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What is Equivalent in an "Equivalent Medium"?

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INTRODUCTION

The static behaviour of a medium consisting of infinite horizontal layers can be described by an equivalent homogeneous medium (Schoenberg and Muir, 1989). In this article we give a simplified notation for the calculation of the equivalent medium and discuss the question of how an equivalent medium should be defined. If stress is applied on the top boundary of a stack of layers and the homogeneous equivalent medium, both media show identical changes in elastic deformation energy within the medium and the same exterior displacements and average forces on the boundaries.

CONTINUOUS AND DISCONTINUOUS QUANTITIES

In a stack of layers certain stress and strain components are continuous across layer boundaries. For a stack of layers perpendicular to the 3-axis (z-axis) the continuous stress components are those acting on the plane normal to the 3-axis, $\sigma_{13}, \sigma_{23}, \sigma_{33}$. The continuous strain components are those tangential to the 3-axis, $\epsilon_{11},\ \epsilon_{12},\ \epsilon_{22}$. The other stress and strain components are discontinuous from layer to layer. In this paper we rearrange the compressed subscript matrix form of Hooke's law by dividing the stress and strain vectors into sub-vectors, the continuous parts,

$$\sigma_N \equiv \begin{pmatrix} \sigma_3 \\ \sigma_4 \\ \sigma_5 \end{pmatrix} \equiv \begin{pmatrix} \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \end{pmatrix}, \quad \epsilon_T \equiv \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{pmatrix} \equiv \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{pmatrix} \tag{1}$$

and the discontinuous parts,

$$\epsilon_N \equiv \begin{pmatrix} \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{pmatrix} \equiv \begin{pmatrix} \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{13} \end{pmatrix}, \quad \sigma_T \equiv \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{pmatrix} \equiv \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}. \tag{2}$$

The relationship between stress and strain for anisotropic media is a generalized version of Hooke's Law. If the medium properties are expressed as stiffnesses it is,

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$$\begin{pmatrix} \sigma_T \\ \sigma_N \end{pmatrix} = \begin{pmatrix} \mathbf{C}_{TT} & \mathbf{C}_{TN} \\ \mathbf{C}_{NT} & \mathbf{C}_{NN} \end{pmatrix} \begin{pmatrix} \epsilon_T \\ \epsilon_N \end{pmatrix} \tag{3}$$

and similarly for the inverse, compliance, matrix

$$\begin{pmatrix} \epsilon_T \\ \epsilon_N \end{pmatrix} = \begin{pmatrix} \mathbf{S}_{TT} & \mathbf{S}_{TN} \\ \mathbf{S}_{NT} & \mathbf{S}_{NN} \end{pmatrix} \begin{pmatrix} \sigma_T \\ \sigma_N \end{pmatrix} \tag{4}$$

Rewriting 3 and 4 we have,

$$\begin{pmatrix} \sigma_T \\ \epsilon_N \end{pmatrix} = \begin{pmatrix} \mathbf{S}_{TT}{}^{-1} & \mathbf{S}_{TT}{}^{-1}\mathbf{S}_{TN} \\ \mathbf{C}_{NN}{}^{-1}\mathbf{C}_{NT} & \mathbf{C}_{NN}{}^{-1} \end{pmatrix} \begin{pmatrix} \epsilon_T \\ \sigma_N \end{pmatrix} \tag{5}$$

The constitutive relation using a hybrid (stiffness/compliance) matrix, $\mathbf{X}$, is

$$\begin{pmatrix} \sigma_T \\ \epsilon_N \end{pmatrix} = \begin{pmatrix} \mathbf{X}_{TT} & \mathbf{X}_{TN} \\ \mathbf{X}_{NT} & \mathbf{X}_{NN} \end{pmatrix} \begin{pmatrix} \epsilon_T \\ \sigma_N \end{pmatrix} \tag{6}$$

Note that since $\mathbf{S} = \mathbf{C^{-1}}$ it can be shown that $\mathbf{X}_{NT} = \mathbf{C}_{NN}{}^{-1}\mathbf{C}_{NT}$ is the transpose of $\mathbf{X}_{TN} = \mathbf{S}_{TT}{}^{-1}\mathbf{S}_{TN}$. The matrix $\mathbf{X}$ is therefore symmetric.

This formulation reflects the special geometry of horizontal layers. It relates quantities, σ_N and ϵ_T, which are continuous to quantities, σ_T and ϵ_N, which are discontinuous across layer interfaces.

STATIC EQUIVALENT MEDIA

When we wish to calculate a static equivalent homogeneous medium we must consider exactly what we mean by an equivalent medium. In the paper by Schoenberg and Muir the following requirements were given. The equivalent medium should match the following properties of the stack of layers,

1. Thickness
2. Mass
3. Displacement of top of stack in response to a given surface traction.
4. Total force acting across planes perpendicular to the layers.

In static equilibrium we have the additional property that the continuous stress and strain components σ_N and ϵ_T are constant throughout the stack.

In the following equations the properties of the equivalent medium will be denoted by the superscript *equiv*. The layered medium can be a general layered medium whose properties vary continuously as a function of z.

Condition (1) is satisfied by the equation,

$$z^{equiv} = \int dz \tag{7}$$

Condition (2) is satisfied if,

$$\rho^{equiv} \int dz = \int \rho(z) dz \tag{8}$$

The total displacement of the top of the stack is given by the integral of the normal strains over the thickness of the stack. The total force acting perpendicular to the layering is given by the integral of the tangential stresses over the thickness of the stack. I.e. to satisfy conditions (3) and (4) we must preserve the integral of the discontinuous components over the thickness of the stack.

$$\int \begin{pmatrix} \sigma_T{}^{equiv} \\ \epsilon_N{}^{equiv} \end{pmatrix} dz = \int \begin{pmatrix} \sigma(z)_T \\ \epsilon(z)_N \end{pmatrix} dz \tag{9}$$

Since the normal components of the stresses and the tangential components of the strains are constant within the medium, we are able to obtain the integrated quantities, as an integral over the material properties with the stresses and stains pulled out of the integral.

$$\int \begin{pmatrix} \sigma(z)_T \\ \epsilon(z)_N \end{pmatrix} dz = \int \begin{pmatrix} \mathbf{X}(z)_{TT} & \mathbf{X}(z)_{TN} \\ \mathbf{X}(z)_{NT} & \mathbf{X}(z)_{NN} \end{pmatrix} dz \begin{pmatrix} \epsilon_T \\ \sigma_N \end{pmatrix} \tag{10}$$

The homogeneous equivalent medium has constant medium properties so that,

$$\int \begin{pmatrix} \sigma_T{}^{equiv} \\ \epsilon_N{}^{equiv} \end{pmatrix} dz = \int dz \begin{pmatrix} \mathbf{X}_{TT}{}^{equiv} & \mathbf{X}_{TN}{}^{equiv} \\ \mathbf{X}_{NT}{}^{equiv} & \mathbf{X}_{NN}{}^{equiv} \end{pmatrix} \begin{pmatrix} \epsilon_T \\ \sigma_N \end{pmatrix} \tag{11}$$

These conditions on the behavior of the homogeneous equivalent medium give a unique solution for that medium.

$$\begin{aligned} z^{equiv} &= \int dz \\ \rho^{equiv} &= \frac{1}{z^{equiv}} \int \rho(z) dz \\ \mathbf{X}^{equiv} &= \frac{1}{z^{equiv}} \int \mathbf{X}(z) dz \end{aligned} \tag{12}$$

ELASTIC ENERGY STORED IN EQUIVALENT MEDIA

The elastic energy stored in a medium when a force is applied on the exterior boundary is calculated by the double dot product of stress and strain integrated over the whole volume. Using our notation we can calculate the deformation energy in a form that incorporates the layer geometry.

$$H = \oint_V \epsilon(z) : \sigma(z) dV = \oint_V \left(\sigma(z)_T \quad \epsilon(z)_N \right) \begin{pmatrix} \epsilon_T \\ \sigma_N \end{pmatrix} dV \tag{13}$$

Thus the total elastic energy H in the medium is given by

$$H = \oint_V \left(\epsilon_T \quad \sigma_N \right) \begin{pmatrix} \mathbf{X}(z)_{TT} & \mathbf{X}(z)_{TN} \\ \mathbf{X}(z)_{NT} & \mathbf{X}(z)_{NN} \end{pmatrix} \begin{pmatrix} \epsilon_T \\ \sigma_N \end{pmatrix} dV \ . \tag{14}$$

We can calculate the energy per unit area for both the heterogeneous and the homogeneous equivalent. If we require that the energy stored in the two media be the same we have the equation,

$$\begin{aligned} \int \left(\epsilon_T \quad \sigma_N \right) \begin{pmatrix} \mathbf{X}_{TT}{}^{equiv} & \mathbf{X}_{TN}{}^{equiv} \\ \mathbf{X}_{NT}{}^{equiv} & \mathbf{X}_{NN}{}^{equiv} \end{pmatrix} \begin{pmatrix} \epsilon_T \\ \sigma_N \end{pmatrix} dz & = \\ \int \left(\epsilon_T \quad \sigma_N \right) \begin{pmatrix} \mathbf{X}(z)_{TT} & \mathbf{X}(z)_{TN} \\ \mathbf{X}(z)_{NT} & \mathbf{X}(z)_{NN} \end{pmatrix} \begin{pmatrix} \epsilon_T \\ \sigma_N \end{pmatrix} dz & \end{aligned} \tag{15}$$

which leads us to the following expression by taking constant components out of the integrals:

$$\begin{aligned} \left(\epsilon_T \quad \sigma_N \right) \begin{pmatrix} \mathbf{X}_{TT}{}^{equiv} & \mathbf{X}_{TN}{}^{equiv} \\ \mathbf{X}_{NT}{}^{equiv} & \mathbf{X}_{NN}{}^{equiv} \end{pmatrix} \begin{pmatrix} \epsilon_T \\ \sigma_N \end{pmatrix} \int dz & = \\ \left(\epsilon_T \quad \sigma_N \right) \int \begin{pmatrix} \mathbf{X}(z)_{TT} & \mathbf{X}(z)_{TN} \\ \mathbf{X}(z)_{NT} & \mathbf{X}(z)_{NN} \end{pmatrix} dz \begin{pmatrix} \epsilon_T \\ \sigma_N \end{pmatrix} & \end{aligned} \tag{16}$$

The requirement of energy equality leads to the same equations as those developed by Schoenberg and Muir. The hybrid matrix of the equivalent medium is the thickness weighted average of the hybrid matrices $\mathbf{X}(\mathbf{z})$.

CONCLUSIONS

When defining the properties required for a homogeneous medium to be the static equivalent of a layered heterogeneous medium the following conditions are sufficient to uniquely define the homogeneous medium.

- Identical thickness (volume).
- Identical mass.
- Either one of the following two equivalent conditions
  1. Identical thickness integral of discontinuous stress and strain components.
  2. Identical thickness (volume) integral of elastic energy.

Either condition leads to the Schoenberg/Muir averaging rule.

REFERENCES

Schoenberg, M., and Muir, F., A calculus for finely layered anisotropic media: Geophysics **54**, 581-589.