MODELING SEISMIC IMPEDANCE BY MARKOV CHAINS -- MODEL PROPERTIES

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Abstract

We model impedance as a special type of Markov chain, one which is constrained to have a purely exponential correlation function. The stochastic model is parsimoniously described by M parameters, where M is the number of states or rocks composing an impedance well-log. The probability mass function of the states provides $M-1$ parameters, and the "blockiness" of the log determines the remaining degree of freedom. Synthetic impedance and reflectivity logs constructed using the Markov model mimic the blockiness of the original logs. Both synthetic impedance and reflectivity are shown to be Bussgang, i.e. if the sequence is input into an instantaneous non-linear device, then the correlation of input and output is proportional to the autocorrelation of the input. The latter property can be used to show convergence of variable norm deconvolution.

Introduction

A property of reflectivity series that has been successfully exploited in minimum entropy and variable norm deconvolution is their spikiness. The corresponding property of impedance logs, which is essentially the integral of reflectivity, is their blockiness. A stochastic model of impedance would be a valuable aid in the development of algorithms that produce blocky impedance logs or spiky reflectivity sequences from seismograms. For instance, maximum likelihood techniques of parameter estimation are based on an *a priori* knowledge of the stochastic model. This paper treats impedance logs as a special type of Markov chain and proposes a measure of blockiness.

A considerable portion of the paper is devoted to properties of the model, especially the Bussgang property. Say a time series undergoes an amplitude distortion in a time-varying instantaneous non-linear device. Then, if the autocorrelation of the input is proportional to the cross-correlation of the input and output, the process from which the time series was realized is said to be Bussgang. Gray (in his forthcoming Ph.D. dissertation) has shown that convergence of the variable norm algorithm is guaranteed if the reflectivity sequence is Bussgang. The paper shows that reflectivity sequences derived from the model exhibit the Bussgang property. Another paper in this report ("Bussgang Processes") discusses the Bussgang property more fully, and a useful theorem for showing whether a process is Bussgang is developed.

Impedance and reflectivity

A review of the equations relating impedance and reflectivity is in order. From plane wave, normal incidence theory, the reflection coefficient c_k at the k-th interface is computed from the impedances i_k and i_{k+1} via

$$c_k = \frac{i_{k+1} - i_k}{i_{k+1} + i_k} \tag{1}$$

Inverting Equation (1) and solving for the i_k gives

$$i_{k+1} = i_1 \prod_{i=1}^{k} \left(\frac{1 + c_i}{1 - c_i} \right) \tag{2}$$

Next, define z_k as the logarithm of impedance [log(imped)]:

$$z_k = \ln \left(\frac{i_{k+1}}{i_1} \right) \tag{3}$$

With this substitution, Equation (2) becomes

$$z_{k+1} = \sum_{i=1}^{k} \ln \left(\frac{1 + c_i}{1 - c_i} \right) \tag{4a}$$

A good approximation of Equation (4a) results by using

$$\ln \left(\frac{1 - c_i}{1 - c_i} \right) = 2c_i + 0(c_i^{\,3})$$

Hence, for "small" c, Equation (4a) becomes

$$z_{k+1} = \sum_{1}^{k} 2c_k$$

$$= z_k + 2c_k \tag{4b}$$

Equation (4b) gives a linear relationship between z and c.

Stochastic model of impedance

A characteristic of most well-logs is their blockiness, that is, given rock A, there is a high probability of remaining in rock A. Furthermore, given that a new rock is encountered, say B, the probability of getting B is independent of the old rock type. For instance, changing from a sand to carbonate is as likely as changing from a shale to carbonate. Of course, the last observation presumes that the length of sedimentary column under study is sufficiently long, e.g. in a sand-shale sequence, there is an obvious correlation in rock change. Figure 1 shows reflectivity and log(imped) plotted for three wells, provided by Chevron Oil Field Research Company. The blockiness of the latter is apparent from the plots.

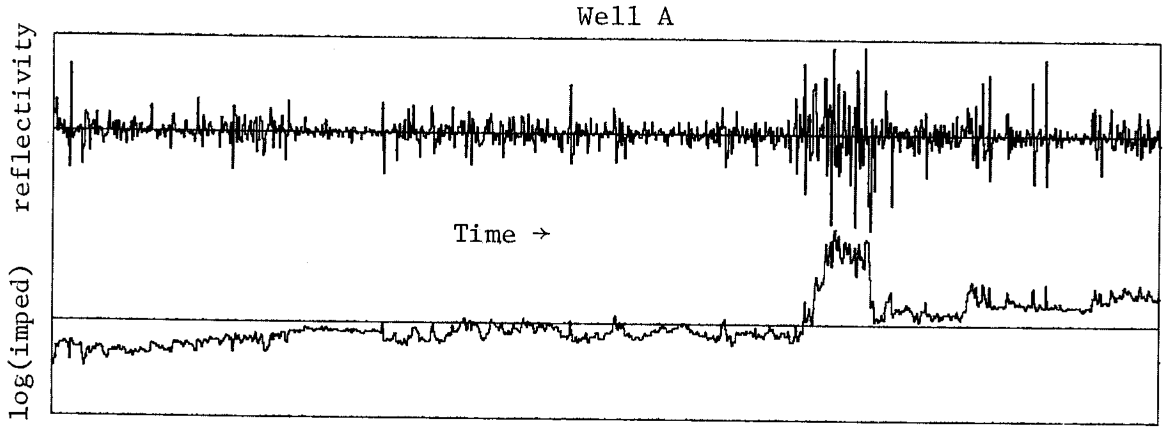
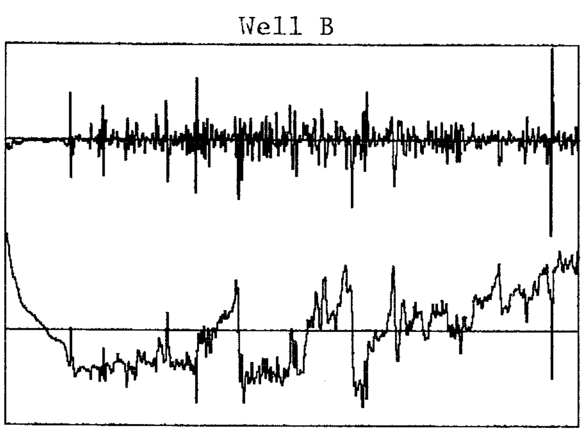
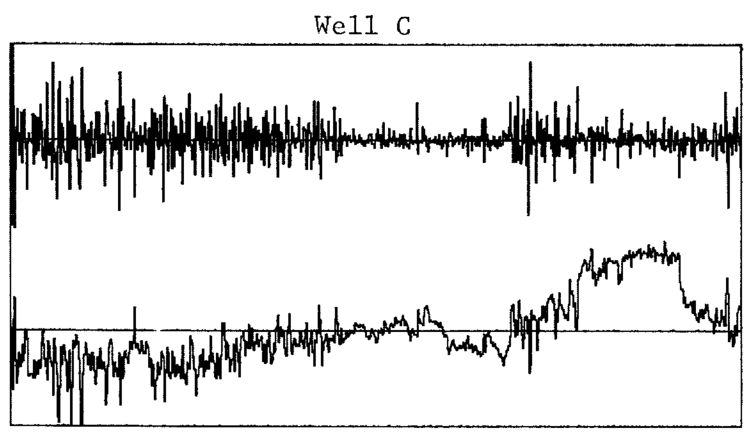


FIGURE 1.--Reflectivity and log(imped) (top and bottom plot in each box) for three wells - A,B,C - provided by Chevron Oil Field Research Company. Number of time samples in Well A = 1153, Well B = 597, Well C = 761.





Another perspective on blockiness is illustrated by plotting scattergrams, x_k vs. x_{k+1}, of reflectivity and log(imped). Figure 2 shows that from a probabilistic point of view, it is much easier to propose a model for log(imped) than reflectivity because of the striking correlation of the former.

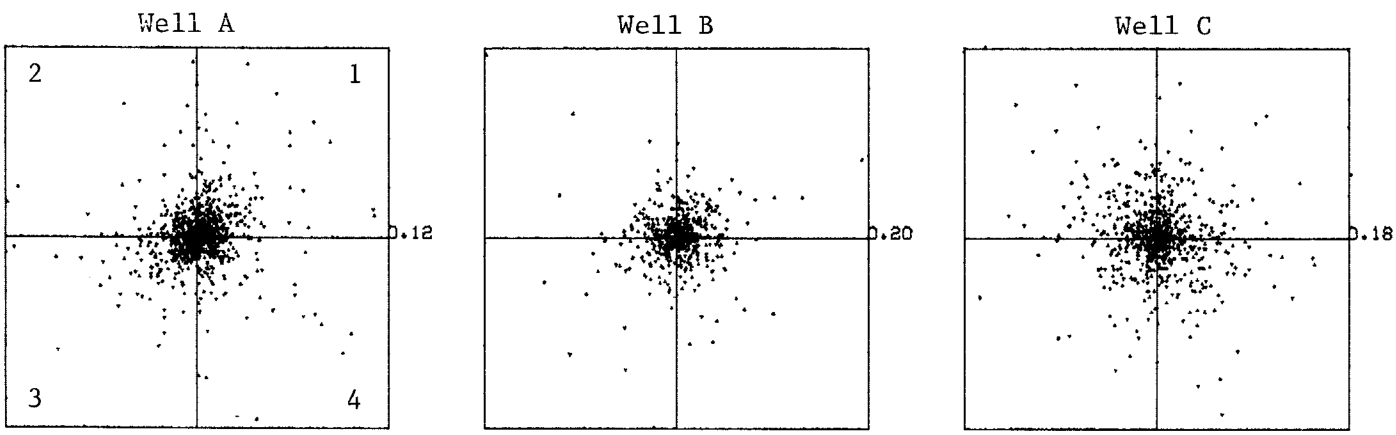


FIGURE 2a.--Scattergrams of reflectivity for three wells of Figure 1. Note presence of many plus/minus doublets in the second and fourth quadrants, perhaps signifying thin beds.

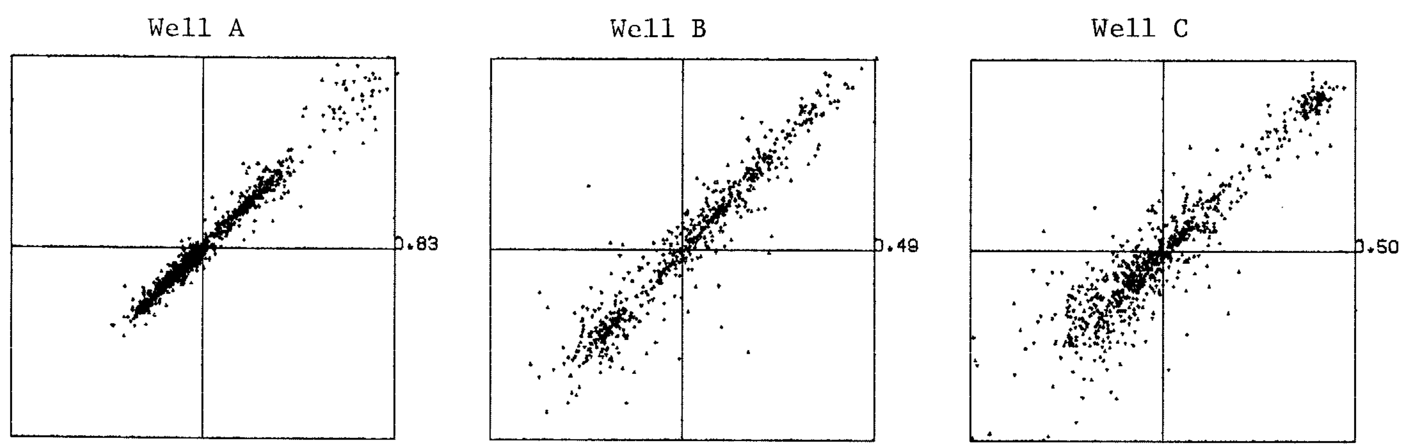


FIGURE 2b.--Scattergrams of log(imped) for three wells of Figure 1. The striking correlation ($\approx$ 1.) suggests a Markov chain for a stochastic model of impedance.

Since the probability of remaining in a rock type depends on the current rock, the model must certainly consider adjacent samples to be dependent, i.e. of second order. The actual order depends on the physics of the process, the length of data available for analysis, and the degree of complexity of the model. For the last two reasons, only one lag of memory was retained, and we model impedance as a Markov chain. A brief review of Markov chains follows.

Markov chains

Many excellent textbooks have been written on Markov chains. We have found the most readable to be Kemeny and Snell (1960). Our notation essentially follows theirs.

A sequence of samples has an underlying probability distribution that is Markov if and only if the probability of obtaining a new sample, given the history of the sequence, depends only on the current sample. The sequence of samples is called a "chain," and the term "state" is used in place of "sample." In the case of a sedimentary column, M rock types define M states (s_i, $i=1,2,\ldots M$). Note that a state is inherently valueless since any sequence of objects can be broken up into states. In our particular situation, we will associate a value of impedance [or log(imped)] to each of the states by constructing the M component value vector x.

The outcome at any particular time n in the chain is a state s_j, and this statement is written compactly as $Q_n = s_j$, where Q_n is the outcome function at time n. The statement that a sequence is Markov follows:

$$\Pr(Q_n = s_j | Q_{n-1} = s_i, Q_{n-2} = s_h, \ldots) = \Pr(Q_n = s_j | Q_{n-1} = s_i) \tag{5}$$

where $\Pr$ is short for probability, $0 \leq \Pr \leq 1$. Equation (5) introduces the concept of a probability transition matrix (PTM), P, for the chain:

$$\{P\}_{ij} = (i,j) \text{ element of matrix } P = P_{ij}$$

$$= \Pr(Q_n = s_j | Q_{n-1} = s_i) \tag{6}$$

$$\sum_j P_{ij} = 1; \quad i = 1,2,\ldots M$$

Note that the rows of P sum to 1, not the columns. Fixing the row index to $i = 3$, for example, the PTM gives the probability of jumping from s_3 to s_j where j is the current column index. A "blocky" chain would have s_{33}, the largest element in the row.

At the n-th step in a chain, the probability of being in any state is given by the probability state vector π_n. It can easily be shown that

$$\pi_n^T = \pi_{n-1}^T P \qquad (7a)$$

$$\text{or} \quad \pi_n^T = \pi_o^T P^n \qquad (7b)$$

$$\sum_i \pi_i = 1$$

Equation (7b) is quite remarkable. Essentially, computing P^n keeps track of how many different ways a state at the n-th step can be initiated from the o-th step. The probability mass vector α (probability density function for continuous states) is the steady-state solution of Equation (7a):

$$\alpha^T = \alpha^T P \qquad (8)$$

A more fundamental matrix than the PTM is the matrix formed from the unconditional probabilities. We call this matrix K for counting matrix:

$$\{K\}_{ij} = K_{ij} = \Pr[Q_n = s_j, Q_{n-1} = s_i]$$

$$= \Pr[Q_{n-1} = s_i] \Pr[Q_n = s_j | Q_{n-1} = s_i]$$

$$= \alpha_i P_{ij}$$

$$K = D P \qquad (9)$$

where D is a diagonal matrix with α along diagonal

$$\sum_{ij} K_{ij} = 1$$

$$\sum_{j} K_{ij} = \alpha_i \qquad (10)$$

We consider K to be the fundamental matrix of Markov chains because both α and P can be calculated from K via (10) and (9).

Algebraically, it is advantageous to consider the sequence $\{X_k\}$ to be zero mean, where $\{X_k\}$ is shorthand for a sequence of random variables (RV) and is formed by assigning a value to each state in the chain. If we assume the sequences to be stationary, zero mean implies:

$$EX = \sum_i Pr(Q_n = s_i) x_i$$

$$= \sum_i \alpha_i x_i$$

$$EX = \alpha^T x = 0 \qquad (11)$$

Since both probability and impedance are positive, $\alpha^T x > 0$. Negative values for x are possible if log(imped) is modeled vs. imped. In both cases, the Markov model is valid, since logarithm is a one-to-one mapping. With an abuse of notation, let x correspond to the value vector for log(imped) and constrain the choice of values so that Equation (11) is satisfied.

Synthetic log computation

A comparison between the actual logs of Figure 1 and those synthesized using a Markov chain approach is shown in Figure 3. Before discussing the comparison, the technique used to generate the synthetic logs will be explained.

The original log was discretized into fifteen states. The discretization scheme need not be linear since the state-value conversion is arbitrary except for the zero-mean constraint. Both linear (uniform intervals) and non-linear (small intervals where most points are concentrated) schemes were tested, with the former being adopted. The latter scheme cannot distinguish between large values, since they have the same state. After discretizing, K is formed via:

$$K_{ij} = \frac{\text{number of } s_i\text{-}s_j \text{ pairs}}{\text{number of samples}} \tag{12}$$

Then, α and P are computed via Equations (10) and (8). To start the chain, a state s_k is chosen from the distribution α (drawing a colored ball from an urn). The state vector π_0 is then formed by inserting the value one at the k-th position in the vector. The new state vector π_1 can then be computed as $\pi_0^T P$ (k-th row of P). As before, a new state s_j is chosen from the distribution π_1 and the whole procedure is re-started. In this way, a synthetic state log is created. The conversion from state to value is accomplished by assigning values from x to the states.

The top plot of Figures 3a-3c is the original log discretized into 15 states. The large number of states has resulted in both large and small-scale features of the original log being retained in the discretized version (compare Figure 1). Three synthetics are shown for each original log, and a cursory examination indicates that all synthetics "look" like logs. Of course, the stochastic model has 225 degrees of freedom (M^2, in general) corresponding to the number of elements in K, and it is not surprising that the synthetics mimic the original logs so well. The next two sections are concerned with reducing the degrees of freedom in the model.

Reversible chains

Most sedimentary sections are reversible, that is, tops and bottoms are difficult to distinguish. Probabilistically, this means

$$\Pr(Q_n=s_j, Q_{n-1}=s_i) = \Pr(Q_n=s_i, Q_{n-1}=s_j)$$

$$K_{ij} = K_{ji}$$

$$K^T = K$$

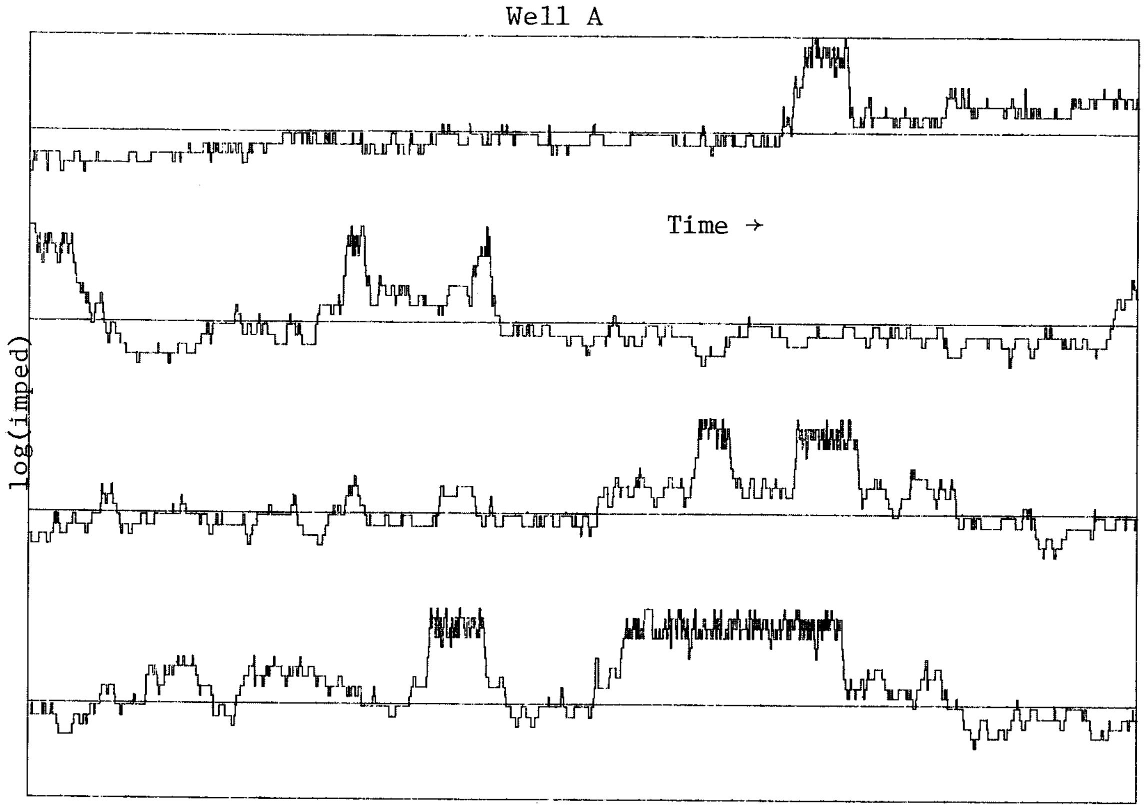
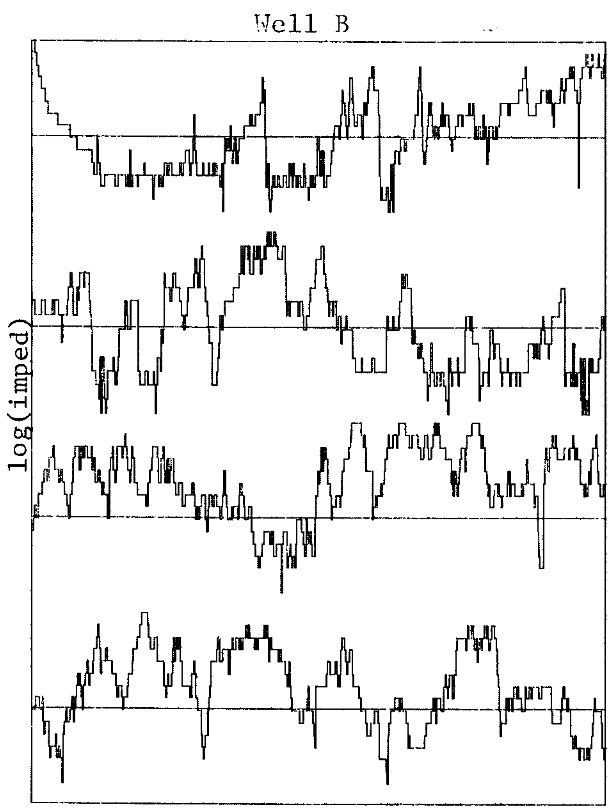
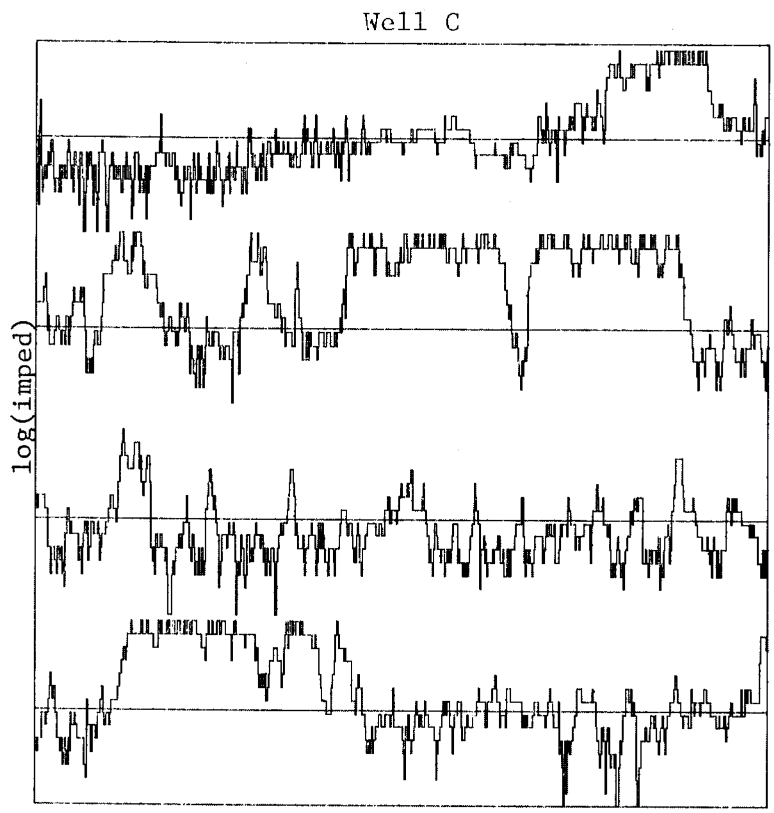


FIGURE 3.--Top plot of each group is original log(imped) of Figure 1 discretized into 15 states. Bottom three plots are synthetics generated using the probability transition matrix derived from the top log. In all cases, both fine-scale and gross features of the discretized logs are preserved in the synthetics.





In all the counting matrices investigated, $K^T \approx K$. Corresponding elements differ by $< 1\%$ on average; hence, constraining K to be symmetric is geologically sound and has the bonus of reducing the degrees of freedom in the model to $M^2/2$ from M^2. A first order model, one that treats samples independently, is completely specified by M-1 parameters, corresponding to the probability mass function. This sets the lower bound for the number of parameters necessary to describe a second order model at M. We would like to force the K matrix to have only M parameters. In doing so, however, we want to retain the blockiness characteristic of the original process. The correlation is one statistic that certainly reflects the blockiness of a time series, and in the next section an expression for the autocorrelation of a Markov chain is derived.

Autocorrelation of the chain

The autocorrelation for the chain $\{X_n\}$, at lag k is computed as follows:

$$
\begin{aligned}
R_{xx}(k) &\triangleq E\, X_n\, X_{n+k} \\
&= \sum_{ij} x_i x_j \; \Pr(Q_n=s_i, Q_{n+k}=s_j) \\
&= \sum_{ij} x_i x_j \; \Pr(Q_n=s_i) \; \Pr(Q_{n+k}=s_j \mid Q_n=s_i) \\
&= \sum_{ij} \alpha_i x_i \; P^k_{ij} x_j \;, \quad \text{where } P^k_{ij} \text{ means } \{P^k\}_{ij} \\
&= x^T\, D\, P^k\, x \qquad (13)
\end{aligned}
$$

Using Equation (13), the autocorrelation for the three discretized logs of Figure 1 (top plots in Figure 3) were computed and the results plotted in Figure 4. A best fit exponential curve is also plotted with the autocorrelation function.

The agreement for large lags is excellent, but for lags close to zero, the exponential lies below the actual curve, indicating the presence

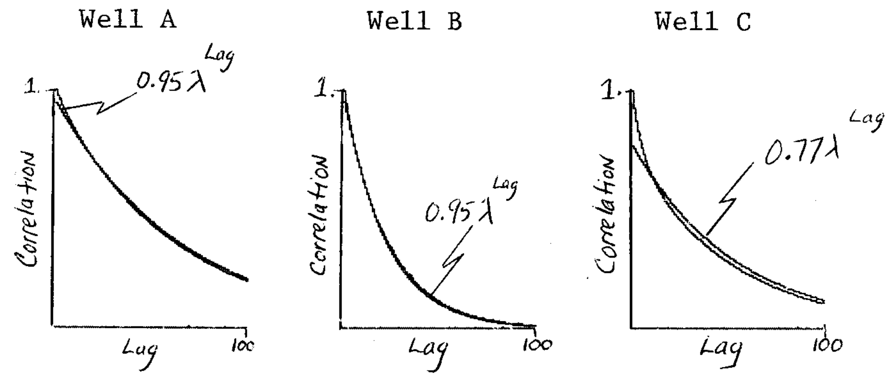


FIGURE 4.--Autocorrelation function of the three logs of Figure 3 plotted with a best fit exponential. In wells A and B, the curves are indistinguishable except at the origin, where the exponential falls below the actual curve. In Well C, the two curves are distinguishable at all lags. The presence of a noise component on the logs is indicated by the difference in the two curves at the origin. For Well A, $\lambda = 0.98$; Well B, $\lambda = 0.96$; Well C, $\lambda = 0.98$, where λ = coefficient of the exponential.

of a noise-like component in the actual logs. We call this noise "geological noise," and the reader is referred to the section on the *telegraph matrix* for a further discussion. The exponential is a parsimonious description of an autocorrelation function, and we pose the question, how can P (or K) be smoothed so that the correlation is purely exponential? Assuming P has a complete set of eigenvectors, we can decompose it into row and column eigenvectors:

$$P = \sum_{1}^{M} \lambda_i \mu_i \beta_i^T \qquad (14)$$

$$\mu_i = \text{column eigenvector}$$

$$\beta_i^T = \text{row eigenvector}$$

$$\beta_j^T \mu_i = \delta_{ij}$$

Using (14), P^k becomes

$$P^k = \sum_{1}^{M} \lambda_i^k \mu_i \beta_i^T \qquad (15)$$

Substituting Equation (15) into Equation (13) gives

$$R_{xx}(k) = x^T D \sum_1^M \lambda_i^k \mu_i \beta_i^T x$$

But $\alpha^T x = 0$ (zero-mean process); hence, calling $\beta_1 = \alpha$ gives

$$R_{xx}(k) = x^T D \sum_2^M \lambda_i^k \mu_i \beta_i^T x \tag{16}$$

For Equation (16) to give a pure exponential decay, there are two alternatives:

(a) $\lambda_i = \lambda$, $i = 2,3,\ldots M$

(b) $\lambda_i = \lambda$, $i = 2,3,\ldots R$

$\lambda_i = 0$, $i = R+1, R+2, \ldots, M$

In case (a), the exponential will be continuous for all lags, whereas in case (b), a discontinuity may exist in going from $k = 0$ to $k = 1$. Since the computed correlation functions were continuous, we consider case (a), and Equation (16) becomes

$$R_{xx}(k) = x^T D \lambda^k \sum_2^M \mu_i \beta_i^T x$$

But $$I = \sum_1^M \mu_i \beta_i^T$$

$$\therefore \quad I_x = \sum_2^M \mu_i \beta_i^T x$$

Hence, $$R_{xx}(k) = \lambda^k x^T D x \tag{17}$$

Therefore, if we force P to have $M-1$ equal eigenvalues, the process will have a purely exponential decay, regardless of what value is assigned to the corresponding state, as long as $\alpha^T x = 0$.

The telegraph matrix P_T

In this section, we describe a class of PTM's having a purely exponential autocorrelation function. We denote these matrices by P_T; the choice of subscript will be apparent later. P_T is completely specified by only M parameters.

A class of PTM's that have a non-zero eigenvalue repeated M-1 times is given by the construction,

$$P_T = \lambda I + (1 - \lambda) \Gamma \alpha^T \tag{18}$$

Γ = column vector of ones

λ = only eigenvalue of P_T not equaling 1.

Proof: We have to show (*i*) α^T is a row eigenvector of eigenvalue 1 and (*ii*) λ is an eigenvalue repeated M-1 times.

$$(i) \quad \alpha^T P_T = \lambda \alpha^T + (1 - \lambda) \alpha^T \Gamma \alpha^T$$

$$= \alpha^T \quad (\alpha^T \Gamma = 1), \quad \text{Q.E.D.}$$

$$(ii) \quad \beta_i^T P_T = \lambda \beta_i^T + (1 - \lambda) \beta_i^T \Gamma \alpha^T$$

There are M-1 vectors β_i, which can be chosen perpendicular to Γ, hence:

$$\beta_i^T P_T = \lambda \beta_i^T ; \qquad i = 2,\ldots,M$$

From this we conclude that λ is repeated M-1 times. Q.E.D.

Equation (18) is easily interpreted if the matrices involved are written out explicitly:

$$P_T = \lambda \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} + (1 - \lambda) \begin{bmatrix} \text{———} \ \alpha^T \ \text{———} \\ \text{———} \ \alpha^T \ \text{———} \\ \text{———} \ \alpha^T \ \text{———} \end{bmatrix} \tag{19}$$

Assume we are in state k. The parameter λ is interpreted as the bias of a coin (biased toward heads). If the result of a coin toss is heads, we stay in state k, otherwise there is an opportunity to change states. The matrix $\Gamma \alpha^T$ represents an independent process, since all rows are identical. If the coin toss resulted in tails, we choose a new state, independent of the previous state. It is quite possible to select state s_k again (especially if α_k is ≈ 1) and the effect is to remain in state s_k. In terms of rocks, this means that at an unconformity, a sand overlies a sand. The independence of $\Gamma \alpha^T$ means the transition from a sand to shale is as likely as a transition from carbonate to shale, given that an unconformity exists. This description is much like a random telegraph signal when M=2; hence, the subscript (T) in P_T, which we call the *telegraph matrix*.

The corresponding counting matrix K_T is:

$$\begin{aligned} K_T &= DP \\ &= \lambda D + (1 - \lambda)\alpha\alpha^T \end{aligned} \tag{20}$$

From (20), we conclude K_T is symmetric; hence, the chains are reversible.

The coefficient of the best-fit exponential is chosen for the value of λ that is regarded as a measure of blockiness. Since λ is an eigenvalue of a PTM, it has a range: $-1 < \lambda \leq 1$. A value of $\lambda = 1$ indicates a chain that never changes state, hence, perfectly blocky. An independent process is represented by $\lambda = 0$, and the corresponding chain has no tendency to be blocky since it is uncorrelated. Negative values of λ indicate a chain that is quite erratic, with a desire to jump out of the current state. While it is hard to imagine an impedance log having $\lambda < 0$, an argument can be made for reflectivity logs having negative values.

The probability mass vectors for both PTM's, P and P_T, are identical. Coupled with the value of λ, chosen by the above method, the M parameters of P_T are completely specified.

When M is large, say 15, the differences between chains computed using P and P_T are considerable since P allows small-scale fluctuations that P_T prohibits. In Figure 5, some synthetic logs

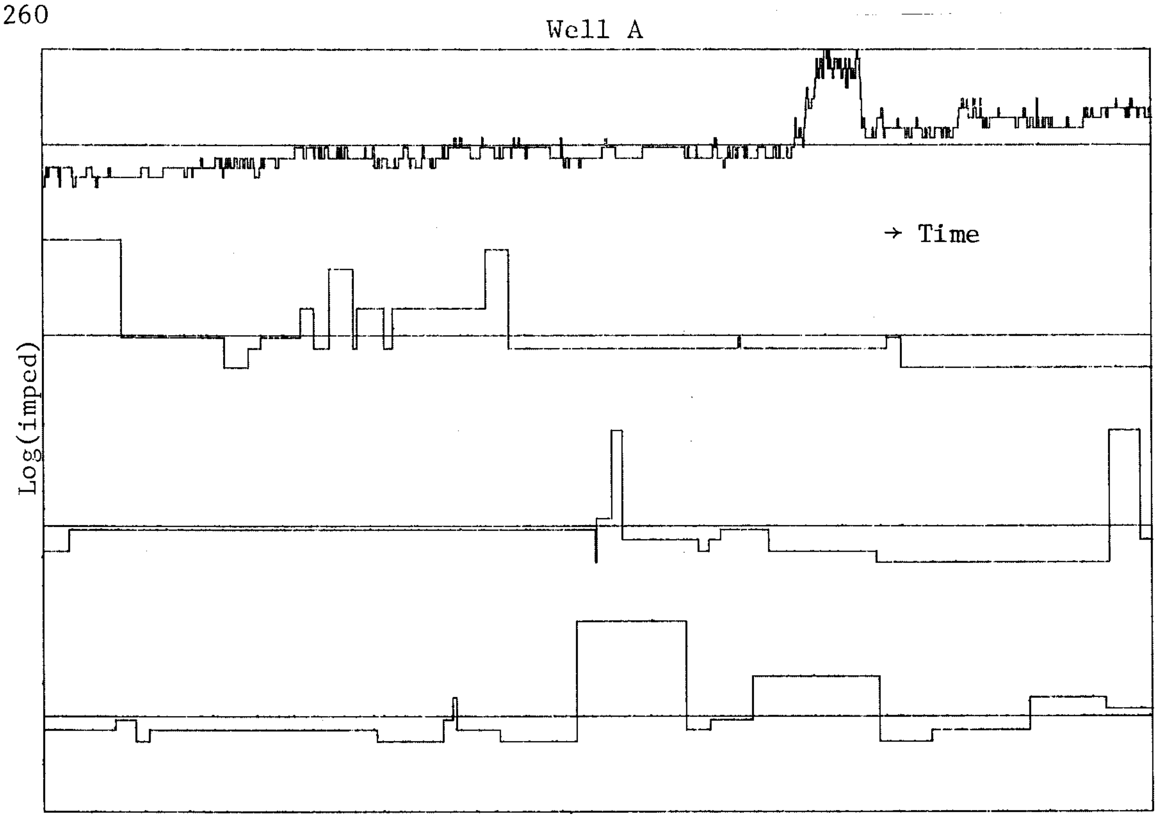
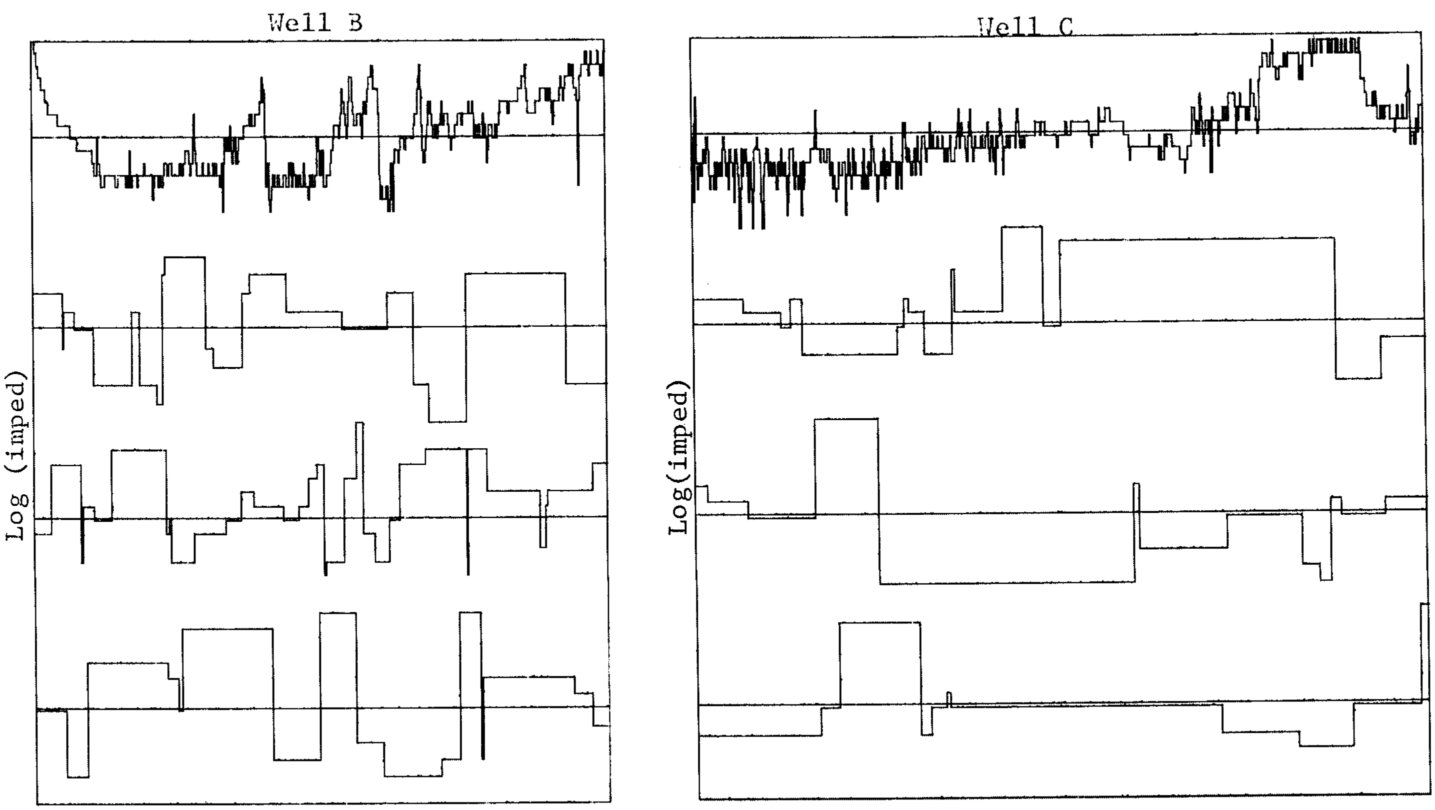


FIGURE 5.--Top plots are identical to top plots of Figure 3. Bottom plots are synthetics generated using the telegraph matrix P_T. All fine-scale structure is lost in the synthetics; however, the gross blockiness of the original logs is retained.



computed using P_T (M=15) are shown, and comparison with the top plot of each group illustrates the previous point. The synthetics can be made to look noisy by adding some "geological noise" onto the P_T chain. This noise, which is independent of the P_T chain, can be either uncorrelated (white) or weakly correlated ($\lambda \approx 0$), and its variance can be computed as the difference in the curves of Figure 4 at zero-lag. The presence of this noise will play an important role when forming a likelihood function for parameter estimation, but this is the subject of another paper.

In summary, the telegraph matrix is the simplest second order stochastic model of impedance possible. The model cannot simulate all features of impedance logs; it will, however, simulate the blockiness characteristic, which a first order model is incapable of reproducing. The next section discusses the Bussgang property ot P_T. Will Gray (Ph.D. dissertation to be published in 1979) noted that variable norm deconvolution will converge (in expectation) if the reflectivity process is Bussgang. Before studying reflectivity, however, we show that if the PTM characterizing log(imped) is P_T, then log(imped) is Bussgang.

Bussgang property of P_T

Consider the process $\{Y_t\}$ formed by passing $\{X_t\}$ through a zero-memory non-linearity (ZNL):

$$y_t = \mathrm{ZNL}(x_t) \tag{21}$$

The process $\{X_t\}$ has the Bussgang property if and only if

$$\frac{EX_nX_{n+k}}{EX_{n+k}Y_n} = \frac{R_{xx}(k)}{R_{xy}(k)} = \text{constant, all } k \tag{22}$$

Restricting the PTM to be of the class P_T, an expression for $R_{xy}(k)$ can be derived in an analogous manner to that giving Equation (13):

$$R_{xy}(k) = \sum_{ij} y_i \; x_j \; \Pr(Q_{n+k}=s_j, Q_n=s_i)$$

$$= \lambda^k \; y^T D \; x, \qquad k \geq 0 \tag{23}$$

Equation (23) is equally valid for negative lags. For instance,

$$R_{xy}(-1) = E \; X_n \; Y_{n+1}$$

$$= x^T \; K \; y \tag{24}$$

Since $R_{xy}(-1)$ is a scalar, transposing Equation (24) does not change the answer:

$$R_{xy}(-1) = y^T \; K^T \; x$$

$$= y^T \; K \; x = R_{xy}(1)$$

A similar symmetry argument can be used for all other lags. Hence,

$$R_{xy}(k) = \lambda^{|k|} \; y^T \; D \; x$$

The process $\{X_t\}$ is Bussgang since Equation (22) holds, i.e.

$$\frac{R_{xx}(k)}{R_{xy}(k)} = \frac{x^T \; D \; x}{y^T \; D \; x} \neq f(k)$$

Reflectivity from impedance

Given two adjacent values of log(imped), either Equation (4a) or Equation (4b) can be used to calculate reflectivity. In either case, the new chain is constructed by combining adjacent states of the old chain. Figure 6 illustrates the construction.

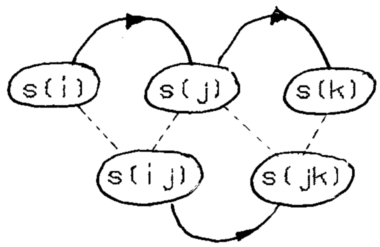


FIGURE 6.--Combining adjacent states in the log(imped) chain produces a new Markov chain of M^2 states -- the reflectivity.

What are the properties of the reflectivity process, given that the log(imped) process was P_T? For instance, do differential processes of P_T retain the Bussgang property of P_T, or more fundamentally, are they even Markov? In this section, it will be shown that reflectivity is both Markov and Bussgang.

The PTM of the new chain $\hat{P}_T$ has M^2 states, and is related to P_T via

$$\begin{aligned}
\hat{P}_{(ij),(k\ell)} &= \Pr[R_n = s_{(k\ell)} | R_{n-1} = s_{(ij)}] \\
&= \Pr(Q_n = s_\ell, Q_{n-1} = s_k | Q_{n-1} = s_j, Q_{n-2} = s_i) \\
&= \Pr(Q_n = s_\ell | Q_{n-1} = s_k, Q_{n-2} = s_i)\delta_{jk} \\
&= \Pr(Q_n = s_\ell | Q_{n-1} = s_k)\delta_{jk} \\
&= P_{k\ell}\, \delta_{jk} \qquad (25)
\end{aligned}$$

Equation (25) is proof that the expanded chain inherits the Markov property of the original chain. The corresponding probability mass vector $\hat{\alpha}$ is constructed from K via

$$\hat{\alpha}_{(ij)} = \Pr[R_n = s_{(ij)}]$$

$$= \Pr(Q_n = s_j, Q_{n-1} = s_i)$$

$$= K_{ij}$$

Figure 7 shows the expansion of a three-state chain into a nine-state chain and should clarify the notation used in the above equations.

$$P_T = \begin{bmatrix} A & b & c \\ a & B & c \\ a & b & C \end{bmatrix} \qquad \hat{P}_T = \begin{bmatrix} A & b & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & B & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & b & C \\ A & b & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & B & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & b & C \\ A & b & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & B & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & b & C \end{bmatrix}$$

$$K_T = \begin{bmatrix} d & e & f \\ e & g & h \\ f & h & i \end{bmatrix} \qquad \hat{\alpha} = \begin{bmatrix} d & e & f & e & g & h & f & h & i \end{bmatrix}$$

FIGURE 7.--The original chain, characterized by the PTM P_T and counting matrix K_T is expanded into M^2 states, giving $\hat{P}_T$ and $\hat{\alpha}$.

The value vector $\hat{x}$ is formed by using either Equation (4a) or (4b) together with the original values x. In either case, it is simple to show that

$$\hat{\alpha}^T \hat{x} = 0$$

i.e. the new chain is zero-mean. The mapping from state to value is *onto* (not one-one). For instance, there are always M zero values corresponding to the M states $s_{(ii)}$, $i = 1,2,\ldots M$. By partitioning the state-space ("lumping states") a mapping that is one-one can be effected. The lumped process is no longer Markov, however. For this reason, we retain M^2 states.

While theoretically advantageous, the PTM $\hat{P}_T$ has dimensions $(M^2 \times M^2)$ that quickly exhaust core storage in a computer. For this reason, we computed synthetic reflectivity sequences (Figure 8) by applying Equation (4b) (differentiating) directly to the synthetic log(imped) logs of Figure 5. The reason for choosing Equation (4b) over (4a) will become apparent in the next section. As expected, the actual reflectivity sequence (top figure) is much "noisier" than the synthetic sequences.

The remainder of the paper is concerned with establishing the Bussgang property of the reflectivity sequences.

Bussgang property of $\hat{P}_T$

Previously, it was shown that chains constructed using P_T exhibit the Bussgang property. Are the reflectivity sequences computed using $\hat{P}_T$ also Bussgang? To answer this, we first derive an expression for the autocorrelation.

Denoting reflectivity by C_k, a random variable, Equation (4b) relates C_k to a differential process involving X_k [a random variable representing log(imped)]:

$$C_k = X_k - X_{k-1} \tag{26}$$

The autocorrelation of C is:

$$\begin{aligned} R_{cc}(j,k) &= E\, C_j\, C_k \\ &= E[(X_j - X_{j-1})(X_k - X_{k-1})] \end{aligned} \tag{27}$$

From Equation (17), however, we have an expression for the A/C of X:

$$R_{xx}(t) = \lambda^{|t|} \tag{28}$$

$$\text{where } x^T D\, x \triangleq 1$$

Expanding Equation (27) and substituting Equation (28) gives

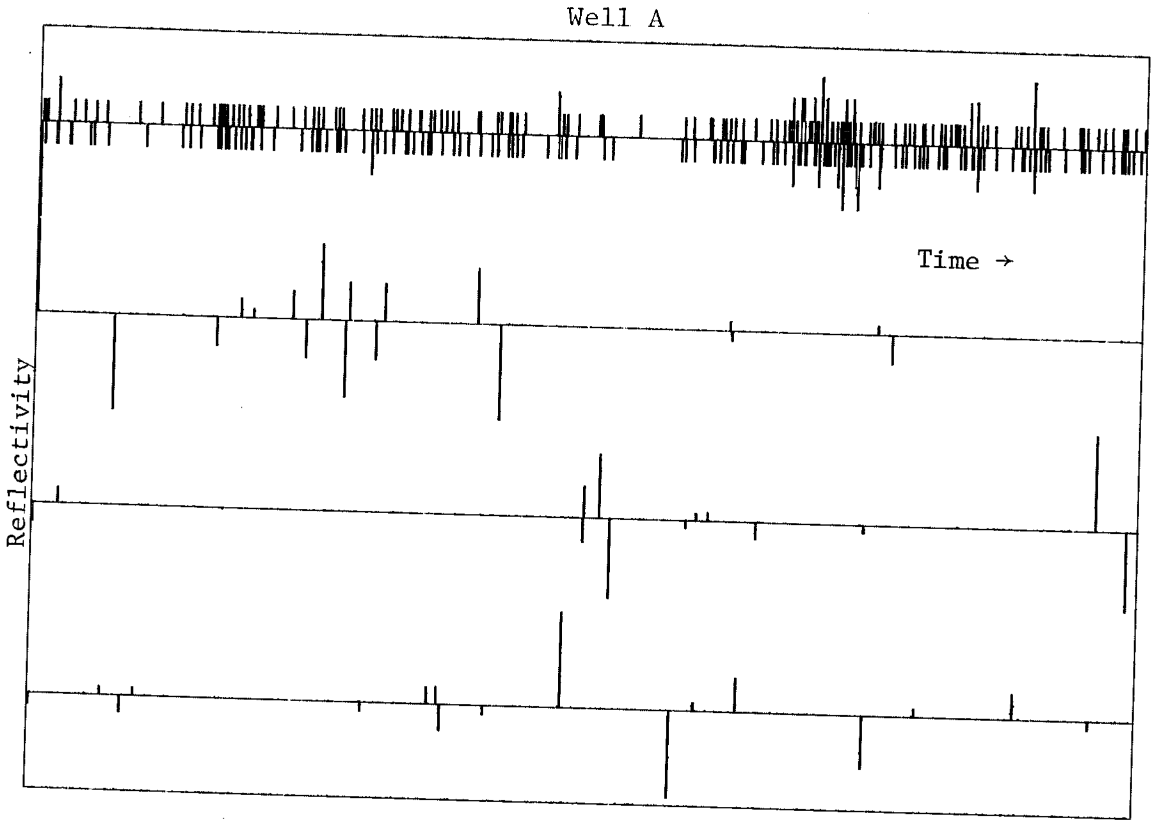
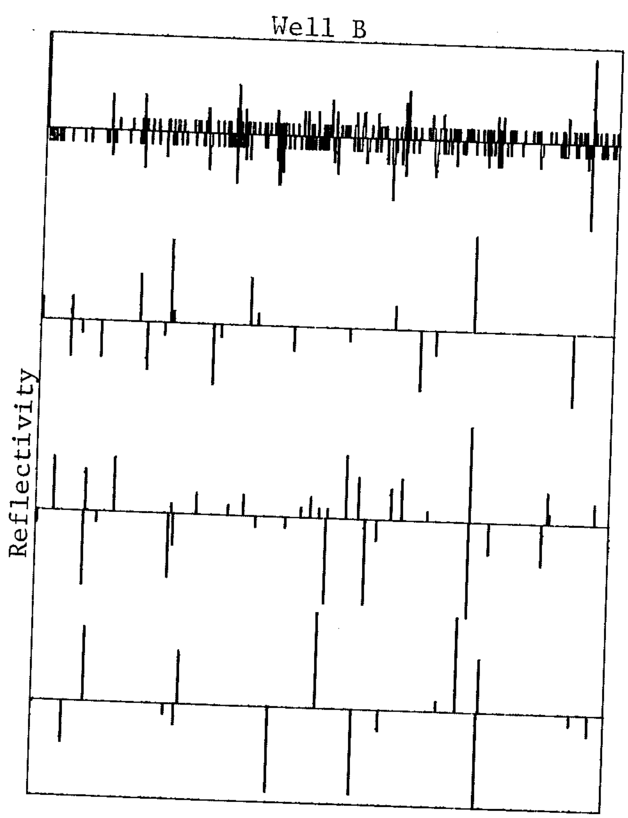
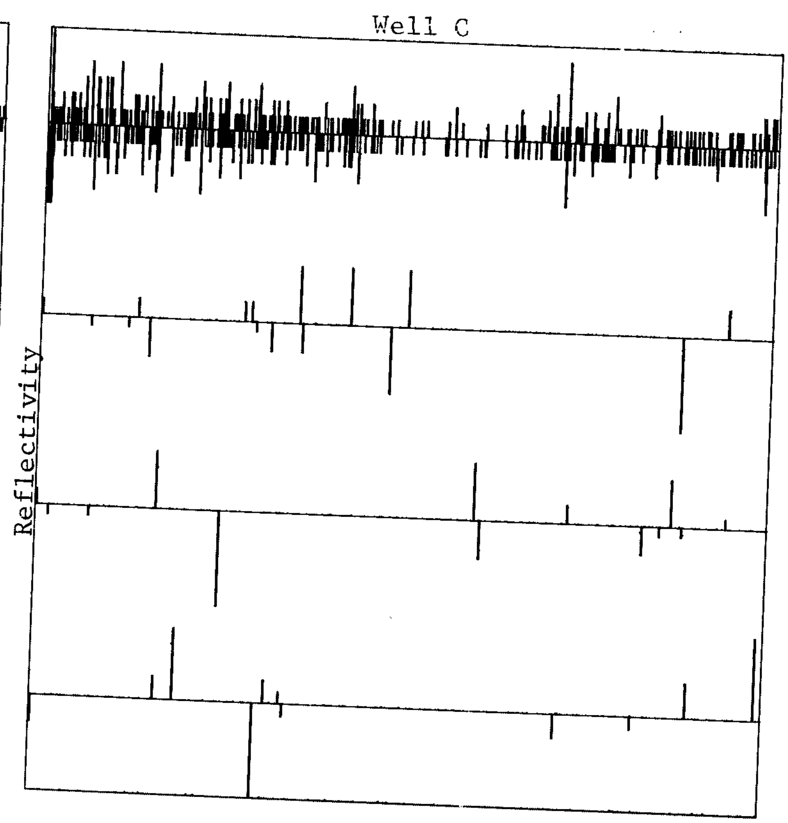


FIGURE 8.--All plots are the differentiated versions of Figure 5. As expected, the synthetics (bottom three plots) contain fewer reflection coefficients than the actual reflectivity sequence (top plot).





$$R_{cc}(0) = 1$$

$$R_{cc}(t) = -\left(\frac{1-\lambda}{2}\right)\lambda^{|t|-1}, \quad |t| > 0 \qquad (29)$$

where the zero-lag has been normalized to one. The values of λ corresponding to the three chains of Figure 8 were substituted into Equation (29) and the results plotted in Figure 9a. The actual A/C of the reflectivity sequences of Figure 1 is plotted in Figure 9b.

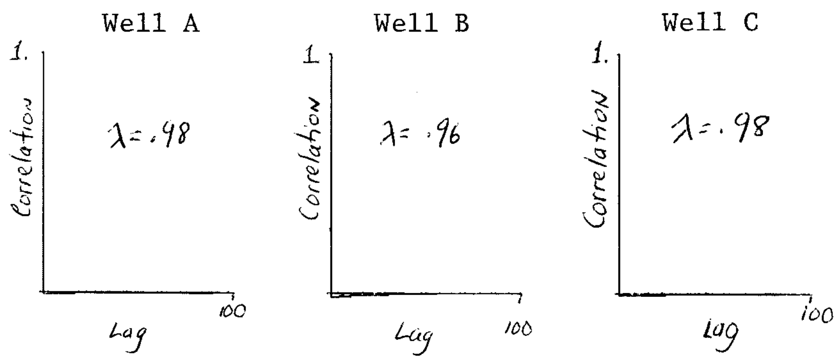


FIGURE 9a.--Autocorrelation function of the three logs of Figure 8. The negative values for lags $\geq$ 1 are hardly visible since $[(\lambda - 1)/2] \cong 0$. One might conclude that reflectivity is an independent process on the basis of these plots.

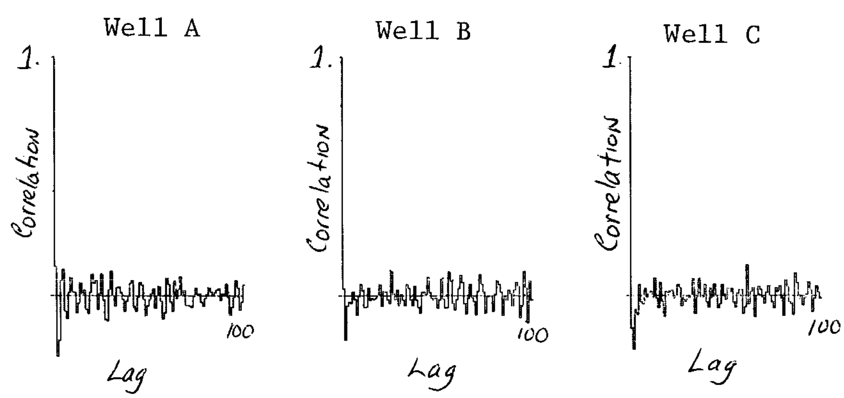


FIGURE 9b.--Actual autocorrelation function of the reflectivity sequences of Figure 1. Note the large negative pulse at lag 1.

The small negative values at lag one in Figure 9a indicate that reflectivity is very weakly correlated, so a valid probabilistic model could treat reflectivity as an independent process. The empirical A/C (Figure 9b), however, has a much larger negative value at lag one, indicating a dependent process. The reason for this discrepancy is two-fold. First, the actual reflectivity sequence has many plus/minus doublets that are absent on the synthetic logs. Second, the presence of white noise in the log(imped) process will give anti-correlated noise in the differential process. Note that the derivation leading up to Equation (29) assumes a purely differential process, and for this reason, Equation (4b) was adopted.

To show that $\{C_k\}$ is Bussgang, it is necessary to derive a relation for the crosscorrelation of C_k with a ZNL. In the appendix, the calculation is carried out. If the ZNL is restricted to being odd, for example,

$$y_t = \operatorname{sgn}(x_t)\ |x_t|^{\alpha}$$

then $\{C_k\}$ is indeed Bussgang. Some preliminary work has indicated that not only variable norm deconvolution, but any method that treats reflectivity as a first order process, will converge if reflectivity is a Bussgang process.

Conclusion

A simple second order model for impedance has been proposed. The model is similar, if not identical, to a Poisson arrival process. Reflectivity sequences synthesized using the model were shown to be Bussgang, a desirable attribute since variable norm deconvolution, along with other gradient techniques of deconvolution, converges to a Bussgang process.

The success of the model is correlated with the number of applications that can be developed from it. Currently, we are developing a maximum likelihood technique for deconvolving seismograms that is based on the hypothesis that reflectivity is the differential of a telegraph process.

REFERENCE

KEMENY, J.G. and J.L. SNELL (1960), *Finite Markov Chains* (New York: Van Nostrand Co., Inc.).

APPENDIX - *Crosscorrelation of reflectivity with ZNL*

Computing the crosscorrelation (C/C) with a ZNL is quite laborious because of the discontinuity in going from the zero-th to first lag. First, there is a restriction on the type of non-linearity allowed, as the following analysis illustrates. Denoting the ZNL by g, we calculate the zero lag of the C/C as follows:

$$
\begin{aligned}
R_{Cg(c)}(0) &= E[(X_k - X_{k-1})g(X_k - X_{k-1})] \\
&= \sum_{pq} (x_p - x_q)g(x_p - x_q)\ \Pr(Q_n{=}s_p, Q_{n-1}{=}s_q) \\
&= \sum_{pq} (x_p - x_q)g(x_p - x_q)\ K_{qp} \\
&= \sum_{pq} x_p g(x_p - x_q)\ K_{qp} - \sum_{pq} x_q g(x_p - x_q)\ K_{qp} \\
&= \sum_{pq} x_p g(x_p - x_q)\ K_{qp} - \sum_{pq} x_p g(x_q - x_p)\ K_{qp}\ , \\
&\qquad\qquad (K_{qp} = K_{pq}) \\
&= 0 \quad \text{if } g \text{ is even} \\
&= 2\sum_{pq} x_p g(x_p - x_q)\ K_{qp} \quad \text{if } g \text{ is odd} \qquad \text{(A1)}
\end{aligned}
$$

From the above, we conclude that if the ZNL is even, the reflection sequence cannot be Bussgang. Assuming the ZNL is odd, consider the crosscorrelation at lag 1:

$$
\begin{aligned}
R_{Cg(c)}(1) &= E[(X_{k+1} - X_k)g(X_k - X_{k-1})] \\
&= \sum_{pqr} (x_p - x_q)g(x_q - x_r)\ \Pr(Q_n{=}s_p, Q_{n-1}{=}s_q, \\
&\qquad\qquad Q_{n-2}{=}s_r)
\end{aligned}
$$

The final thing is to show that the C/C is symmetric. As before, the discontinuity creates problems, and the first lag must be computed separately:

$$R_{Cg(c)}(-1) = E[(x_k - x_{k-1})g(x_{k+1} - x_k)]$$

$$= \sum_{pqr} (x_q - x_r)g(x_p - x_q)K_{rq}P_{qp}$$

$$= \sum_{pqr} x_q g(x_p - x_q)K_{rq}P_{qp} - \sum_{pqr} x_r g(x_p - x_q)K_{rq}P_{qp}$$

$$= A - B \qquad \text{(A4)}$$

Each term in (A4) will be developed:

$$A = \sum_{pqr} x_q g(x_p - x_q)K_{rq}P_{qp}$$

$$= \sum_{pq} x_q g(x_p - x_q)\alpha_q P_{qp} \qquad \left(\sum_r K_{rq} = \alpha_q\right)$$

$$= \sum_{pq} x_q g(x_p - x_q)K_{qp} \qquad \left(\alpha_q P_{qp} = K_{qp}\right)$$

$$= -\sum_{pq} x_q g(x_q - x_p)K_{pq} \qquad \text{(A5)}$$

$$B = \sum_{pqr} x_r g(x_p - x_q)K_{rq}P_{qp}$$

$$= \sum_{pq} g(x_p - x_q)P_{qp} \sum_r x_r K_{rq}$$

But $K_{rq} = \lambda\alpha_r\delta_{rq} + (1 - \lambda)\alpha_r\alpha_q$

$$\therefore \quad B = \lambda \sum_{pq} g(x_p - x_q)P_{qp}x_q\alpha_q \qquad (\Sigma\, x_r\alpha_r = 0)$$

$$= \sum_{pqr} (x_p - x_q) g(x_q - x_r) K_{rq} P_{qp}$$

$$= \sum_{pqr} x_p g(x_q - x_r) K_{rq} P_{qp}$$

$$- \sum_{pqr} x_q g(x_q - x_r) K_{rq} P_{qp}$$

$$= \sum_{pqr} g(x_q - x_r) K_{rq} \sum_{p} x_p P_{qp}$$

$$- \sum_{qr} x_q g(x_q - x_r) K_{rq} \left(\sum_{p} P_{qp} = 1 \right)$$

But $$\sum_{p} x_p P_{qp} = \sum_{p} x_p [\lambda \delta_{qp} + (1 - \lambda)\alpha_p] = \lambda x_q$$

Hence, $$R_{Cg(c)}(1) = \lambda \sum_{qr} x_q g(x_q - x_r) K_{rq} - \sum_{qr} x_q g(x_q - x_r) K_{rq}$$

$$= \frac{(\lambda - 1)}{2} R_{Cg(c)}(0) \qquad \text{(A2)}$$

The ratio of the zero and first lags of the crosscorrelation match the ratio of the zero and first lags of the autocorrelation [see Equation (34)]. The remaining lags of the C/C can be determined from knowledge of the first lag if an eigenvalue expansion of $\hat{P}_T$ is used. It can be shown that $\hat{P}_T$ has M-1 non-zero eigenvalues equal to λ and M^2-M zero eigenvalues (the remaining eigenvalue is 1, of course). Using a development analagous to that used for P_T, we can show that the C/C is a pure exponential for lags ≥ 1. Hence, normalizing the zero-lag to one gives

$$R_{Cg(c)}(0) = 1$$

$$R_{Cg(c)}(k) = -\frac{(1 - \lambda)}{2} \lambda^{k-1} \qquad k \geq 1 \qquad \text{(A3)}$$

$$= \lambda \sum_{pq} x_q g(x_p - x_q) K_{qp}$$

$$= -\lambda \sum_{pq} x_q g(x_q - x_p) K_{pq} \qquad \text{(A6)}$$

Using Equations (A5) and (A6) in Equation (A4), and making note of (A1), we get

$$R_{Cg(c)}(+1) = R_{Cg(c)}(-1)$$

The same holds for the remaining negative lags, so Equation (A4) is equally valid for negative lags:

$$R_{Cg(c)}(0) = 1 \qquad \text{(A7)}$$

$$R_{Cg(c)}(k) = -\left(\frac{(1 - \lambda)}{2}\right)\lambda^{|k| - 1} \qquad |k| \geq 1$$

From Equation (A7), we conclude that the reflectivity sequence is Bussgang if the ZNL is restricted to being odd.