can also be written as

$$(\mathbf{L}^*_{MT} - \mathbf{L}_h)\sum v_i\mathbf{T}^{hi} = \sum v_i(\mathbf{L}_i - \mathbf{L}_h)\mathbf{T}^{hi}, \tag{25}$$

where I have retained the redundant terms $\mathbf{L}_h\sum v_i\mathbf{T}^{hi}$ on both sides of the equation. This form of the equation for $\mathbf{L}^*_{MT}$ is convenient for comparison with the KT explicit scheme to be considered next.

Kuster-Toksöz scheme

The second explicit scheme considered is that due to Kuster and Toksöz (1974). To obtain this approximation, first recall, from the definitions (17) of the tensors relating the reference material stresses and strains to those of the constituents, that the average strain and average stress are formally related to reference material quantities by

$$e = \sum v_i\mathbf{G}^{ri}e_r \quad \text{and} \quad \sigma = \sum v_i\mathbf{H}^{ri}\sigma_r. \tag{26}$$

So (19) and (20) may be rewritten as

$$(\mathbf{L}^* - \mathbf{L}_r)e = \sum v_i(\mathbf{L}_i - \mathbf{L}_r)\mathbf{G}^{ri}e_r \tag{27}$$

and

$$(\mathbf{M}^* - \mathbf{M}_r)\sigma = \sum v_i(\mathbf{M}_i - \mathbf{M}_r)\mathbf{H}^{ri}\sigma_r. \tag{28}$$

Now, consider the relationship between the average strain e in the composite and the strain e_r in the reference material, supposing in particular that the composite is imbedded in the reference material so

$$e = \mathbf{G}^{r*}e_r \quad \text{and} \quad \sigma = \mathbf{H}^{r*}\sigma_r. \tag{29}$$

Note that it follows from (26) and (29) that

$$\mathbf{G}^{r*} = \sum v_i\mathbf{G}^{ri} \quad \text{and} \quad \mathbf{H}^{r*} = \sum v_i\mathbf{H}^{ri}. \tag{30}$$

Substituting (29) into (27) and (28), taking the reference material to be the host $r \to h$, and making the usual approximation that $\mathbf{G} \simeq \mathbf{T}$ gives the KT approximation

$$(\mathbf{L}^*_{KT} - \mathbf{L}_h)\mathbf{T}^{h*} = \sum v_i(\mathbf{L}_i - \mathbf{L}_h)\mathbf{T}^{hi}. \tag{31}$$

The corresponding equation for the compliance is

$$(\mathbf{M}^*_{KT} - \mathbf{M}_h)\mathbf{W}^{h*} = \sum v_i(\mathbf{M}_i - \mathbf{M}_h)\mathbf{W}^{hi}, \tag{32}$$

where $\mathbf{W}^{h*} \equiv \mathbf{L}^*_{MT}\mathbf{T}^{h*}\mathbf{M}_h$ and $\mathbf{W}^{hi} \equiv \mathbf{L}_i\mathbf{T}^{hi}\mathbf{M}_h$. Again multiplying (32) on both the left and right by $\mathbf{L}_h$, (31) is recovered.

Equations (31) and (32) may at first appear to be implicit rather than explicit formulas. However, Kuster and Toksöz (1974) made the further assumption that tensor $\mathbf{T}^{h*}$ on the left had side of (31) is always the tensor for spherical inclusions. Then, (31) reduces to the two equations

$$(K^*_{KT} - K_h)\frac{K_h + \frac{4}{3}\mu_h}{K^*_{KT} + \frac{4}{3}\mu_h} = \sum v_i(K_i - K_h)P^{hi} \tag{33}$$

and (note that ζ is defined in Table 1)

$$(\mu^*_{KT} - \mu_h)\frac{\mu_h + \zeta_h}{\mu^*_{KT} + \zeta_h} = \sum v_i(\mu_i - \mu_h)Q^{hi}, \tag{34}$$

which are easily rearranged into explicit formulas for K^*_{KT} and μ^*_{KT}.

Kuster and Toksöz (1974) consider their approach to be a low concentration approximation based on a "noninteraction" assumption. They translate this assumption into a quantitative condition stating that the aspect ratio of spheroidal inclusions should not exceed the corresponding volume fraction. Such a condition clearly places severe constraints on models containing oblate spheroids and disks.

COMPARING MT AND KT SCHEMES

Since the right hand sides of (25) and (31) are exactly the same if the host medium h is taken to be the same, note that

$$(\mathbf{L}^*_{MT} - \mathbf{L}_h) \sum v_i \mathbf{T}^{hi} \equiv (\mathbf{L}^*_{KT} - \mathbf{L}_h)\mathbf{T}^{h*}, \tag{35}$$

showing that these two explicit approximations are very closely related but not generally equal. To achieve equality between the two approximations ($\mathbf{L}^*_{MT} = \mathbf{L}^*_{KT}$), it is both necessary and sufficient that

$$\mathbf{T}^{h*} = \sum v_i \mathbf{T}^{hi} \tag{36}$$

— which is certainly not satisfied in general, but is true for spherical inclusions. To check satisfaction of (36), I find, using the Ps and Qs for spheres from Table 1, that (36) implies

$$\frac{K_h + \frac{4}{3}\mu_h}{K^* + \frac{4}{3}\mu_h} = \sum v_i \frac{K_h + \frac{4}{3}\mu_h}{K_i + \frac{4}{3}\mu_h} \tag{37}$$

must be satisfied by the bulk modulus, and

$$\frac{\mu_h + \zeta_h}{\mu^* + \zeta_h} = \sum v_i \frac{\mu_h + \zeta_h}{\mu_i + \zeta_h} \tag{38}$$

by the shear modulus. When the common factors depending only on host properties in the numerators are divided out of these two equations, the remaining formulas are exactly the results of the MT and KT theories for spherical inclusions. Since $\mathbf{T}^{hi} = \mathbf{I}$ when the host and inclusion properties are the same regardless of the assumed host particle shape, the results (36), (37), and (38) are also valid for both MT and KT whenever all the inclusions except the host are sphere shaped. In general, equation (36) is just an approximation to (30) that is not always valid.

EXAMPLES

Porous materials have high contrasts in their constituents' properties and therefore serve as good examples to highlight practical differences among the various approximation schemes.

Spherical inclusions

Porous glass foam: Walsh *et al.* (1965) made measurements on the compressiblity $(1/K)$ of a porous glass over a wide range of porosities. The samples they used were fabricated from borosilicate powder die-pressed and then sintered in alumina crucibles. Depending on the thermal history of the samples, they obtained porous

glass with porosities ranging from 0.70 to near zero. Porosity measurements were stated to be accurate to ±0.01.

The resulting glass samples were foam, with microgeometry such that large open dry pores were dispersed in a continuous glass background that thinned to a network of needle-like filaments at the highest porosities. Linear compressibility of the samples was measured by Walsh *et al.* (1965) using electrical resistance strain gages attached to sample surfaces in a high-pressure medium. The two explicit methods under consideration (MT and KT) produce the same estimates for composites with spherical inclusions and Figure 1 shows that these estimates are quite good at lower concentrations of voids, but they serve more as upper bounds than as estimates for the higher concentrations. Ferrari and Filipponi (1991) and Zimmerman (1991) have also demonstrated that the MT method produces good estimates of the compressibilities of these foam samples.

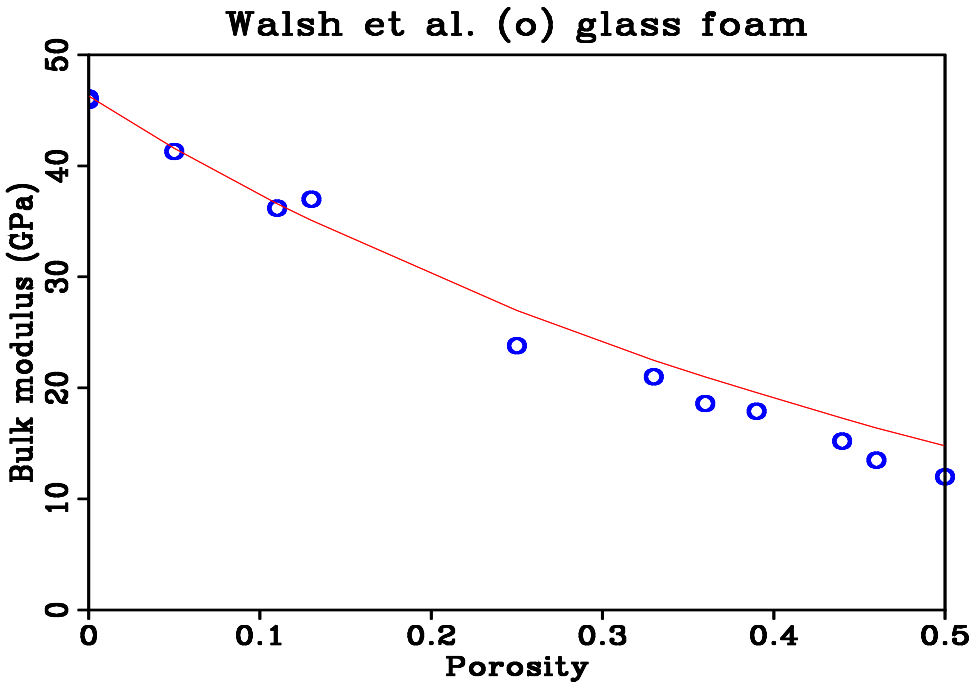

Figure 1: Bulk moduli of porous glass foam computed from compressibilities measured by Walsh *et al.* (1965). Measured values are compared to theoretical predictions of the Mori-Tanaka theory and the Kuster-Toksöz (1974) theory, which happen to coincide for composites with spherical inclusions. The same curve is also obtained when either theory assumes the voids are spherical while the glass particles take any other ellipsoidal shape. berryman1-walsh [ER]

Sintered glass beads: Synthetic sandstones were made by sintering glass beads made from soda-lime plate glass at peak temperatures of 700-760° C to achieve porosities of ∼1-40% (Berge *et al.*, 1993). Samples were cored with diameters of 2.54±0.002 cm and ground to right circular cylinders with lengths of 2.54±0.002 cm. The sample volumes, masses, and glass density were used to compute porosities with an uncertainty of ∼0.6% for samples with porosities below ∼30%, and an uncertainty of ∼1% for the higher porosity samples. A pulse transmission technique was used with 0.5 MHz P and 5 MHz S transducers to determine the compressional (P) and shear (S) wave velocities of both dry and saturated samples. Velocity measurements had an uncertainty of ∼1%. All measurements were made at atmospheric pressure and room temperature. Berge *et al.* (1993) showed that the SC approximation produced good estimates of the observed velocities of these synthetic sandstones for all porosities because the granular structure of these synthetic sandstones is compatible with this theory. The results of these calculations are displayed here in Figure 2.

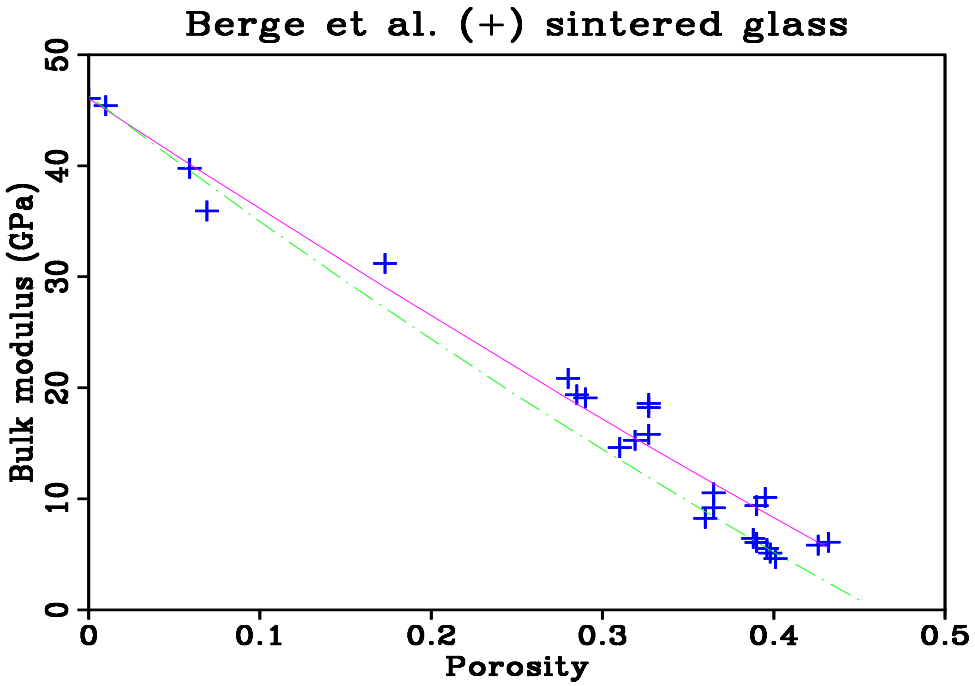

Figure 2: Bulk moduli of sintered glass bead samples measured by Berge *et al.* (1993). Measured values are compared to predictions of the SC theory (Budiansky, 1965; Hill, 1965; Berryman, 1980) for spherical inclusions (solid line). Dot dash line is the result of the SC method assuming spherical glass particles and needle shaped voids, which is a reasonable model of a granular material with connected porosity. berryman1-berge [ER]

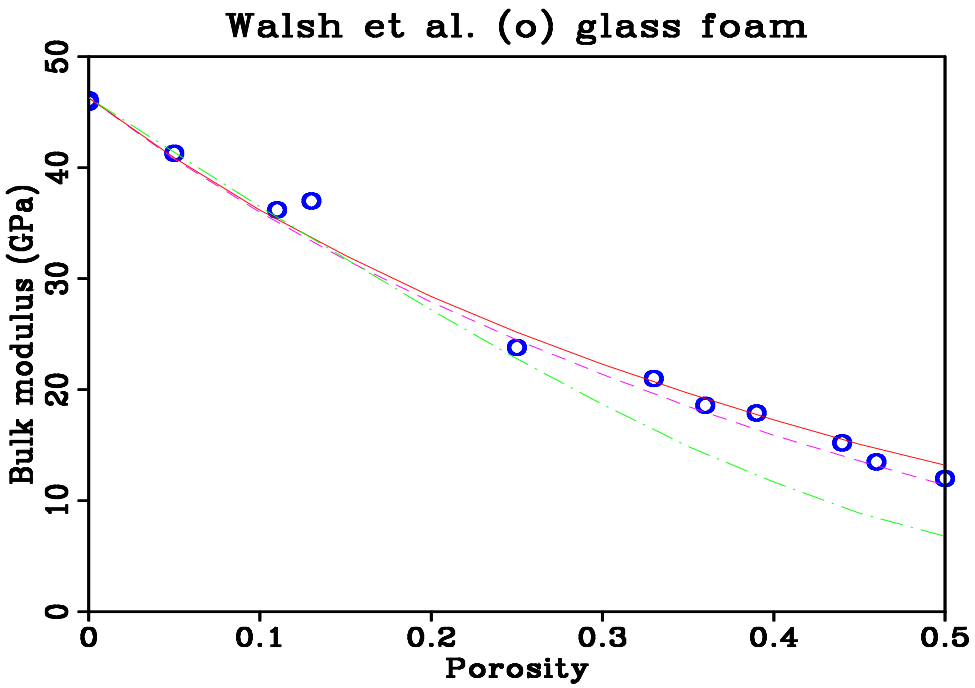

Figure 3: Bulk moduli of porous glass foam computed from compressibilities measured by Walsh *et al.* (1965). Measured values are compared to theoretical predictions of the Mori-Tanaka (1973) theory (solid line) and the Kuster-Toksöz theory (dashed line) assuming the voids are needle shaped. Dot dash line is the result of the SC method assuming needle shaped glass and spherical voids. berryman1-walsh2 [ER]

Nonspherical inclusions

If it is desired to model a material by assuming different shapes for the inclusions and host materials, the Mori-Tanaka and Kuster-Toksöz schemes may be used. Again considering the data of Walsh *et al.* (1965) for glass foam, suppose that one reasonable model would be spherical glass particles and needle shaped voids. The result of such a calculation is presented in Figure 3. The agreement is somewhat better than that for the spherical inclusions model, although the correspondence between this model and the physical composite seems unclear. In fact, this should be a better model for the sintered glass beads, than for the foam. It seems therefore that a more physically realistic model of the glass foam should be just the opposite: spherical voids and needle shaped glass inclusions. However, when I do this computation for MT and KT, I find that the results in both cases are identical to the results for spherical voids and spherical glass inclusions illustrated in Figure 1 and therefore this model gives no improvement in the estimates. I consider this to be a failure of these theories, because it shows they are not flexible enough to capture differences in the microgeometry of the host. For comparison, I also show in Figure 3 the result obtained using the SC theory assuming needle shaped glass and spherical voids. Although the voids are indeed approximately spherical in the glass foam, it seems that needles should be a poor

approximation of the actual glass structure until very high porosities are achieved. This intuition seems to be borne out by the results obtained, which underestimate the strength of the foam at intermediate values of porosity but have the right trend overall.

Considering the data of Berge *et al.* (1993) again, a reasonable model of a finite permeability granular sample is this: spherical grains and needle shaped voids. This result computed using the SC theory is also plotted in Figure 2. The agreement with the data is comparable to that achieved by the theory assuming spherical grains and spherical voids.

CONCLUSIONS

Mori-Tanaka and Kuster-Toksöz give the same results for composites with spherical inclusions as illustrated in Figure 1. In fact, host microgeometry cannot contribute to the results for either Mori-Tanaka and Kuster-Toksöz estimates since $\mathbf{T}^{hi} = \mathbf{I}$ for any ellipsoidally shaped particle if $i = h$. Although this result is reasonable for low concentrations of inclusions in a host medium, it clearly limits the usefulness of these theories. In particular, I have no control (*i.e.*, I cannot specify the limiting host geometry) over the behavior of either theory when the limit of vanishing host volume fraction is approached. Viewing these approximations as interpolation schemes, both theories have no means of adjusting the slope of the elastic constant trajectories as the host volume fraction vanishes. This failure has some serious consequences: For example, Norris (1989) has shown that MT can give results violating the Hashin-Shtrikman bounds for multiphase composites. Similarly, Ferrari (1991) has shown that for anisotropic composites the MT scheme produces the absurd result that the composite can still depend on the host elastic constants even in the limit of vanishing host volume fraction. Berryman (1980) shows that the KT scheme violates the Hashin-Shtrikman bounds when the inclusions are either disks at any finite concentration or needles at volume fractions greater than about 60%.

I conclude that MT and KT have limited ranges of usefulness and in particular should not be used when the nominal host material does not have the dominant volume fraction. A reasonable recommendation for the use of these schemes then is to limit calculations to situations where the host occupies at least 50% of the overall volume, as I have done in this paper.

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APPENDIX

Wu (1966) developed another type of self-consistent estimator, distinct from the approximation (21) and (22) that I call the SC theory. It can also be derived from (19) and (20). First, remove the redundant terms from these equations and then recall (26). The resulting formulas are

$$\mathbf{L}^* e = \sum v_i \mathbf{L}_i \mathbf{G}^{hi} e_h \tag{39}$$

and

$$\mathbf{M}^* \sigma = \sum v_i \mathbf{M}_i \mathbf{H}^{hi} \sigma_h, \tag{40}$$

where I have taken the reference material to be the host material $r \to h$.

Now the strains and stresses of the host material are related to those of the composite by

$$e_h = \mathbf{G}^{*h} e \quad \text{and} \quad \sigma_h = \mathbf{H}^{*h} \sigma. \tag{41}$$

Then, substituting (41) into (39) and (40) and recalling the identities

$$\mathbf{G}^{*i} = \mathbf{G}^{hi} \mathbf{G}^{*h} \quad \text{and} \quad \mathbf{H}^{*i} = \mathbf{H}^{hi} \mathbf{H}^{*h}, \tag{42}$$

together with

$$\sum v_i \mathbf{G}^{*i} = \mathbf{I} = \sum v_i \mathbf{H}^{*i}, \tag{43}$$

I find

$$(\mathbf{L}^* - \mathbf{L}_h) e = \sum v_i (\mathbf{L}_i - \mathbf{L}_h) \mathbf{G}^{*i} e \tag{44}$$

and

$$(\mathbf{M}^* - \mathbf{M}_h) \sigma = \sum v_i (\mathbf{M}_i - \mathbf{M}_h) \mathbf{H}^{*i} \sigma. \tag{45}$$

So far no approximations have been made in arriving at (44) and (45).

Now, to obtain Wu's scheme, I make the usual substitution $\mathbf{G} \to \mathbf{T}$. Then, for arbitrary e and σ, I have

$$\mathbf{L}^*_{Wu} - \mathbf{L}_h = \sum v_i (\mathbf{L}_i - \mathbf{L}_h) \mathbf{T}^{*i} \tag{46}$$

and

$$\mathbf{M}^*_{Wu} - \mathbf{M}_h = \sum v_i (\mathbf{M}_i - \mathbf{M}_h) \mathbf{W}^{*i}. \tag{47}$$

These two formulas are exactly the multiphase generalizations of Wu's formulas (Berryman, 1980), and they are consistent in the sense that either one can be derived from the other using the uniqueness of the matrix inverses.

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