Eisner's reciprocity paradox and its resolution part II: the resolution

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Please read this part of "Eisner's Reciprocity Paradox and its Resolution" only *after* you have already read the beginning, which starts on page 99.

RESOLUTION OF EISNER'S PARADOX

Integrals on a small sphere

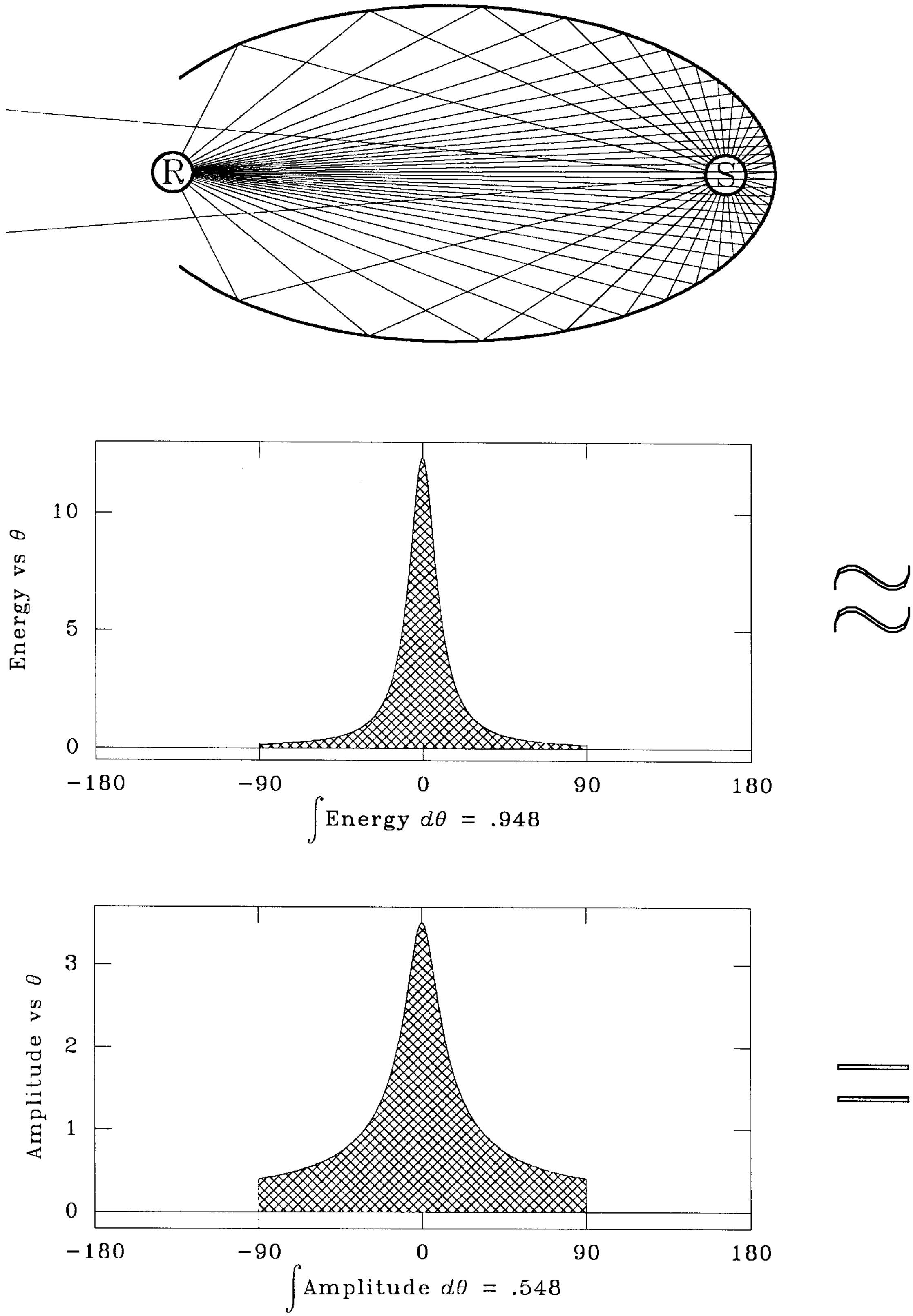
To avoid infinities, replace each focus by a small sphere. Take the outgoing wave at each sphere to have unit amplitude in all directions.

Reciprocity says that the integral of the received wave over the cut sphere in experiment A equals the integral of the received wave over the uncut sphere in experiment B. Observe that the integrand over the uncut sphere is nonzero over almost all angles, whereas the integrand over the cut sphere is nonzero over only half of all angles. This does not necessarily mean that one integral is twice that of the other, because the arriving energy varies with angle. In fact, we know that it is the *cut* sphere that receives *more* energy.

Amplitude versus energy

Reciprocity says the two integrals of wave ***amplitude*** are equal. Since each ray represents the same amount of energy, the density of rays (shown in Figures 2 and 3) gives the density of energy arriving as a function of angle. Each ray carries a local plane wave and the square root of the energy of each plane wave is its amplitude. Graphing the energy and the amplitude versus angle, and numerically integrating the areas, shows that the amplitudes are the same and that the energy in one case is almost twice that in the other.

You saw a paradox only if you confused energy integrals with amplitude integrals. Such a "paradox" is harder to understand if coupled with a breakdown of geometrical optics, but in fact these two errors in our intuition have nothing to do with each other.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∫Energy dθ = .948
Amplitude vs θ
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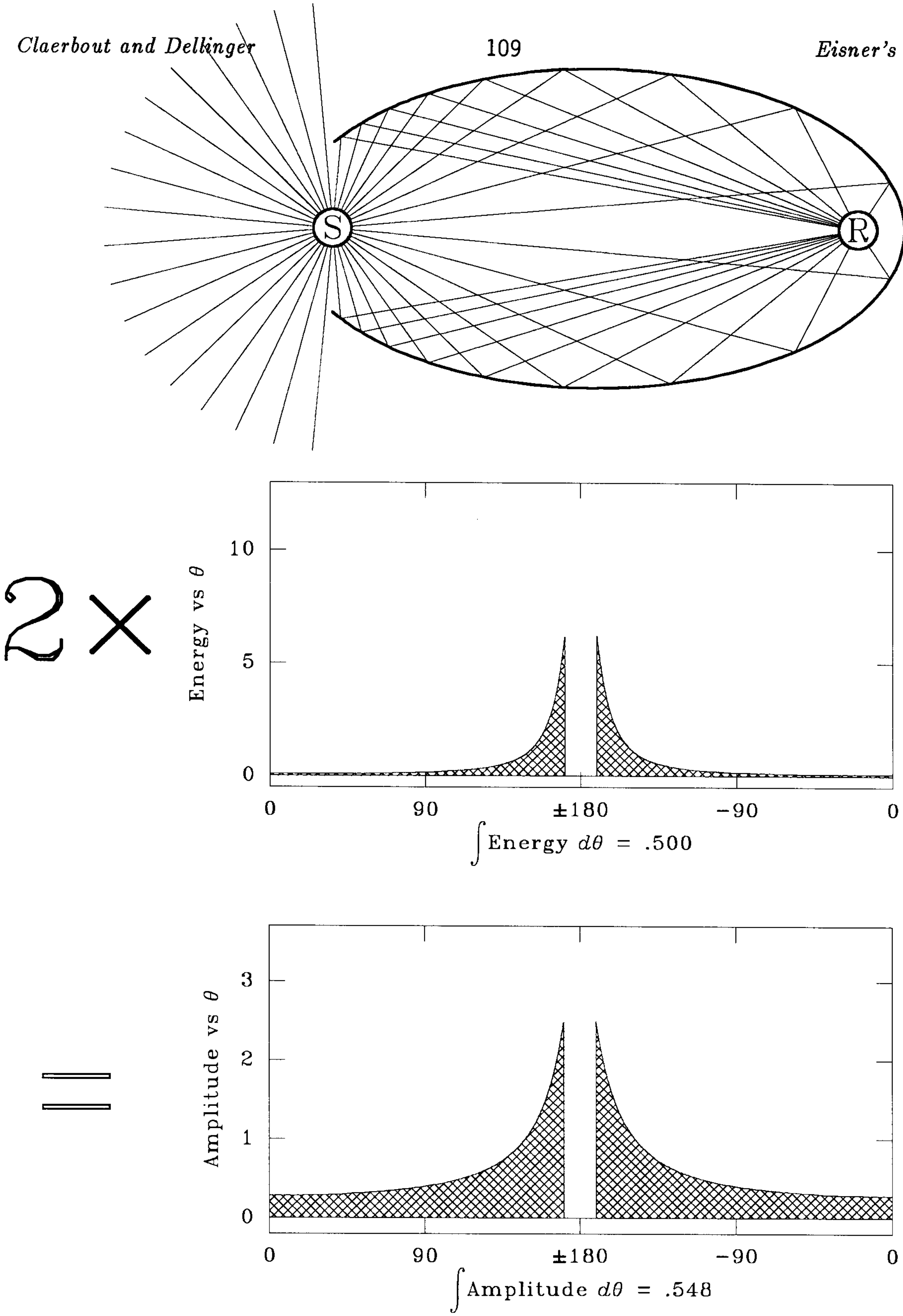
∫Amplitude dθ = .548

FIG. 2. Case A.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$\int$ Amplitude $d\theta$ = .548

FIG. 3. Case B.

The sophomore paradox

Sharpen up Eisner's paradox again by letting the small spheres collapse to points. If the energies differ by about a factor of two, how can the amplitudes be the same?

The energy in a plane wave is proportional to amplitude squared. Imagine a collision between two plane waves, one moving positively along the x-axis, the other moving negatively along the x-axis. The principle of linear superposition says the waves add. For acoustic waves, the pressures add. If the plane waves each carry a triangular pulse of unit amplitude, then at the point of collision there is a triangular pulse of amplitude two. Two squared is four, but the energy cannot have doubled.

If one wave had a negative polarity, the collision would give zero pressure, but the energy cannot have disappeared. What happened?

Resolution of the sophomore paradox

Please study Figure 4 as you read the following section.

Acoustical energy can be defined at a point as the sum of kinetic (velocity squared times density) plus potential (pressure squared times compressibility).

Where the positive pressure pulses collide (left side of the figure), the potential energy is twice as great but the kinetic energy vanishes (by symmetry because the motion of a wave moving positively is opposite to one moving negatively). Similarly, in the second case (right side) the kinetic energy doubles but the potential energy (and amplitude) vanishes.

Newton's law, mass times acceleration equals force, tells us how to compute the velocity from the pressure.

$$\text{density} \times \frac{d}{dt}\,\text{velocity} \quad = \quad -\frac{d}{dx}\,\text{pressure}$$

To get velocity from pressure, it is only necessary to differentiate with respect to x and integrate with respect to t. Observe how this equation relates to Figure 4.

So, the energy at a point is not just the square of the pressure. The velocity must be accounted for as well.

Although the two cases in Eisner's paradox have the same potential energy, they do not have the same kinetic energy. As can be seen in Figures 2 and 1, at the focus in case B it is more like plane waves colliding head-on, where the kinetic energy extinguishes, and at the focus in case A it is more like a single plane wave with all the energy going in one direction.

In the end, the lesson of Eisner's paradox is that "energy equals amplitude squared" is not true when more than one wave is involved. We fall into the habit of thinking it is always true because we so often see single plane waves, where it *is* true.

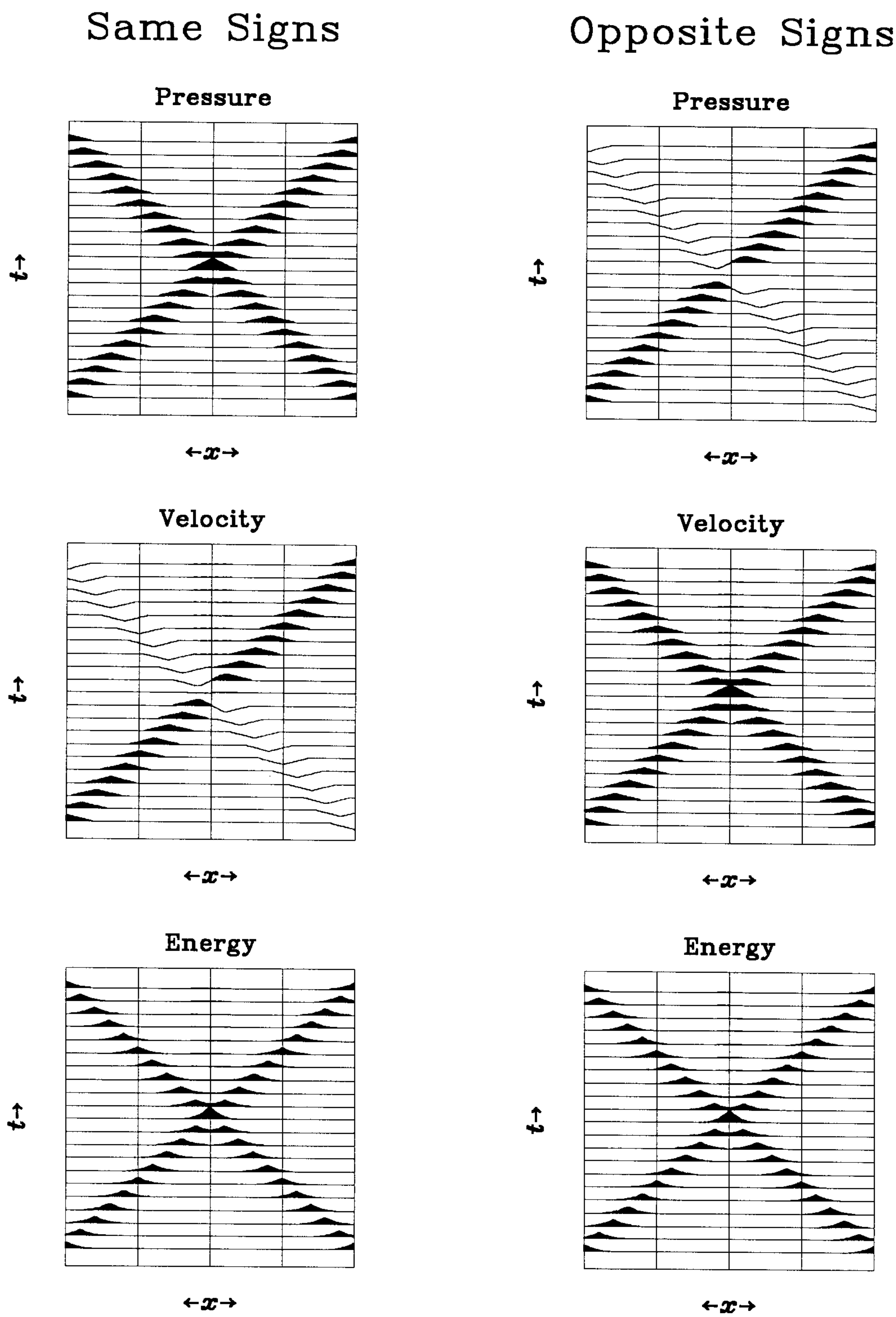

FIG. 4. The sophomore paradox, and its resolution. Left: two positive colliding pulses. Right: pulses of opposite sign colliding. Bottom: Energy conservation says that each trace has the same area. The velocity and energy fields were calculated numerically from the corresponding pressure fields by the method stated in the text.

DISCUSSION

Hydrophones and geophones

Does seismic data record pressure (potential energy) or velocity (kinetic energy)? It depends on whether it is land or marine. Geophones measure velocity, and thus kinetic energy, while hydrophones measure pressure, and thus potential energy. As you move a hydrophone towards the surface, its sensitivity to reflections gets less and less. This is because the upgoing wave is canceled by a downgoing wave of opposite polarity. On the other hand, the vertical geophone measures best on the surface, just where we put it.

Seismic reciprocity in practice

Despite its obvious bearing on the cost of land seismic data acquisition, little has been published on seismic reciprocity in the field. We know of only two published references that include field data, the first, a paper of Fenati and Rocca [1984], and the second, several pages from a textbook (Claerbout [1985]). The persistent worker is advised to return to the abstracts of the 1980 meeting (published in 1981) and contact authors directly. One particularly memorable talk (unpublished we believe) was by D. Michon.

Some exercises

Put the cap back on the end of Eisner's ellipse. Now the energy reaching the focus is constant, regardless of the eccentricity of the ellipse. Can you predict how the amplitude varies with the eccentricity of the ellipse? What happens in the limiting case when the eccentricity tends towards 1?

An advanced exercise is to show how a complex variable can have pressure as the real part and velocity as the imaginary part, and the Hilbert transform envelope as the energy. For a start, note the symmetries present in Figure 4.

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