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A simple modeling scheme for layered anisotropic media

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ABSTRACT

Wave propagation in an anisotropic medium can be modeled by a two-way phase shift scheme. The purpose of the method I present is to calculate the particle velocities at the surface of a stack of anisotropic plane layers generated by a traction at the surface. This method is simple but flexible. It can be used to generate primary-only sections or to model any order of multiples. The user can also choose to ignore propagation of waves of a particular wavetype in any layer. This permits, for example, the creation of output sections containing only P-waves even though the full elastic reflection coefficients have been modeled.

INTRODUCTION

Modeling of elastic wave propagation in layered anisotropic media is usually done by a reflectivity method (e.g., Mandal and Toksoz, 1990, or Fryer & Frazer, 1984,1990). In the reflectivity method, the amplitudes of the waves propagating in all the layers are solved for in one large system of equations. In the phase shift method that I propose, a step-by-step approach is used. Although this method is probably less efficient than the reflectivity method it does allow me to modify the amplitudes of the different wavetypes at each step. I can zero all the P-wave amplitudes at each step of the algorithm to produce an S-wave only section or zero all the S-wave amplitudes to create a P-wave section. Since my method generates one order of multiples for each pass through the heart of the algorithm, I can produce a primary only section or a specific order of multiples.

While all of these things may be done in a reflectivity algorithm, it is simpler and, I hope, more intuitive with a phase shift algorithm.

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TWO WAY PHASE SHIFT MODELING

Two way phase shift modeling is a step-by-step approach to calculating the recorded particle velocities. The input to the process is a layered medium and a traction applied at the surface. The sequence of steps in the modeling algorithm is shown in figures **??** to **??**.

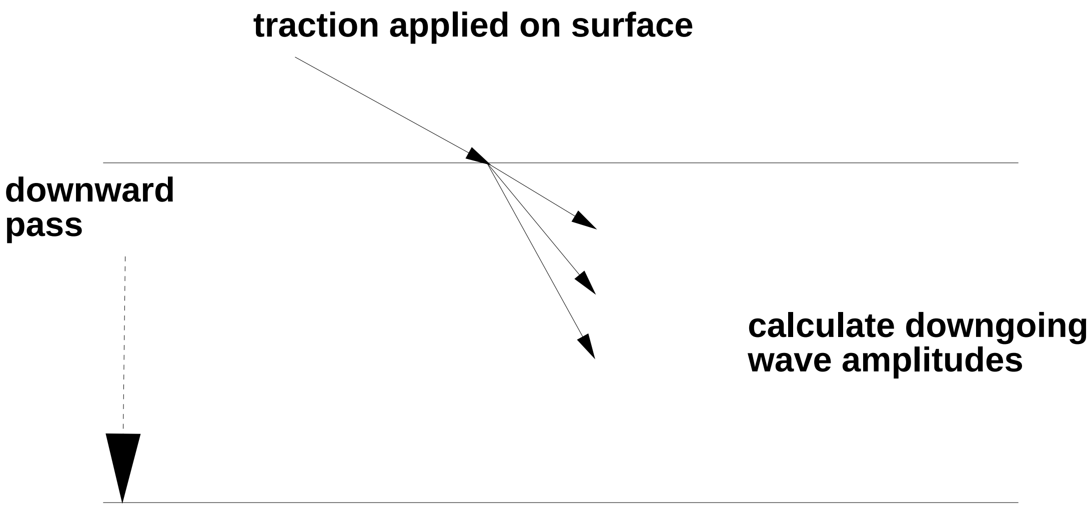

Figure 1: The first step is to calculate the (complex) amplitudes of the three downgoing waves and the particle velocities generated by the traction. Each phase shift uses the vertical slowness appropriate for the wavetype. dave1-pict1 [ER]

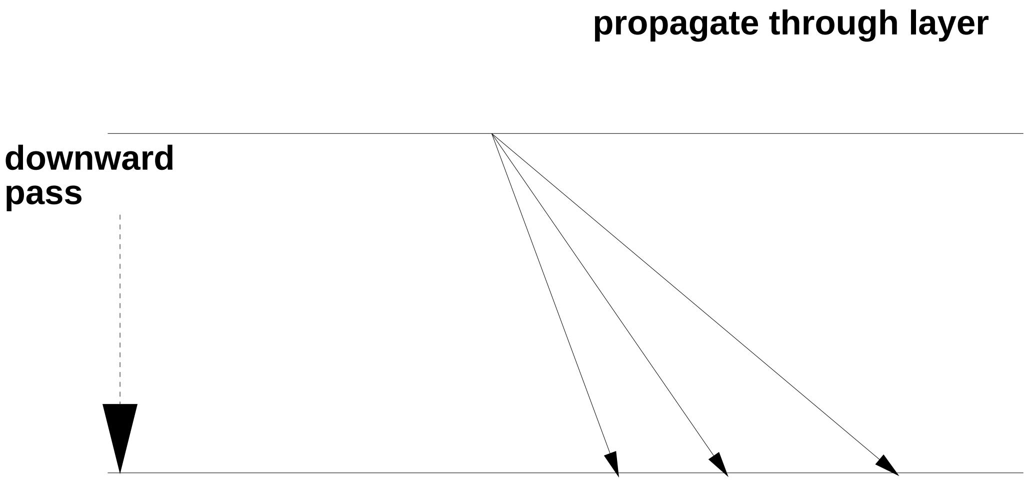

Figure 2: The second step propagates the waves to the bottom of the layer. Each component of the downgoing wavefield is phase shifted using the Appropriate vertical slowness. dave1-pict2 [ER]

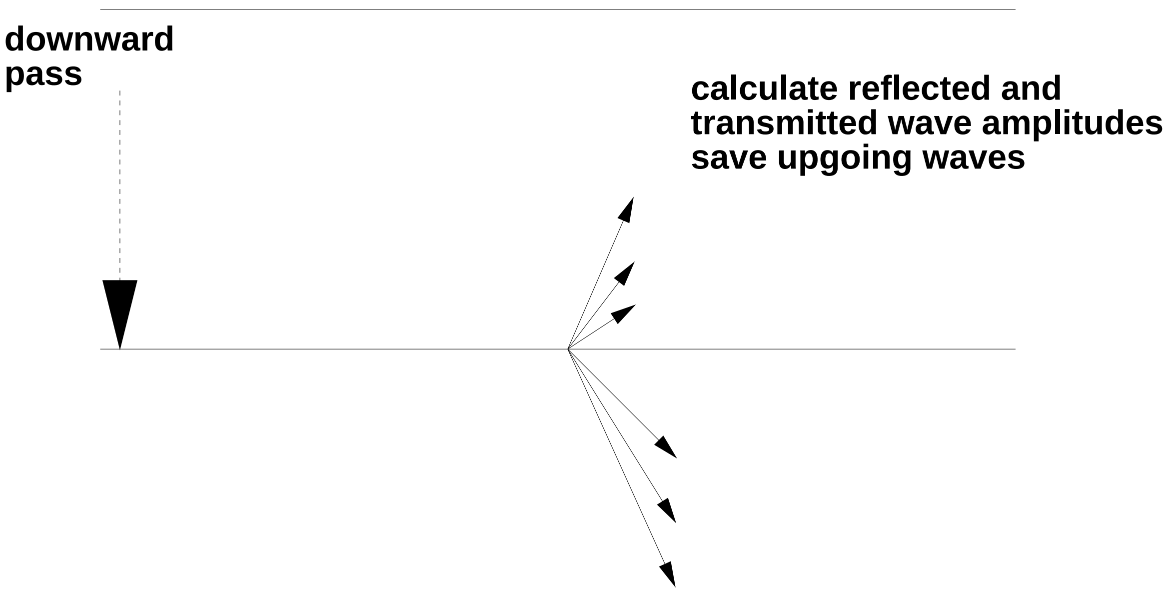

Figure 3: The third step calculates the transmitted downgoing wave amplitudes and the reflected upgoing wave amplitudes. The upgoing amplitudes are saved to be used later. Steps 2 and 3 are repeated until the bottom of the model is reached. dave1-pict3 [ER]

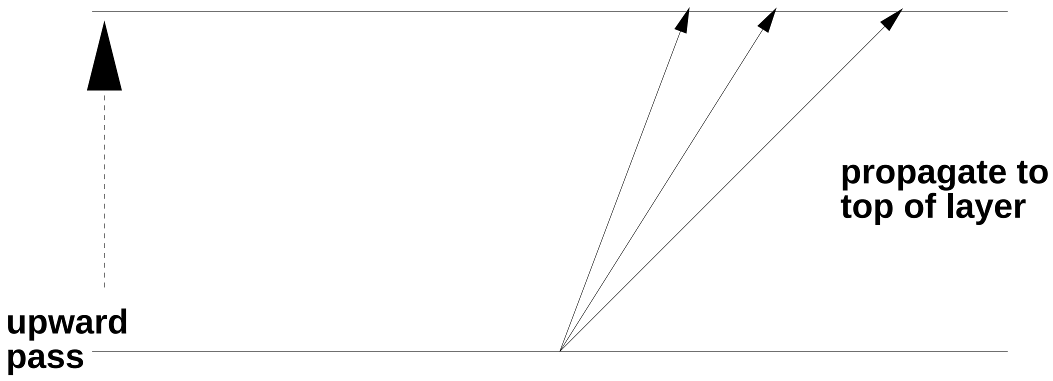

Figure 4: The fourth step propagates the upgoing waves in the last layer to the top of that layer. dave1-pict4 [ER]

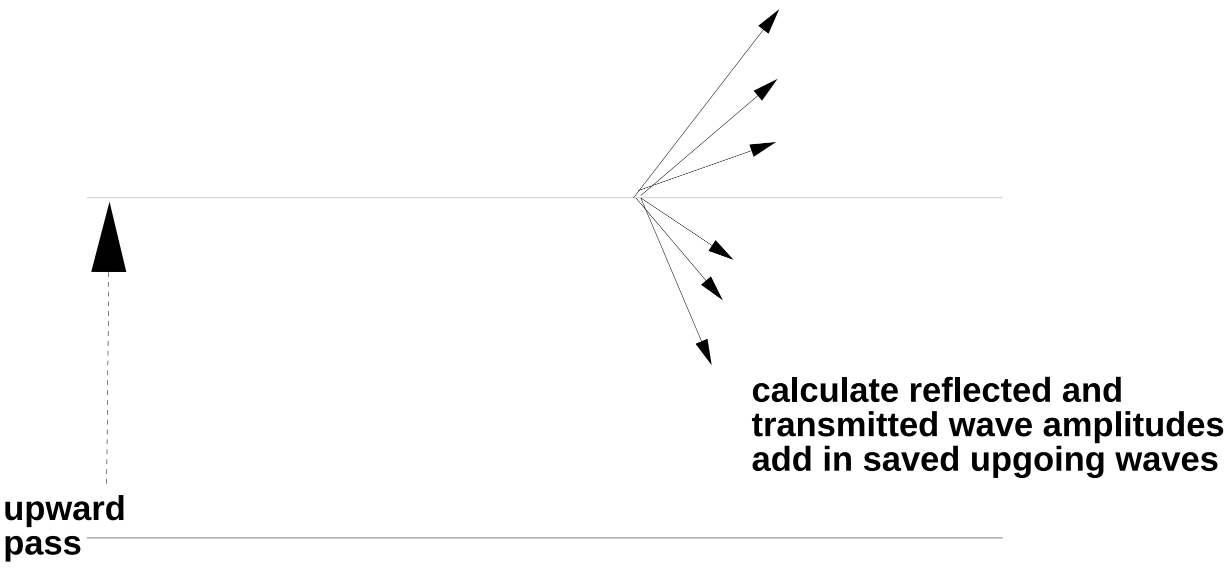

Figure 5: The fifth step calculates downgoing reflected wave amplitudes and upgoing transmitted wave amplitudes at the top of a layer. The upgoing reflected amplitudes at this interface that were saved is step 3 are added into the upgoing wavefield. The downgoing reflected amplitudes are saved to be used later. Steps 4 and 5 are repeated until the wavefield reaches the surface. dave1-pict5 [ER]

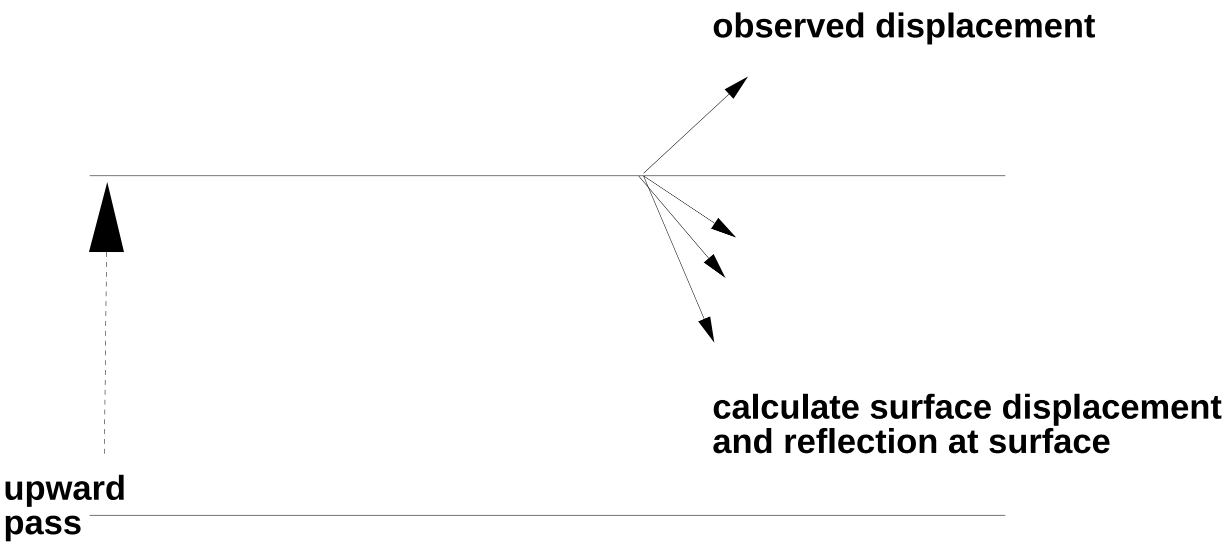

Figure 6: At the surface the reflected downgoing wavefield and the particle velocities at the free surface are calculated. The velocities are summed into the output particle velocity field. If multiples are to be modeled the downgoing wave field is used as the input to step 2. As the wavefield is propagated downwards the saved downgoing reflected amplitudes saved at each interface are added into the wavefield. Each pass down and up through the stack of layers adds one more set of multiples to the output displacement field. dave1-pict6 [ER]

The main loop of the program runs over the two horizontal slownesses p_x and p_y. For each (p_x, p_y) pair I calculate the amplitude of the recorded particle velocities at each frequency. The operators related to free surface and boundary effects are frequency independent as is the vertical slowness p_z. The only place that the frequency enters into the calculation is in the phase shift through each layer. The vertical wavenumber is given by ωp_z. The frequency independence makes the process relatively cheap. The most expensive step is the calculation of the vertical slownesses and operators for each layer. This calculation is only performed once for all frequencies and for all orders of multiples. The application of the operators and the phase shifts is fully vectorized over frequency. I will now discuss each stage of the calculation. Most of the mathematical detail is in the appendix.

Propagation of elastic waves in a homogeneous medium

I compute the propagation of waves in (ω, p_x, p_y) space in a homogeneous medium by phase shifting in depth. Given a wavefield at some depth z_0, $P(\omega, p_x, p_y, z_0)$ the wavefield at another depth, z, can be expressed as a phase shifted version of this wavefield.

$$P(\omega, p_x, p_y, z) = P(\omega, p_x, p_y, z_0)e^{i\mathbf{A}(z-z_0)}$$

In general the operator $\mathbf{A}$ is a matrix operator but if the wavefield, P is expressed as a vector of amplitudes of the different wavetypes the operator A is diagonal. The different wavetypes are the eigensolutions of the Christoffel equation (see appendix) so the operator is diagonalized. For each of the six wavetypes the extrapolation in depth can be expressed as,

$$P_i(\omega, p_x, p_y, z) = P_i(\omega, p_x, p_y, z_0)e^{i\omega p_{z_i}(z-z_0)} \qquad i = 1, \ldots, 6$$

Thus, once I have calculated the wave amplitudes at the top of any layer I can calculated the amplitudes at the bottom of the layer by phase shifting the data. The value of p_z is the same for all frequencies but the phase shift through the layer for each frequency is given by.

$$\mathbf{d_{bottom}} = \begin{pmatrix} e^{i\omega p_z(1)(z_{bottom}-z_{top})} & 0 & 0 \\ 0 & e^{i\omega p_z(2)(z_{bottom}-z_{top})} & 0 \\ 0 & 0 & e^{i\omega p_z(3)(z_{bottom}-z_{top})} \end{pmatrix} \cdot \mathbf{d_{top}}$$

This calculation is different for each frequency but on a vector computer it is fully vectorized over frequency.

If I am "turning off" propagation of a particular wavetype it is at this stage that I do it. I can simply zero the amplitude of the wave instead of applying the shifting operator.

Calculation of reflection and transmission operators

The vector of velocities and normal stresses in a layer can be related to the amplitudes of the up- and downgoing waves by a block 3×3 operator as shown in the appendix.

$$\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \begin{pmatrix} \mathbf{d} \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} \mathbf{v} \\ \sigma_{\mathbf{N}} \end{pmatrix}$$

At a horizontal boundary between two layers the vectors $\mathbf{v}$ and $\sigma_{\mathbf{N}}$ are continuous so we can write.

$$\begin{pmatrix} E^1_{11} & E^1_{12} \\ E^1_{21} & E^1_{22} \end{pmatrix} \begin{pmatrix} \mathbf{d}^1 \\ \mathbf{u}^1 \end{pmatrix} = \begin{pmatrix} E^2_{11} & E^2_{12} \\ E^2_{21} & E^2_{22} \end{pmatrix} \begin{pmatrix} \mathbf{d}^2 \\ \mathbf{u}^2 \end{pmatrix}$$

I wish to calculate four reflection and transmission operators: In the appendix I show how to calculate the four reflection and transmission operators that my algorithm needs for steps three and five.

The operators $downrefl$ and $downtrans$ are the reflection and transmission operators for a downgoing wave incident on an interface.

$$\mathbf{u}^1 = downrefl \cdot \mathbf{d}^1$$
$$\mathbf{d}^2 = downtrans \cdot \mathbf{d}^1$$

The operators $uprefl$ and $uptrans$ are the reflection and transmission operators for an upgoing wave incident on an interface.

$$\mathbf{d}^2 = uprefl \cdot \mathbf{u}^2$$
$$\mathbf{u}^1 = uptrans \cdot \mathbf{u}^2$$

Conversions at the surface

At a free surface the stress free boundary condition must be satisfied. As discussed in the appendix this results in 3×3 matrix operators that convert applied tractions to particle velocities and downgoing wave amplitudes, and operators that convert upgoing wave amplitudes into velocities and downgoing amplitudes.

$$\mathbf{d}^1 = tracdown \cdot \sigma_N$$
$$\mathbf{v}^1 = tracdisp \cdot \sigma_N$$

and

$$\mathbf{v}^1 = surfdisp \cdot \mathbf{u}^1$$
$$\mathbf{d}^1 = surfrefl \cdot \mathbf{u}^1$$

These four operators are required in performing steps one and six of the algorithm.

Results in the (p, τ) domain

All the results that I will show in this paper are for a simple two layer model. The bottom half space is isotropic and of very low density so that there is a very high reflectivity for most angles. The surface layer is highly anisotropic. It has orthorhombic symmetry with a horizontal plane of symmetry. The medium is expected to exhibit triplication in the propagation of shear waves. Most of the examples are for the response to a horizontal traction applied along the vertical symmetry plane. In figure **??** I show the (p, τ) domain response for a run with no multiples and no direct arrivals. The (p, τ) domain response is found by taking the inverse fourier transform of the (p, ω) domain data that I have calculated. The plot shows the x-component of displacement. The uppermost event is the P-P wave reflection. The second event is the P-S and S-P reflections. The third event which exhibits a downwards concavity in the wavefront is the S-S wave reflection. The range of p values used did not extend far enough to observe all of the propagating S-waves. Figure **??** shows the same model but in this case the direct arrivals have been calculated. Figure **??** shows the same model with the first order multiples calculated. The P-P-P multiple is almost hidden behind the S-S arrival. The final event is the S-S-S-S arrival, in between there are multiples with mixed wavetypes, P-P-P-S, P-P-S-S, and P-P-P-S. This figure is already somewhat confusing. One advantage of the phase shift method is that I can easily "turn off" some of these events to obtain a clearer picture.

CALCULATION OF THE TIME/SPACE DOMAIN RESPONSE

Given data in the slowness,frequency domain $P(\omega, p_x, p_y)$ I wish to calculate the response in the time,offset domain $P(t, x, y)$. This response can be calculated by a double inverse slant stack and inverse fourier transform of the input data.

$$
\begin{aligned}
P(\omega, x, y) &= \int\int P(\omega, p_x, p_y) e^{i\omega(p_x x + p_y y)} dp_x dp_y \\
P(t, x, y) &= FT^{-1} P(\omega, x, y)
\end{aligned}
$$

In the discrete modeling scheme the integral over p_x and p_y is replaced by a discrete summation and the inverse fourier transform is replaced by an inverse discrete fourier transform.

There are problems with the slowness integration caused by singularities in the modeling operation. Whenever the modeling algorithm is performed for a slowness where p_z is zero, or nearly zero, for some wavetype there is a singularity or instability in the modeling algorithm. These points occur when the waves are traveling horizontally. The integration may be performed in a stable manner by choosing to integrate in the complex slowness domain. I can choose some other contour between p_{min} and p_{max} along which to evaluate the integral. In my program I choose the contour lying along the line between $p_{min}(1 + \epsilon i)$ and $p_{max}(1 + \epsilon i)$. This contour is

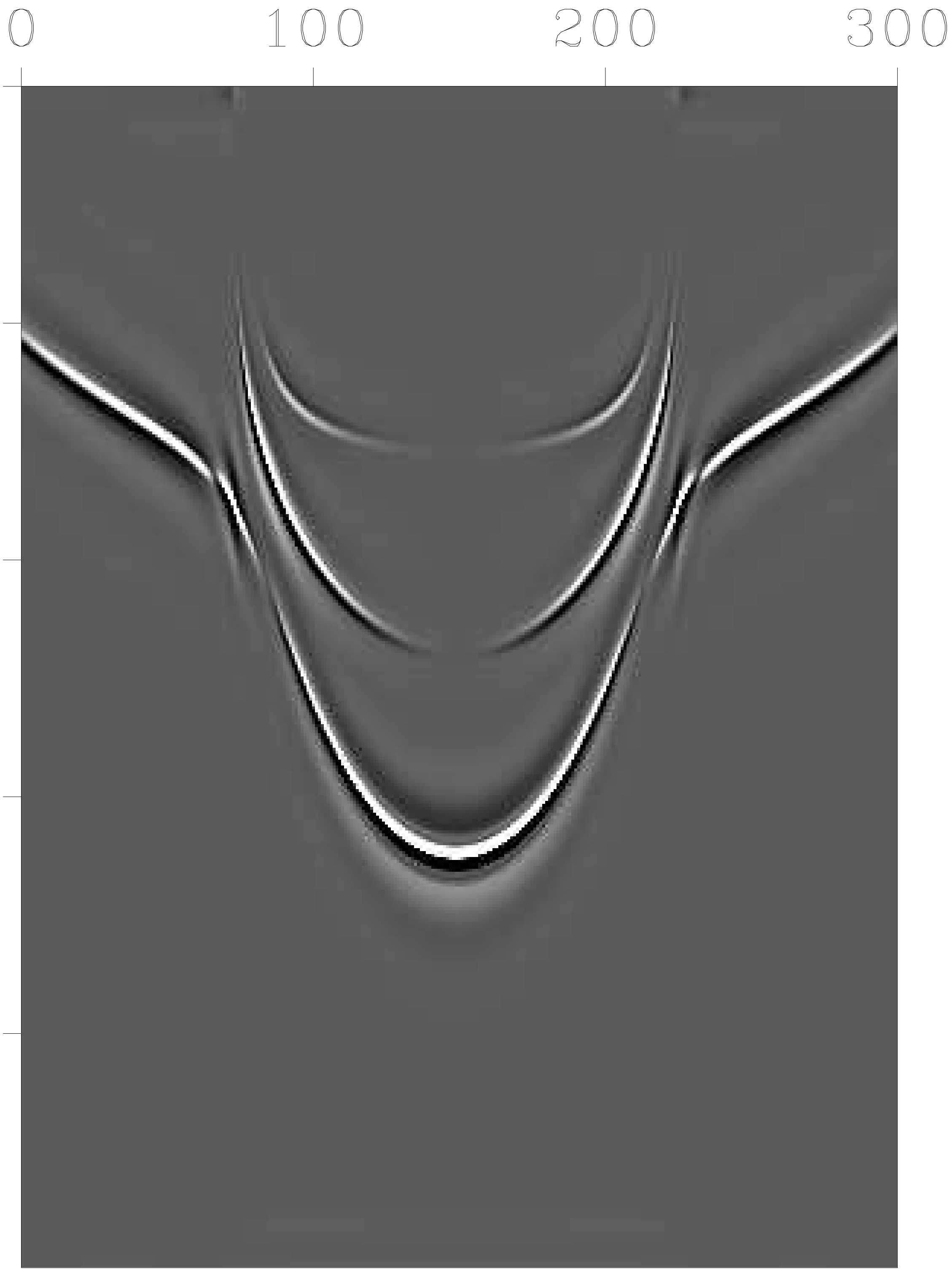

Figure 7: Primaries only data in the $p - \tau$ domain. Traction applied along x-axis, velocities measured along x-axis. dave1-pt1 [NR]

Figure 8: Primaries + direct arrivals in the $p - \tau$ domain. Traction applied along x-axis, velocities measured along x-axis. dave1-pt2 [NR]

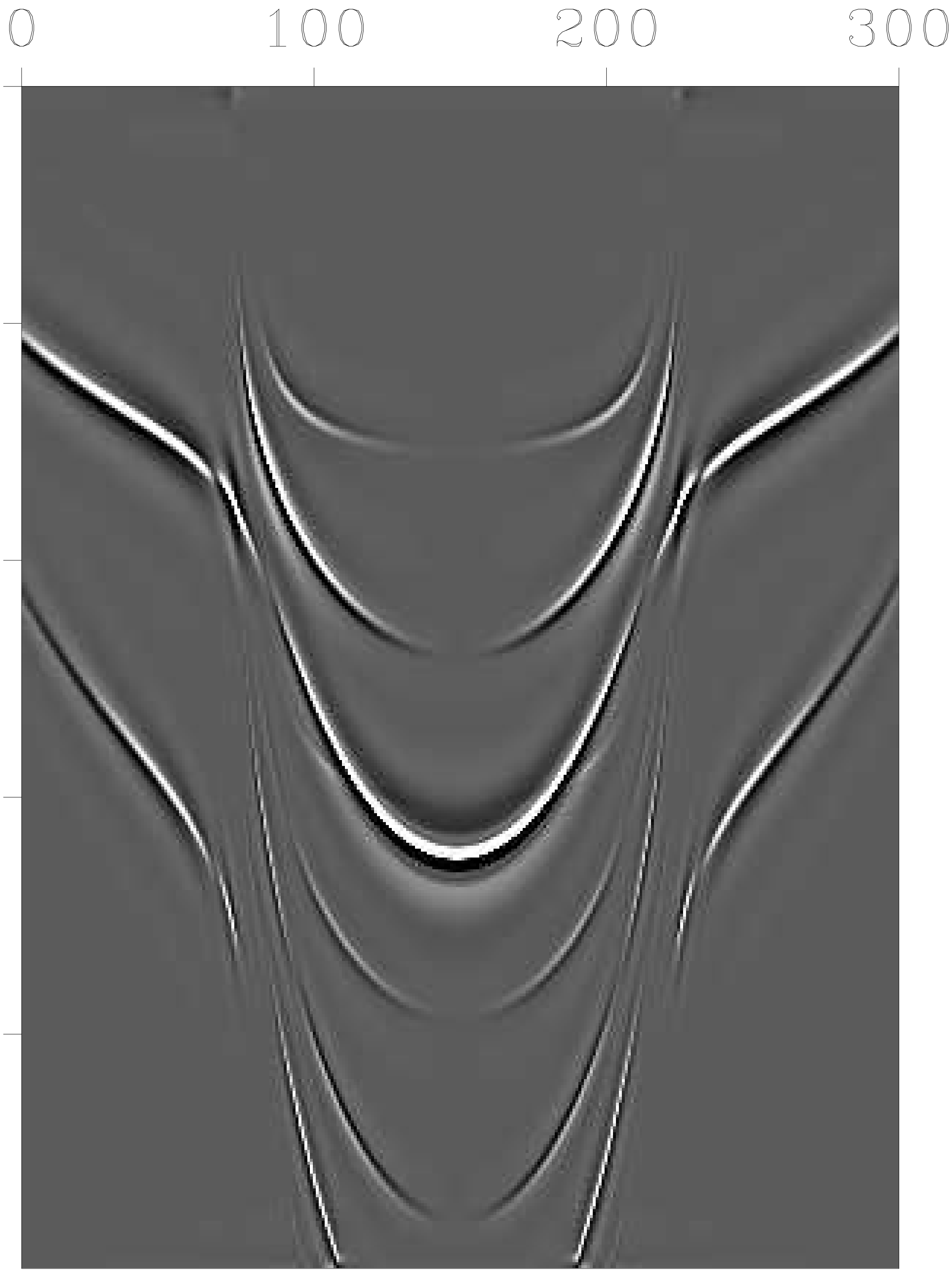

Figure 9: Primaries and first order multiples in the $p-\tau$ domain. Traction applied along x-axis, velocities measured along x-axis. dave1-pt3 [NR]

shown in figure ??. In practice I choose $\epsilon = 10^{-3}$ and I do not bother to sample the contour along the small vertical parts of the contour.

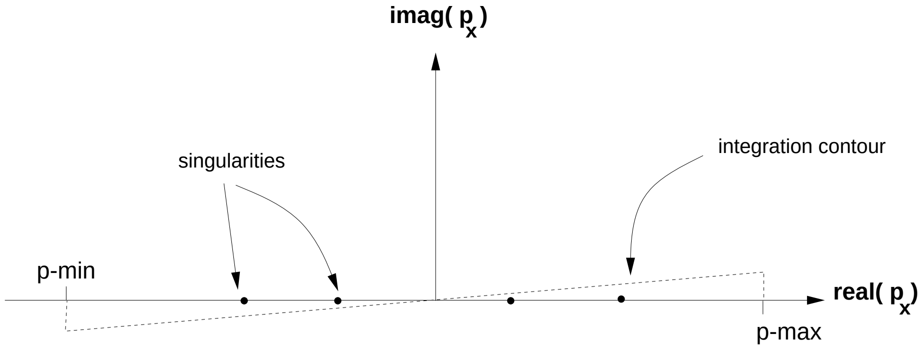

Figure 10: Integration contour is complex slowness domain. The contour avoids the singularities. dave1-contour [ER]

Figures ?? to ?? show the same data examples as before but now they are calculated in the (x, t) domain. The most noticeable feature in the (x, t) domain is the triplication of the shear wave arrival. The amplitudes at the cusps of the triplication are relatively high. This matches well with theory and modeling results using other methods.

The P-wave direct arrival is clear in figure ?? but no direct shear wave is seen because the integration in the slowness domain did not cover a large enough range.

Figure ?? shows the results of modeling a full nine-component shot gather. The shot is oriented at 30° to the symmetry axis of an orthorhombic medium. There are clear shear wave conversions and shear wave splitting in the data. The range of offsets and dips modeled was not as great as the previous examples so the shear wave triplications are not visible. The top row shows the x,y and z velocities for an x-traction. The second rows shows the results for a y-traction and the third row for a z-traction.

CONCLUSIONS

The two way elastic phase shift modeling scheme that I have presented can be used to efficiently calculate synthetic seismograms for layered anisotropic media. The scheme is closely related to reflectivity methods but differs in that it takes a stepwise approach

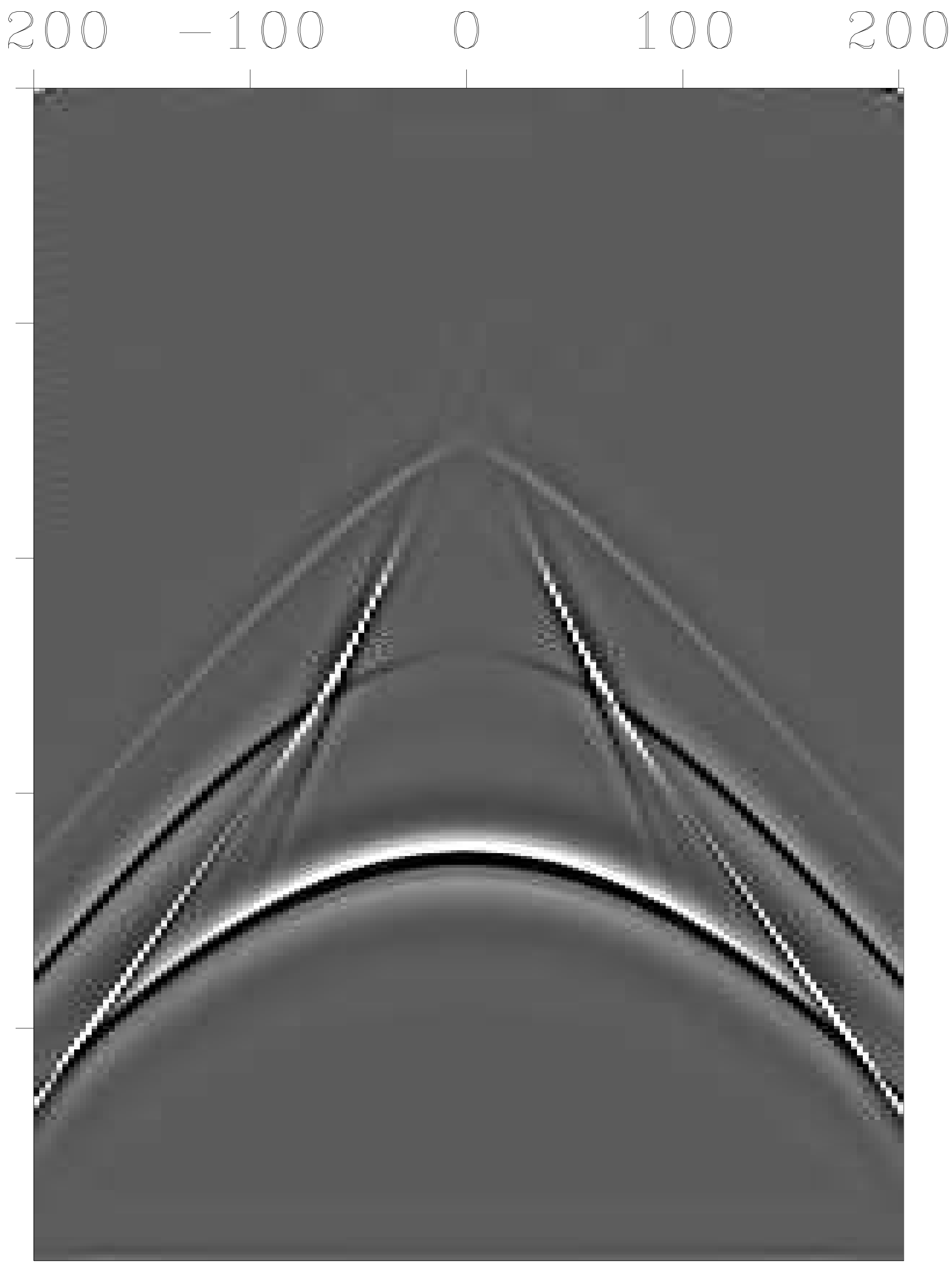

Figure 11: Primaries only data in the $x - t$ domain. Traction applied along x-axis, velocities measured along x-axis. $\boxed{\text{dave1-xt1}}$ [NR]

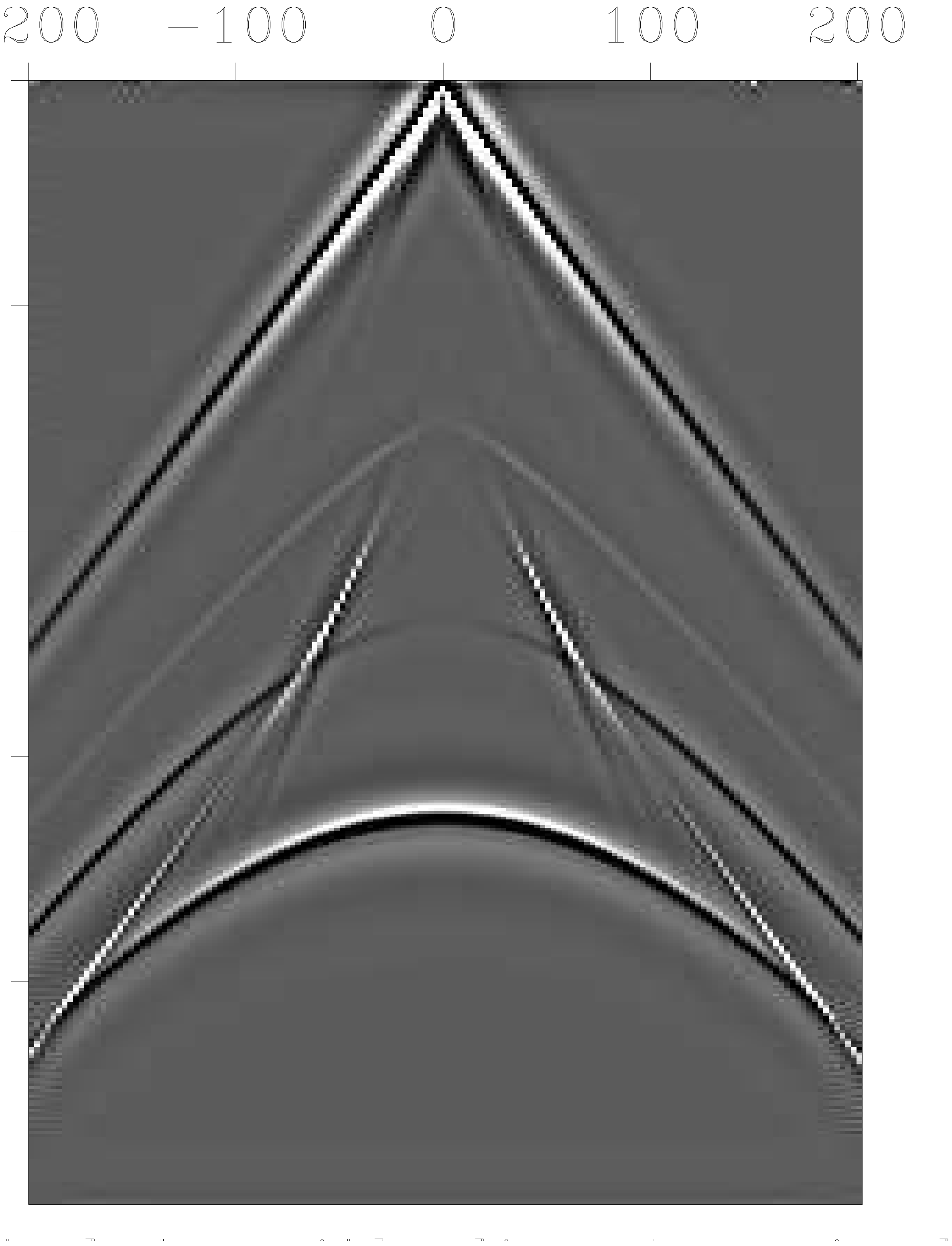

Figure 12: Primaries + direct arrivals in the $x-t$ domain. Traction applied along x-axis, velocities measured along x-axis. $\boxed{\text{dave1-xt2}}$ [NR]

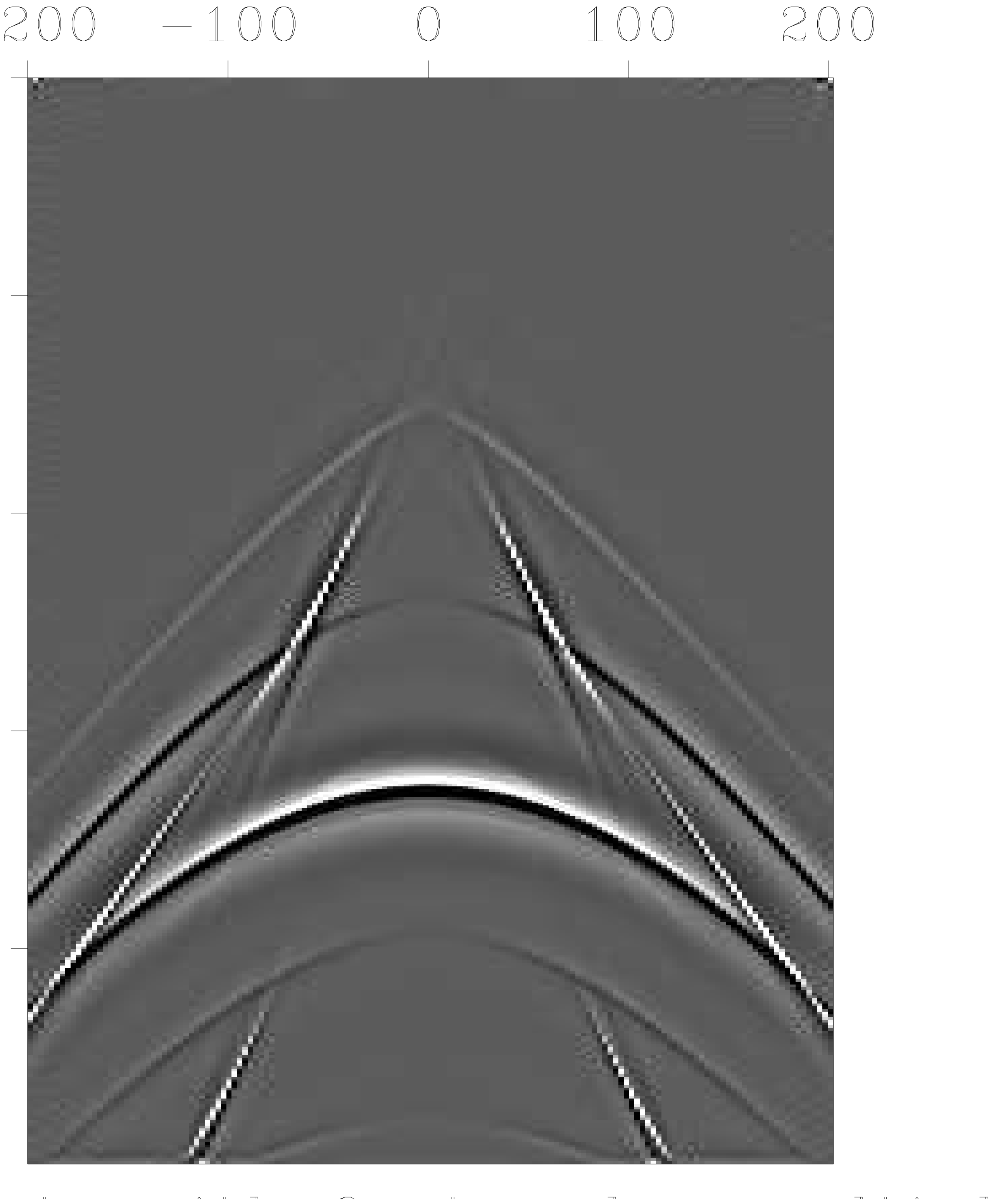

Figure 13: Primaries and first order multiples in the $x - t$ domain. Traction applied along x-axis, velocities measured along x-axis. dave1-xt3 [NR]

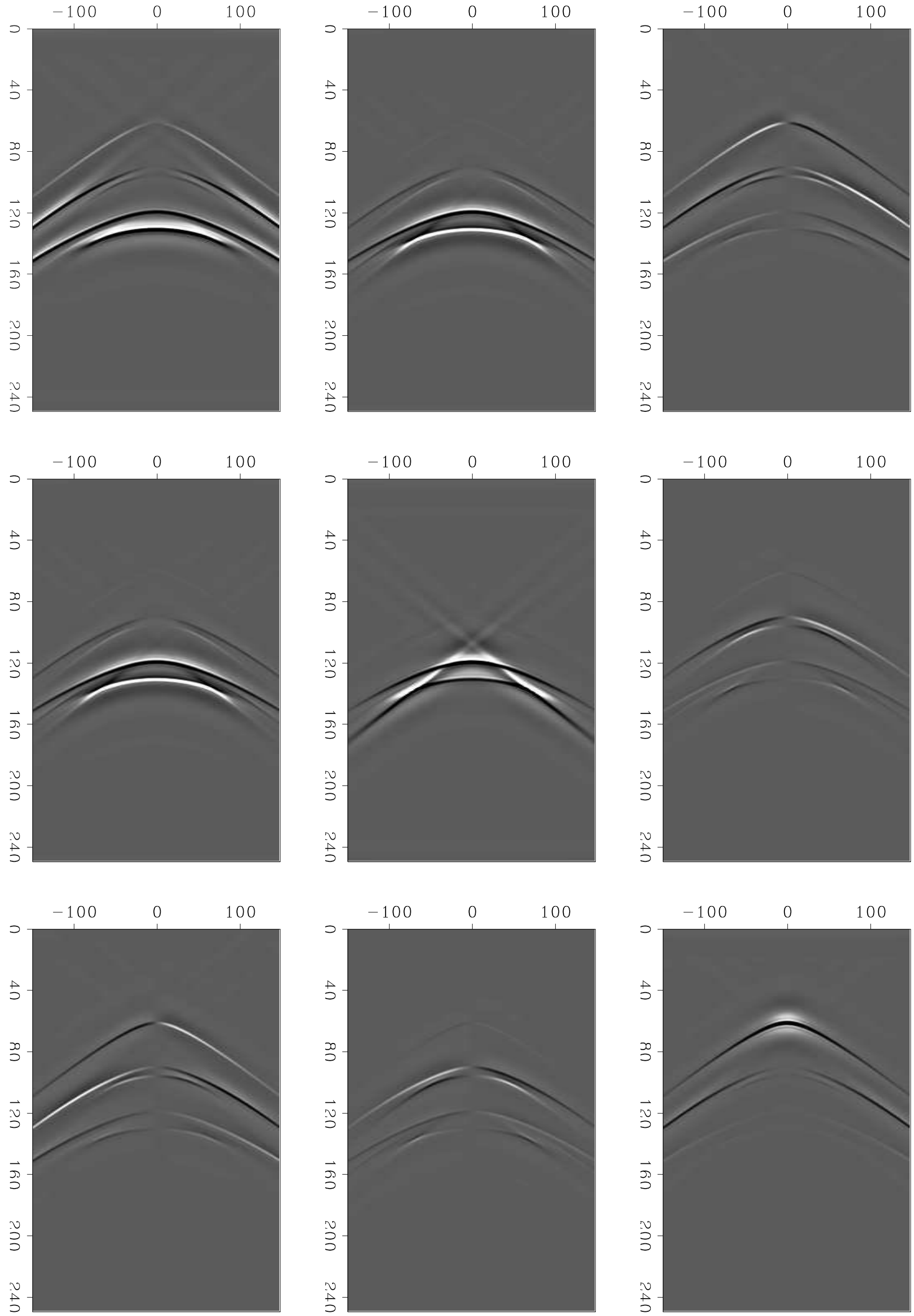

Figure 14: Synthetic nine component shot gather. Top row is response to x-traction, middle = y-traction, bottom = z-traction. First column = x-displacement, second = y-displacement, third = z-displacement. $\boxed{\text{dave1-gridplot}}$ [NR]

to modeling the wave propagation. Although this may be less efficient it allows the user select which parts of the total wavefield they wish to see in the output data. It does this without any simplifying assumptions (e.g. no conversions) that would lead to incorrect amplitude information.

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ACKNOWLEDGMENTS

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APPENDIX: WAVE PROPAGATION IN ANISOTROPIC MEDIA

Finding phase velocities and polarization vectors

When considering plane waves propagating an a homogeneous anisotropic medium it is convenient to use the Christoffel equations for an anisotropic medium. These equations:

$$\{\ k_{iM}\, C_{ML}\, k_{Lj} \quad - \quad \rho\omega^2\ \}\ \ u_j \quad = \quad 0\ ,$$

are the equations of motion of an elastic anisotropic medium, represented in the Fourier domain. Here C_{ML} is the stiffness matrix in compressed notation and k_{iM}, k_{Lj} ,ω denote the Fourier representations of spatial and temporal derivatives. ρ_{ij} is the density tensor, u_j is the particle displacement. If we write the derivative matrices as functions of slowness rather than wavenumber we have,

$$(\, p_{iM} \; C_{ML} \; p_{Lj} \quad - \quad \rho \,) \; \omega^2 \; u_j \quad = \quad 0 \; ,$$

For the general anisotropic case C_{ML} has 21 independent elements. In the monoclinic case C_{ML} has 13 independent elements.

In equation we can write $p_{iM} \; C_{ML} \; p_{Lj}$ as a product of matrices:

$$\begin{pmatrix} p_x & 0 & 0 & 0 & p_z & p_y \\ 0 & p_y & 0 & p_z & 0 & p_x \\ 0 & 0 & p_z & p_y & p_x & 0 \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & \dots & c_{16} \\ c_{21} & c_{22} & \dots & c_{26} \\ \vdots & \vdots & \ddots & \vdots \\ c_{61} & c_{62} & \dots & c_{66} \end{pmatrix} \begin{pmatrix} p_x & 0 & 0 \\ 0 & p_y & 0 \\ 0 & 0 & p_z \\ 0 & p_z & p_y \\ p_z & 0 & p_x \\ p_y & p_x & 0 \end{pmatrix} .$$

The result of the product is a 3x3 matrix a_{ij}. The Christoffel equation can be written now in the form:

$$\begin{pmatrix} a_{11} - \rho & a_{12} & a_{13} \\ a_{12} & a_{22} - \rho & a_{23} \\ a_{13} & a_{23} & a_{33} - \rho \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

This system of equations has solutions when it's determinant vanishes. In general this requires solution of a sixth order polynomial in p_x, p_y, p_z. To find the vertical slowness for a given pair of horizontal slownesses p_x, p_y I must find the roots of a sextic in p_z. If the system has monoclinic symmetry the equations are somewhat simpler. The matrix elements in equation only have terms in p_z^2 so the roots may be found by solving a cubic equation in p_z^2.

Solving the system equation using the previously calculated roots gives the corresponding polarization vectors $\mathbf{a}$. I separate the wavetypes into up- and downgoing solutions by examining the energy flow direction, the Poynting vector.

Expressing waves in terms of wave amplitudes

For a plane wave with horizontal slownesses p_1 and p_2 and frequency ω the displacement vector $\mathbf{u}$ at some point in 3-space $\mathbf{x}$ is given by,

$$\mathbf{u} = \sum_{n=1}^{6} f(n) \mathbf{a}(n) e^{i\omega(t - \mathbf{x} \cdot \mathbf{p}(n))}$$
$$\mathbf{p}(n) = (p_1, p_2, p_3(n))^T$$

The summation is over the six different wavetypes in the medium. These correspond to the six different roots of the sextic polynomial in p_3. They have slownesses vectors $\mathbf{p}(n)$, polarization vectors $\mathbf{a}(n)$ and amplitudes $f(n)$. The vertical slownesses and displacement vectors are calculated by solving the Christoffel equation.

The strains in the medium are given by,

$$\epsilon_{jk} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)$$

So the strains due to the n^{th} wavetype are,

$$\epsilon^n_{jk} = \frac{1}{2} f(n) e^{i\omega(t-\mathbf{x}\cdot\mathbf{p}(n))}[a_j(n)p_k(n) + a_k(n)p_j(n)]$$

And the stresses are given by,

$$\sigma^n_{ef} = c_{efjk}\epsilon^n_{jk}$$

Boundary conditions at a horizontal interface

At a horizontal interface the conserved quantities are the displacements, or velocities, and the tractions on the interface. I choose to work in terms of velocities as this simplifies some later expressions. It is convenient to write these quantities as a function of the amplitudes of the six wavetypes in the layer.

The vector, $(v_1, v_2, v_3, \sigma_{31}, \sigma_{32}, \sigma_{33})$ is conserved across a horizontal interface.

This vector can be written as

$$\begin{pmatrix} \mathbf{v} \\ \sigma_{\mathbf{N}} \end{pmatrix} = E \cdot f,$$

where the elements of E are,

$$E_{jn} = a_j(n) \qquad\qquad j = 1,2,3 \quad n = 1,6$$

$$E_{jn} = -c_{(j-3)3lm}[a_m(n)p_l(n) + a_l(n)p_m(n)]/2 \qquad j = 4,5,6 \quad n = 1,6$$

and the elements of f are a function of the wave amplitudes,

$$f_n = f(n) i\omega e^{i\omega(t-\mathbf{x}\cdot\mathbf{p}(n))}$$

To calculate the amplitude partitioning at an interface between two layers we need to solve the equation,

$$E^1 \cdot f^1 = E^2 \cdot f^2.$$

The superscript denotes the layer.

The coordinate frame can be translated so that the interface is at $x_3 = 0$. If we do this the exponential terms in f_n are the same in both layers and we can write,

$$E^2 \cdot f^2(n) = \cdot E^1 \cdot f^1(n).$$

This gives the relationship between the amplitudes of the six wavetypes propagating in the layers. The quantities that are required for the modeling program are the four 3×3 reflection and transmission matrices. If we partition f(n) so that $\mathbf{d} = (f(1), f(2), f(3))$ is a vector of the amplitudes of downgoing waves and $\mathbf{u} = (f(4), f(5), f(6))$ is a vector if the amplitudes of upgoing waves we can write the following equation,

$$\begin{pmatrix} E^1_{11} & E^1_{12} \\ E^1_{21} & E^1_{22} \end{pmatrix} \begin{pmatrix} \mathbf{d}^1 \\ \mathbf{u}^1 \end{pmatrix} = \begin{pmatrix} E^2_{11} & E^2_{12} \\ E^2_{21} & E^2_{22} \end{pmatrix} \begin{pmatrix} \mathbf{d}^2 \\ \mathbf{u}^2 \end{pmatrix}$$

This is the block matrix form of the layer matrix operator discussed in the text.

For a downward propagating wavefield incident on the boundary from above I wish to calculate an operator that gives the upgoing reflected wavefield and one that gives the downgoing transmitted wavefield. The equation I need to solve is,

$$\begin{pmatrix} E^1_{11} & E^1_{12} \\ E^1_{21} & E^1_{22} \end{pmatrix} \begin{pmatrix} \mathbf{d}^1 \\ \mathbf{u}^1 \end{pmatrix} = \begin{pmatrix} E^2_{11} & E^2_{12} \\ E^2_{21} & E^2_{22} \end{pmatrix} \begin{pmatrix} \mathbf{d}^2 \\ \mathbf{0} \end{pmatrix}$$

After some manipulation I obtain,

$$\mathbf{u}^1 = downrefl \cdot \mathbf{d}^1$$
$$\mathbf{d}^2 = downtrans \cdot \mathbf{d}^1$$

where,

$$downrefl = [E^2_{21}(E^2_{11})^{-1}E^1_{12} - E^1_{22}]^{-1}[E^1_{21} - E^2_{21}(E^2_{11})^{-1}E^1_{11}]$$
$$downtrans = [E^2_{11} - E^1_{12}(E^1_{22})^{-1}E^2_{21}]^{-1}[E^1_{11} - E^1_{11}(E^1_{22})^{-1}E^1_{21}]$$

The 3×3 matrices $downrefl$ and $downtrans$ convert the 3 vector of downgoing wave amplitudes in layer 1 into a three vector of upgoing reflected wave amplitudes in layer 1 and a three vector of downgoing transmitted amplitudes in later 2.

For an upward propagating wavefield incident on the boundary from below I wish to calculate an operator that gives the downgoing reflected wavefield and one that gives the upgoing transmitted wavefield. On solving,

$$\begin{pmatrix} E^1_{11} & E^1_{12} \\ E^1_{21} & E^1_{22} \end{pmatrix} \begin{pmatrix} bf0 \\ \mathbf{u}^1 \end{pmatrix} = \begin{pmatrix} E^2_{11} & E^2_{12} \\ E^2_{21} & E^2_{22} \end{pmatrix} \begin{pmatrix} \mathbf{d}^2 \\ \mathbf{u}^2 \end{pmatrix}$$

The two operators that I require are,

$$uprefl = [E^1_{11}(E^1_{22})^{-1}E^2_{21} - E^2_{11}]^{-1}[E^2_{12} - E^1_{12}(E^1_{22})^{-1}E^2_{22}]$$
$$uptrans = [E^1_{22} - E^2_{21}(E^2_{11})^{-1}E^1_{12}]^{-1}[E^2_{22} - E^2_{21}(E^2_{11})^{-1}E^2_{12}]$$

These are the reflection and transmission operators for upgoing waves.

These operators allow me to calculate the reflected and transmitted wavefields at any horizontal boundary.

Free surface boundary conditions

At the free surface we have to consider different boundary conditions. If an upcoming wave hits the free surface it will be measured as a particle velocity vector at the surface. Reflected downgoing waves will also be produced. The free surface boundary condition is that there are no normal stresses on the free surface. For a wave hitting the free surface we have,

$$(E_{11} \quad E_{12}E_{21} \quad E_{22}) \cdot \begin{pmatrix} \mathbf{d} \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} \mathbf{v} \\ \mathbf{0} \end{pmatrix}$$

This gives the solutions,

$$\mathbf{d} = surfrefl \cdot \mathbf{u}$$
$$\mathbf{v} = surfdisp \cdot \mathbf{u}$$

where,

$$surfrefl = -E_{21}^{-1}E_{22}$$
$$surfdisp = E_{12} - E_{11}E_{21}^{-1}E_{22}.$$

For a traction applied to the free surface, there is no upgoing wave. To solve for the downgoing waves produced by a source traction, we have

$$(E_{11} \quad E_{12}E_{21} \quad E_{22}) \cdot \begin{pmatrix} \mathbf{d} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{v} \\ \sigma_N \end{pmatrix}$$

This gives the solutions,

$$\mathbf{d} = tracdown \cdot \sigma_N$$
$$\mathbf{v} = tracdisp \cdot \sigma_N$$

where,

$$tracdown = E_{21}^{-1}$$
$$tracdisp = E_{11}E_{21}^{-1}.$$