# IMPEDANCE, REFLECTANCE, AND TRANSFERENCE FUNCTIONS

*Jon F. Claerbout*

In describing stable physical processes rarely is much attention given to the stability of the modeling equations. The common feeling is that since the physical process is stable, so must be any correct and reasonably accurate modeling equations. So often is this true that concern with stability, like concern with existence proofs, is frequently regarded as highly academic.

Quite the opposite circumstance applies in geophysical data processing where we are involved with the *inverse* of physical modeling. Modeling, the way nature does it, is extrapolating forward in time. Extracting information about the earth's interior from surface measurements is inverse modeling. Such extraction is really extrapolating information in depth. Nature does boundary value problems in depth, not initial value problems, so we can consider ourselves lucky when we are able to extrapolate downward. When a depth extrapolation is unstable then we simply cannot determine the information we seek.

The instability may arise from either of the following two causes:

1) Mathematical equations may have a unique solution, but there may be a ridiculous sensitivity to data accuracy.

2) Approximations which are reasonable and valid in the frequency range of interest might violate causality outside that range.

In any practical situation there is obviously a great need to know which of the above two situations is applicable. Luckily in seismic imaging we are usually in case (2). To regain stability the main requirement is that we learn some stability analysis and use it. Of all the virtues a computational algorithm can have - stability, accuracy, clarity, generality, speed, modularity, etc. - the most important seems to be stability.

Early chapters of my book, *Fundamentals of Geophysical Data Processing* (FGDP), develop the basics of stability in time domain calculations. These are causality, Z-transform analysis, and minimum phase. Additional techniques stem from the mathematical properties of impedance, reflectance, and transference functions, which will be more fully developed here. The present development will be self-contained for frequency domain calculations, but will rely heavily on FGDP for time domain causality ideas.

*Review of impedance filters*

Use Z-transform notation to define a filter $R(Z)$, its input $X(Z)$ and output $Y(Z)$. Then

$$Y(Z) = R(Z)\, X(Z)$$

The filter $R(Z)$ is said to be causal if the series representation of $R(Z)$ has no negative powers of $Z$. In other words, $y_t$ is determined from present and past values of $x_t$. The filter $R(Z)$ is said to be minimum phase if $1/R(Z)$ has no negative powers of $Z$. This means that $x_t$ can be determined from present and past values of $y_t$ by straightforward polynomial division in

$$X(Z) = \frac{Y(Z)}{R(Z)}$$

Given that $R(Z)$ is already causal and minimum phase, it can additionally be an impedance function if positive energy or work is represented by

$$0 \leq \text{work} = \sum_t \text{force x velocity} = \sum_t \text{voltage x current}$$

$$= \frac{1}{2} \sum_t (\bar{x}_t y_t + \bar{y}_t x_t)$$

$$= \text{coef of } Z^0 \text{ of } \bar{X}\left(\frac{1}{Z}\right) Y(Z) + \bar{Y}\left(\frac{1}{Z}\right) X(Z)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \text{Re } (\bar{X}\, Y)\ d\omega$$

$$\cong \int \text{Re } (\bar{X}\, R\, X)\ d\omega = \int \bar{X}\, X\, \text{Re } (R)\ d\omega$$

It therefore follows that $\mathrm{Re}[R(\omega)] \geq 0$ for all real $\omega$ so that impedance functions are also called *causal positive real* functions. It is shown in FGDP, page 31, that every impedance function is minimum phase, though not the other way around. Adding an impedance function to its Fourier conjugate we get a purely positive function (no imaginary part) like a power spectrum, say

$$R(Z) + \bar{R}(Z^{-1}) \geq 0 \quad \text{for real } \omega$$

which is the basis for the statement that the impedance time function is one side of an autocorrelation function.

*Rules for compounding impedance functions*

One of the difficulties in applied geophysics is this: Results may have physical utility only in a certain limited range of frequencies and reasonable approximations may be made in that range. But if a spectrum or impedance becomes negative outside the applicable range, say near the Nyquist folding frequency, then the calculation (by Murphy's Law) will be unstable and hence useless. Thus Muir's rules* for compounding impedance functions deserve careful attention. Let $R'$ denote a new impedance function generated from old known impedance functions $R$, $R_1$, or $R_2$.
Muir's rules are:

| | | |
|---|---|---|
| i1: | Multiplication by positive scalar $a$ | $R' = aR$ |
| i2: | Inversion | $R' = \frac{1}{R}$ |
| i3: | Addition | $R' = R_1 + R_2$ |

Rules i1 and i3 self-evidently preserve causality and positivity of the real part of the Fourier transform. Causality for Rule i2 was already mentioned as having been developed in FGDP. The positivity of the real part for Rule i2 follows by considering $R$ at each frequency $\omega$ to be the complex number $a + ib$ where $a \geq 0$. Then $1/(a+ib) = (a-ib)/(a^2+b^2)$ also has a positive real part.

---

*personal communication

*Isomorphism with reflectance function C*

Given any impedance function R, then the following equation defines an associated reflectance function C

$$C = \frac{1 - R}{1 + R} \tag{1}$$

We will see that the reflectance function is also causal and that it is less than unity in magnitude, say

$$|C|^2 = \bar{C}\left(\frac{1}{Z}\right) C(Z) < 1$$

Causality follows because the numerator $1 - R$ is causal and the denominator $1 + R$ is positive real (since R is positive real), hence minimum phase. That the magnitude of C is less than unity follows from noting that the magnitude of the numerator is less than the magnitude of R and the magnitude of the denominator is greater. Unlike the impedance function $R(Z)$, the reflectance function $C(Z)$ is not necessarily minimum phase. An example is $R = 1 + Z/2$, $C = -.5Z/(2 + Z/2)$. Equation (1) may be solved for R:

$$R = \frac{1 - C}{1 + C} \tag{2}$$

We may now inquire if $C$ = causal and $|C| < 1$ alone will ensure that R is an impedance function.

Multiply (2) on top and bottom by $1 + \bar{C}$

$$R = \frac{(1 - C)(1 + \bar{C})}{|1 + C|^2}$$

$$= \frac{(1 - \bar{C}C) + (-C + \bar{C})}{\text{positive}}$$

$$= (\text{real}) + (\text{imag})$$

Clearly the positive reality is ensured by $|C| < 1$. The causality follows since the numerator of (2) is assumed causal and the denominator is causal with positive real part (since $1 > |C|$). In summary, then, Equation (2) will reliably produce an impedance function from any apparent reflectivity function.

An obvious feature of reflectivity functions is that the product of two of them (in the frequency domain) will produce another. Let two reflectivity functions be denoted by B and C. Then a new reflectivity function is $C' = BC$. The corresponding impedance function from (2) is

$$R' = \frac{1 - BC}{1 + BC} = \frac{1 - B\,\dfrac{1 - R}{1 + R}}{1 + B\,\dfrac{1 - R}{1 + R}}$$

$$= \frac{(1 + R) - B(1 - R)}{(1 + R) + B(1 - R)}$$

$$= \frac{(1 - B) + R(1 + B)}{(1 + B) + R(1 - B)}$$

$$= \frac{\dfrac{1 - B}{1 + B} + R}{1 + \dfrac{1 - B}{1 + B}R}$$

So denoting $R_1 = (1-B)/(1+B)$ and $R_2 = R$ we have another rule for impedance functions

$$\text{i4:} \quad R' = \frac{R_1 + R_2}{1 + R_1 R_2}$$

When $R_2$ goes to infinity Rule i4 reduces to Rule i2.

We have just taken the reflectivity rule $C' = C_1 C_2$ into a rule in the domain of the impedances. This is known as isomorphism. Let us now take the first three impedance rules into the domain of the reflectivities. In sequence they become:

$$\text{c1:} \quad C' = \frac{c + C}{1 + cC} \quad \text{where } c \text{ is a number such that } |c| < 1$$

$$\text{c2:} \quad C' = -C$$

$$\text{c3:} \quad C' = \frac{-1 + C_1 + C_2 + 3C_1C_2}{3 + C_1 + C_2 - C_1C_2}$$

$$\text{c4:} \quad C' = C_1C_2$$

Rule c1 follows by an algebraic process almost identical to that which derived i4. Rule c2 states the obvious fact that the negative of a reflectance is a reflectance. Rule c3 does not seem to be very useful and Rule c4 states the obvious. Rules 1 and 3 are particularly self-evident in the impedance domain, and Rules 2 and 4 are particularly self-evident in the reflectance domain. In summary:

$$\begin{aligned} &1: \quad R' = aR \qquad\qquad a > 0 \\ &2: \quad C' = -C \\ &3: \quad R' = R_1 + R_2 \\ &4: \quad C' = C_1C_2 \end{aligned}$$

After this excess of formalism the reader may welcome some examples, and we shall have them.

*Example: Causal integration and differentiation*

Define a reflectivity function which is almost $-1$ with a unit delay, say

$$C = -\rho Z \quad \text{where} \quad \rho = 1 - \varepsilon, \quad 1 \gg \varepsilon > 0$$

Now look at half the associated impedance function

$$\begin{aligned} I &= \frac{1}{2}\frac{1 + \rho Z}{1 - \rho Z} \\ &= \frac{1}{2}(1 + \rho Z)\,[1 + \rho Z + (\rho Z)^2 + (\rho Z)^3 + \cdots] \\ &= \frac{1}{2} + \rho Z + (\rho Z)^2 + (\rho Z)^3 + \cdots \end{aligned}$$

Extracting the coefficients of $Z^t$ we find the time response of the filter. As $\varepsilon$ tends to zero this is the discrete step function $(\cdots 0,0,0,\frac{1}{2},1,1,1,\cdots)$ Convolution with it approximates integration from minus infinity to time $t$. For small non-zero $\varepsilon$ this operation is called *leaky integration*. It is a simple exercise to show that

$$R \approx \frac{1}{-i\omega\Delta t + \varepsilon} \quad \text{where} \quad Z = e^{i\omega\Delta t} \quad \text{and} \quad 0 < \varepsilon << \omega << \pi$$

As multiplication by $-i\omega$ in the frequency domain is associated with differentiation $d/dt$ in the time domain, so is division by $-i\omega$ associated with integration. Now the surprising thing is that people usually associate the asymmetric operator $(1,-1)$ with differentiation, but the inverse to the causal integration operator, namely

$$R^{-1} = 2\,\frac{1-\rho Z}{1+\rho Z}$$

$$= 2 - 4\rho Z + 4(\rho Z)^2 - 4(\rho Z)^3 + \cdots$$

is completely causal, not at all asymmetric, and also represents differentiation. That is to say, when the time sampling $\Delta t$ tends to zero or, what is the same thing, when the frequency is sufficiently far from the folding frequency (where there is a pole), the operator $R^{-1}$ represents differentiation. In fact, in linear systems analysis this is often the preferred discrete representation of differentiation. By analogy with the words *definite integral* this operator may be called the *definite derivative*. As we will see, the construction of higher order stable differential equations must now be subject to the rules which we developed for combining impedance functions.

Occasionally it will be necessary to have a *negative* real part for the differentiation operator. This can be achieved by taking $\varepsilon$ negative which means taking $\rho > 1$ and doing the infinite series expansion in powers of $Z^{-1}$, that is, anticausally instead of causally with positive powers of $Z$. In either case the imaginary part will be $-i\omega$ but the real part has opposite sign.

*Example: Waves crossing an interface*

At any level within the earth we can think of the downgoing wave D which we have initiated as an input signal, and the returned upcoming wave U as the output of an earth filter. The ratio U/D is the earth filter. Clearly the earth filter is causal and also some of the energy must escape into the earth's interior so we must have $|U| \leq |D|$ at all frequencies. Thus the earth's response should be a reflectance C = U/D. The reflectance C vanishes beneath the deepest reflector but it is non-zero above, so it is obviously a function of depth. FGDP, page 149, Equation (8-2-2), shows how to extrapolate vertically propagating scalar plane waves across a flat horizontal interface:

$$\begin{bmatrix} U \\ D \end{bmatrix}_{\text{below}} = \frac{1}{(1 + c)} \begin{bmatrix} 1 & c \\ c & 1 \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_{\text{above}}$$

Here c is a reflection coefficient so that $|c| < 1$. By manipulating this equation we can find a procedure to get U/D below the interface from U/D above (or vice versa). The ratio of the first equation to the second is

$$C_{\text{below}} = \left(\frac{U}{D}\right)_{\text{below}} = \frac{U + cD}{cU + D}$$

$$= \frac{(U/D)_{\text{above}} + c}{c(U/D)_{\text{above}} + 1}$$

$$= \frac{c + C_{\text{above}}}{1 + c\, C_{\text{above}}}$$

This can be seen to agree with Rule c1, so we can rest assured that $C_{\text{below}}$ has the correct properties of a reflectance. Rule c1 can also be thought of as Rule i1, multiplying the impedance by the positive constant $(1 - c)/(1 + c)$.

*Example: Waves crossing a thin layer*

The very next equation in FGDP shows how waves cross a layer of unit traveltime thickness. It is

$$\begin{bmatrix} U \\ D \end{bmatrix}_{bottom} = \begin{bmatrix} Z^{-1/2} & 0 \\ 0 & Z^{1/2} \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_{top}$$

Forming the same ratio $C = U/D$ we get

$$C_{bottom} = \frac{C_{top}}{Z}$$

Now $Z^{-1}$ is a non-causal operator, so there appears to be some danger of developing non-causal reflectances when downward continuing, but there seems to be no danger when going up. How can the danger be avoided? Clearly, if we are inside a layer but near its top, we should get no reflection until after one time unit. It follows that if $C_{top}$ has been properly computed then its coefficient of $Z^0$ should vanish so that division by $Z$ does not cause $C_{bottom}$ to be non-causal. In a downward continuation, alternating between layers and interfaces, the reflection coefficients can be chosen so that $C$ remains causal.

This wave extrapolation involves Rule 4, which is almost trivial in the reflectance domain but which makes little intuitive sense in the impedance domain.

*Example: Wide angle wave extrapolation*

Let $s = -i\omega$ denote the causal positive real discrete representation of the differentiation operator, say

$$s = -i\omega\Delta t = 2\,\frac{1 - \rho Z}{1 + \rho Z}$$

Consider the following recursion starting from $S_0 = s$

$$S_{n+1} = s + \frac{X^2}{s + S_n}$$

F. Muir introduced this recursion as a means of developing wide angle square root approximations for migration and developed his three rules i1,2,3 to show that every $S_n$ is an impedance function. To see why this works, first note that the denominator $s + S_n$ is, for $n=0$, the sum of two impedance functions. Then its inverse is an impedance function, and multiplication by the real positive constant $X^2$ and addition of another $s$ all preserve the properties of impedance functions. As $n$ becomes large this recursion either converges or it does not. Supposing that it does, we can see to what it converges by setting $S_{n+1} = S_n = S_\infty = S$. We have

$$S = s + \frac{X^2}{s + S}$$

$$S(s + S) = s(s + S) + X^2$$

$$S^2 = s^2 + X^2$$

$$S = (s^2 + X^2)^{1/2}$$

$$S = s \left(1 + \frac{X^2}{s^2}\right)^{1/2}$$

In wave extrapolation problems $X^2$ is $v^2 k_x^{\ 2}$ where $v$ is the wave velocity and $k_x$ is horizontal spatial frequency, namely, the Fourier dual to the horizontal x-axis. The quantities $S_n$ are $ik_z$ where $k_z$ is the Fourier dual to the depth z-axis. The cases $n = 0$, 1, and 2 are commonly referred to as the 5-degree, 15-degree, and 45-degree equations respectively. The desirability of $S$ being positive real is related to the fact that it is acceptable for $e^{ik_z z}$ to decay with $z$ (when $k_z$ is complex) but growth is almost certainly not acceptable. But this leads us to our next topic - exponentials of impedance functions.

*Transmittance Functions*

We will define a transmittance function $E$ as the exponential of the negative of an impedance function

$$E(Z) = e^{-R(Z)} \tag{3}$$

This function occurs in many circumstances, but it occurs particularly as the function that can Extrapolate waves across a region of homogeneous space.

First, the magnitude of E, like C, is always less than unity by the positivity of the real part of R. Second, E will be causal since R is causal, and R may be substituted into the power series for exponential (which always converges) and no powers of $Z^{-1}$ arise. Third, the transmittance function E, unlike C, will always be minimum phase since for any $|Z| \leq 1$, R is not infinity, and exp(-R) cannot be zero anywhere inside the unit as it would have to be for E to be non-minimum phase. The inverse to (3) is easily written

$$R(Z) = -\log_e E(Z) \tag{4}$$

Now we can ask whether Equation (4), when used upon an arbitrary E(Z) satisfying only the condition of causality, minimum phase, and $|E| \leq 1$ will always produce a valid impedance function R(Z). If E is expressed in polar form $re^{i\phi}$ then its logarithm is $\log |r| + i\phi$, so since $|r| < 1$ we see that a positive real function is indeed produced by (4). Proof of causality is found elsewhere in this report ("Powers of Causal Operators," by Claerbout and Kjartansson). It shows that (4) recovers a valid impedance function from any minimum phase E with $|E| < 1$. Next we can develop the four rules for compounding transmittances that correspond to the previously discussed rules. In sequence, these are

$$\text{e1:} \quad E' = e^{-Rz} = E^z \qquad z > 0$$

$$\text{e2:} \quad E' = \text{unlikely function}$$

$$\text{e3:} \quad E' = e^{-(R_1 + R_2)} = E_1 E_2$$

$$\text{e4:} \quad E' = \text{unlikely combination}$$

*Wave extrapolation operators*

Causal wave extrapolation problems in depth z for the wavefield P generally seem to take the form

$$\frac{dP}{dz} = -RP \tag{5}$$

where R is an impedance function. There is a lot of physics buried in the impedance function R but no matter what the problem, if the material properties are independent of depth z, then R will be independent of z and the solution will be

$$P(z) = e^{-R(z - z_0)} P(z_0) = E\,P(z_0)$$

where E is evidently a transference function. For stability, the minus sign must be present and R must have a positive real part. But we have said nothing about the imaginary part of R which is the real part of $k_z$. It could be either sign corresponding to either up- or downgoing waves. We already know that the transference function E has magnitude less than unity $|E| < 1$, so that stability is ensured.

Now what happens if the material properties are depth-dependent so that (5) does not have an analytic solution? Then we revert to numerical techniques, the most common of which is known as the Crank-Nicolson Method. Letting P(z) be represented at discrete intervals, say $P(j\Delta z)$, and abbreviating this by $P_j$, Equation (5) is represented by

$$P_{j+1} - P_j = -\frac{R\Delta z}{2}(P_{j+1} + P_j)$$

Absorb the positive scale factor $\Delta z/2$ into R, solve for $P_{j+1}$, and recognize that the extrapolation operator is by Equation (1) a reflectance function C,

$$P_{j+1} = \frac{1 - R}{1 + R} P_j = C\,P_j$$

Since both reflectances and transferences do not exceed unity, we have not lost the stability which we seek to achieve. The reflectance function need not be minimum phase but that does not seem to make any difference here.

*Physics of wave numbers and impedances*

It is curious that we have been exponentiating the impedance function R and the vertical spatial frequency $ik_z$ but we know that these two are not physically the same thing. Hopefully, when applying the foregoing mathematical techniques to physical problems, we will find that the two are related by the mathematical rules for combining impedance functions. Let us check into the definitions for acoustics which are found in FGDP, pages 171 and 172. Specializing the equations given there to the vertically incident case we have

$$ik_z = \frac{i\omega}{v}$$

$$R_{acoustic} = \rho v$$

where $\rho$ is material density and $v$ is velocity. Since we can choose $i\omega$ to have a positive real part it seems that both are mathematically impedance functions, although only one is physically called the acoustic impedance.

With more general physics or numerical analysis situations, discussions will center on modifications to some $R$ in Equation (5). Often this will be by addition of more terms, in which case we may speak of superposition, not of solutions, but of impedances or of physical effects in the equations. Other times it may be by means of the full set of rules for compositing impedances. Inclusion of anything by any means other than the compositing rules is asking for a computer program with instabilities. Such a program could easily be stable for some numerical parameters, but unpredictably unstable for others.

*Redundancy of rule 4*

After reading the first version of this manuscript, F. Muir commented that rule i4 is redundant since it is derivable from i2 and i3 as follows:

$$\frac{R_1 + R_2}{1 + R_1 R_2} = \frac{\frac{R_1}{R_2} + 1}{R_1 + \frac{1}{R_2}} = \frac{1}{R_2 + \frac{1}{R_1}} + \frac{1}{R_1 + \frac{1}{R_2}}$$

*Exercises*

1. Take $\varepsilon < 0$ and expand the integration operator for negative powers of $Z$. Explain the sign difference.

2. Rule c4 will obviously still be valid under the slightly less restrictive condition $|C_1| \leq 1$, $|C_2| < 1$ (we have allowed $|C_1|$ to equal unity).

   a. How does this affect Rule i4?

   b. How can Rule 3 be made less restrictive?

3. Consider the fourth order Taylor Expansion for square root in an extrapolation equation

$$\frac{dP}{dz} = i\omega\left[1 - \frac{1}{2}\left(\frac{vk}{\omega}\right)^2 - \frac{1}{8}\left(\frac{vk}{\omega}\right)^4\right] P$$

   a. Will this equation be stable for the complex frequency $-i\omega = -i\omega_0 + \varepsilon$? Why?

   b. Consider causal and anticausal time domain calculations with the equation. Which, if any, is stable?

4. Consider material velocity which may depend on frequency $\omega$ and on horizontal x-coordinate as well. Suppose that luckily the velocity can be expressed in factored form $v(x,\omega) = v_1(x)\ v_2(\omega)$. Obtain a stable 45-degree wave extrapolation equation. Hints: Try

$$s = -\frac{i\omega}{v_2}$$

$$X^2 = \text{positive eigenvalue of } (v_1\partial_x)(v_1\partial_x)^T$$

5. Is the Levinson Recursion in FGDP related to the rules in this paper? If so, how?