Slippery Edges

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ABSTRACT

Equations are formulated for pressure waves on an irregular mesh.

INTRODUCTION

Immediate goal

A rectangular mesh of masses and springs has no shear strength, so shear waves are automatically excluded. A triangular mesh has shear strength, so it is not so easy to set up a model that supports compressional waves only. That is the goal here.

Long-range goal

We presume that a mesh of triangular faces has been already laid out. An icosahedron is one example. A 3-D survey is another. In both cases the mesh is nominally uniform with 10-20% fluctuations.

In finite-difference analysis we know how to obtain 4-th order accuracy in spatial operators. Fourth order accuracy is important in modeling and in data processing. We want to get such accuracy with irregular meshes.

To assure stability, meticulous attention must be paid to certain symmetries. In one dimension the discrete Laplacian is usually represented by (1, −2, 1), which is the autocorrelation of the factor (1, −1). This factor be approximated, by say (1., −.9) or (1, −1.1, +.1), but approximating the Laplacian itself, say by (1, −1.99, 1) usually leads to disaster. The reason is that the autocorrelation — more generally, the form $A^{\mathrm{T}}A$, is positive semidefinite for *any* A. The required symmetries were established for representations of the Laplacian of higher-order accuracy in one dimension in my earlier paper, *Mass-spring networks to simulate the 4-th order Laplacian.* In this paper stable equations are established in two dimensions.

A later step will be to augment the two-dimensional results here by longer range forces (in the style of the earlier paper) needed for fourth-order spatial behavior.

DEFINITIONS

Faces and edges

Let f be a subscript to point to each face and e be a subscript to point to each edge. Each face is surrounded by three edges. Each edge is surrounded by two faces.

Let $\mathbf{S} \equiv S_{ef}$ be a matrix containing a one when edge e is a boundary of face f and zero when it is not. Any row of the matrix S_{ef} has exactly three ones because each face has three edges. Any column of this matrix has exactly two ones because each edge is surrounded by two faces. Operating on an edge vector (c.f. pressure p_e) by this matrix sums around the edges of each triangle, say $\sum_e^3$. The transpose matrix operating on vector (components a function of f) sums over faces surrounding each edge, say $\sum_f^2$. The matrix $\mathbf{S} = S_{ef}$ is a rectangular matrix. Multiplying by $\mathbf{S}$ takes you from one space to the other. Multiplying by $\mathbf{S}^{\mathrm{T}}$ does the reverse.

The nodes of an icosahedron are mostly 6 cornered with some 5-cornered. The present formulation doesn't use nodes. It uses only edges and faces.

Duality and the Laplacian operator

The velocity of a face is $\mathbf{v}_f$. The pressure at an edge will be p_e. So the duality of pressure and velocity is shared by the duality of faces and edges. By "dual" I mean that we use rectangular matrices to relate the space of one to the space of the other.

I will put velocity on the faces and pressure on the edges. I suppose it could be done the other way around. In any case, from the final pair of first-order equations, a single second-order equation can be obtained.

The pressure on an edge is surrounded by velocities of the two surrounding faces which are in turn each surrounded by three pressures. Eliminating duplications we see there are *four* edges surrounding any given edge. So the second-order scalar equation for pressure expresses the Laplacian on an edge and its four surrounding edges. The differencing star looks like "<=>".

I have noticed that there are three faces surrounding each face, but I am not yet prepared to claim I understand the Laplacian of a –3 surrounded by three ones.

Masses and forces

Each face has a mass m_f and a center of mass. The edges are taken to be infinitely slippery, so that forces can act only perpendicularly to the edges. (Corners of the triangles can be considered to be chewed off so that triangles can slide short distances along their edges). Let the length of edges be L_e . On each face f draw a line from the center of mass to each edge e perpendicular to that edge. The lengths of these lines are denoted L_{ef} . Pressure forces act along the lines L_{ef} .

DYNAMICS

Rotation

Observe the L_{ef} line of one triangular face does not continue into the L_{ef} line of the adjoining triangular face. Forces are not acting on the line that connects masses. So the forces that triangles exert on one another could cause them to rotate, which we do not want because it would constitute a shear wave. Rotation is avoided by insisting on one of two conditions. Either the triangles are constrained somehow against rotation, or else they have an infinite moment of inertia so that if rotation occurs, it is infinitely slowly compared to the frequency of our waves.

Should we later decide to model shear waves but not pressure waves, we would allow only tangential forces. An axle would pass through the center of each face. The faces would have infinite mass but a finite moment of inertia.

Normals

Let $\mathbf{n}_{ef}$ be a unit vector normal to edge e pointing outward from face f .

RESULTING EQUATIONS

To ensure stability we want the symmetry properties noted in my earlier paper, i.e. looking ahead to (1) and (2), the right side operators in [ ]'s are transposes whereas the left side operators are diagonal matrices times d/dt .

We know that summing (a row vector of ones) is transpose to spraying (a column vector of ones). Likewise the dot product operator $[\mathbf{n}\cdot]$ onto a vector, is transpose to multiplying the vector $\mathbf{n}$ with a scalar, thus expanding the scalar into a vector. The S_{ef} matrix illustrates that summing over the three edges of a face, for all faces, $\sum_e^3$, is transpose to summing over the two faces adjoining an edge, for all edges, $\sum_f^2$.

The area of a triangle is half the base times the height. The area of a face is half of $\sum_e^3 L_{ef} \ \ L_e$.

So, here are the dynamical equations.

$$\rho(\frac{1}{2}\sum_e^3 L_{ef} \ \ L_e)\ \frac{d}{dt}\,\mathbf{v}_f \quad = \quad -\,[\sum_e^3 L_e\ \mathbf{n}_{ef}\ ]\ p_e \tag{1}$$

$$K^{-1}\,L_e\,(\sum_f^2 L_{ef}\ )\ \frac{d}{dt}\,p_e \quad = \quad [L_e\ \sum_f^2 \mathbf{n}_{ef}\ \cdot\ ]\ \mathbf{v}_f \tag{2}$$

The equations are have the correct symmetry for energy conservation, but is the geometry of the mechanics correct? Does the use of $\mathbf{n}$ and L contain the correct metric? Let us examine the parts of the equations.

Equation (1) expresses the mass exactly correctly on the left. On the right the force is directed by the normal vector $\mathbf{n}_{ef}$ and its magnitude is the pressure times the area.

In equation (2), I first cancel the L_e from both sides. It was needed only to make the right sides of (1) and (2) mutual transposes. Pressure arises from compressional strain times an intrinsic incompressibility K . Compressional strain is relative deformation, i.e. *stretch/distance*. Let $\mathbf{u}_f$ be the displacement of each mass center, i.e. the time integral of $\mathbf{v}_f$. The *stretch* is $\sum_f^2 \mathbf{n}_{ef} \cdot \mathbf{u}_f$. The *distance* over which it acts is $\sum_f^2 \ L_{ef}$. Equation (2) contains this stretch and distance.

If the model is made one dimensional and (1) is substituted into (2) we find the denominator contains L_{ef}^2 . It is like the Δx^2 divisor in a discrete Laplacian.

Although I feel that (1) and (2) are correct, a program demonstration is really desirable. Perhaps as an exercise I should develop the shear wave equations. Besides intrinsic interest, shear waves may exhibit a different dispersion relation near the Nyquist.