NUMERICAL VISCOSITY CONSIDERATIONS
FOR THE MONOCHROMATIC 45-DEGREE EQUATION

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Abstract

An analysis of a class of positive real, viscous, 45-degree extrapolation operators shows that the customary procedure of transforming the frequency ω to $\omega + i\varepsilon$ does not give an operator with very desirable attenuation properties. Much better dip filtering characteristics can be realized in the evanescent part of F-K space by damping only the higher order terms in the 45^o expansion for the exact dispersion relation. As a bonus, it turns out that an operator constructed in this manner has less phase distortion everywhere in the passband for typical values of the viscosity parameters.

From a practical standpoint this means that 45^o migration is best carried out by incorporating viscous terms directly in the migration operator. This no-cost option gives better results than premultiplying the input dataset by exponential gain before doing a pure all pass migration.

Introduction

When frequency-wavenumber $(\omega - k_x)$ spectral analysis is carried out on a seismic section, some energy appears in the evanescent region, $|vk_x| > |\omega|$. Although an earth of velocity v does not propagate this energy, there will invariably be some noise power in this zone. This could be due to recording noise or, quite commonly, multiple reflections. Whatever the origin of this type of energy, it is undesirable to downward continue it when migrating. To get rid of it we can either apply a "once only" $\omega - k_x$ dip filter followed by all pass migration, or we can simulate what the earth itself does and use a viscous continuation operator. Since the signal-to-noise ratio decreases with increasing reflection event times, the second choice is preferable.

Similar difficulties arise in diffraction modeling if the source or reflector structure has any energy in the evanescent region of $\omega - k_x$ space. Figure 1 is a good example of this type of problem. This synthetic models a point diffractor using a monochromatic 45^o program without viscosity. Since a single point in $t - x$ space has a uniform power distribution in $\omega - k_x$ space, a pure all pass continuation filter is clearly inappropriate.

The 45-degree problem

Implementation of viscosity in the $\omega - x$ domain is not as straightforward for the 45^o equation as it is for the 15^o equation. In the latter case, the transformation $-i\omega \rightarrow -i\omega + \varepsilon$ (FGDP, p. 225) yields a propagation number whose imaginary (or attenuating) part is given by

$$\mathrm{Im}(k_x) = \frac{v\varepsilon k_x^2}{2(\omega^2 + \varepsilon^2)} \tag{1}$$

For $\omega > \varepsilon$ (ε assumed small), attenuation is basically a Gaussian function of the dip, vk_x/ω. For $\omega < \varepsilon$, attenuation still goes as $\exp(-\alpha k_x^2)$. If we use this same transformation approach for the (retarded) 45^o equation we obtain

$$\partial_z - \frac{\dfrac{\frac{v}{2} k_x^2}{(-i\omega + \varepsilon)}}{1 + \dfrac{\frac{v^2}{4} k_x^2}{(-i\omega + \varepsilon)^2}} = 0 \tag{2}$$

The shifting theorem of Laplace Transforms tells us that if $F(s) = F(-i\omega)$ is the Laplace transform of $f(t)$, then $\exp(-at)f(t)$ has the transform $F(s + a)$. This means that this type of viscosity can be implemented by premultiplying the data by $\exp(-\varepsilon t)$, carrying out a completely all pass migration, and postmultiplying by $\exp(\varepsilon t)$.

It can be seen from (2) that this approach does not work well for the region of $\omega - k_x$ space for which $(vk_x/\omega)^2 >> 1$. In this area, attenuation is proportional only to $\exp(-\varepsilon)$, but is essentially independent of k_x. This means that our only recourse to suppressing super high-dip noise is to choose a fairly large value of ε. This has the potentially disastrous side effect of attenuating moderately dipping energy lying in the signal passband. We can do better.

Generalized 45-degree viscosity

A more general viscous 45° equation is obtained by rewriting (2) in the form:

$$\partial_z - \frac{\frac{v}{2} k_x^2}{(-i\omega + \varepsilon_1) + \frac{\frac{v^2 k_x^2}{4}}{(-i\omega + \varepsilon_2)}} = 0 \tag{3}$$

Equations (2) and (3) are not identical but are equivalent for the case $\varepsilon_1 = \varepsilon_2$. Equation (3) is unconditionally stable for $\varepsilon_1, \varepsilon_2 > 0$. This is easily shown using Muir's rules for combining positive definite operators (see, for example, Claerbout's "Impedance Functions," this report). The most useful feature of (3), however, is that it gives us an extra viscosity parameter to control the shape of the attenuation surface in $\omega - k_x$ space.

Before proceeding further it will be useful to develop Equation (3) in some appropriate dimensionless coordinates. One useful normalizing parameter is ω_ℓ, the lowest frequency of interest on the data.

Defining $m_\ell = \omega_\ell / v$, we can rewrite (3) as

$$\frac{k_z}{m_\ell} = \frac{\frac{1}{2}\left(\frac{k_x}{m_\ell}\right)^2}{\left(\frac{-im + \varepsilon_1}{m_\ell}\right) + \frac{\frac{1}{4}\left(\frac{k_x}{m_\ell}\right)^2}{\frac{(-im + \varepsilon_2)}{m_\ell}}} \tag{4}$$

The maximum value of dimensionless viscosity that one would want to use if the ε's are equal is $\varepsilon/m_\ell \simeq 0.1$. Such a choice would mean that the exponential premultiplication decreases the amplitude by a factor of $1/\varepsilon$ for every 10 wavelengths of the lowest frequency of interest on the data. (If ε/m_ℓ was significantly greater than this, ω_ℓ would not be the lowest frequency of interest.)

Figure 2 is a plot of ln(-real k_z) = ln(attenuation) vs. (k_x/m_ℓ) and (m/m_ℓ) for $\varepsilon_1/m_\ell = \varepsilon_2/m_\ell = 0.1$. The maximum value for $|k_x/m_\ell|$ and $|m/m_\ell|$ was chosen to be 10 since most seismic data is bandpass-filtered at recording time with a useful bandwidth on the order of one decade. The main feature of this plot is that this choice of viscosity tends to concentrate all the attenuation along a ridge given approximately by $|m/k_x| = 0.5$.

Choice of ε's

If we make the choice $\varepsilon_1/m_\ell = 0$, $\varepsilon_2/m_\ell = 0.5$, then the attenuation is distributed more evenly (see Figure 3). Maximum attenuation still occurs for $|m/k_x| = 0.5$ but is not as strong there as it was for the case $\varepsilon_1 = \varepsilon_2 = 0.1\ m_\ell$. For $|m/k_x| < 0.5$, however, attenuation is much greater for the (0,.5) choice of ε's. This comparison is made more directly in Figure 4, which is a plot of ln[real $k_z(1)$/real $k_z(2)$].

From this point on we will refer to the propagation number associated with the "$(0,\varepsilon_2)$" type of viscosity as $k_z(2)$. To designate a propagation number of the "$(\varepsilon,\varepsilon)$" type of viscosity, $k_z(1)$ will be used.

A comparison of phase error for $k_z(2)$ and $k_z(1)$ in the passband $(|k_x| < |m|)$ shows that $k_z(2)$ has less phase error everywhere in the passband than $k_z(1)$ (see Figure 5). Thus $k_z(2)$ gives both improved rejection in the evanescent band and less phase error in the passband. The only thing we have to sacrifice to gain this improvement is some attenuation about the narrow ridge $|m/k_x| = 1/2$, but this is where attenuation is maximal anyway.

The same general conclusions still hold if we discretize the x and t coordinates (see Figures 6 & 7). Here again if $\varepsilon_1 = 0$, we can choose ε_2 to be fairly large and obtain a better distribution of attenuation.

Figures 8 and 9 demonstrate the superiority of $k_z(2)$ over $k_z(1)$ for removing the artifacts of Figure 1. The operator $k_z(2)$ removes the offending energy completely but retains as much signal as the $k_z(1)$ operator.

Summary

We have demonstrated that if viscosity is placed only in the higher order terms of the 45^o equation, we obtain a wave extrapolation filter with attenuation and phase characteristics that are an improvement over schemes which damp the 15^o and 45^o terms equally. This improvement is realized only at the expense of getting a little less attenuation along the ridge of maximum attenuation in $\omega - k_x$ space. This means that 45^o migration is best done with viscosity built directly into the migration operators rather than by premultiplying the data by exponential gain.

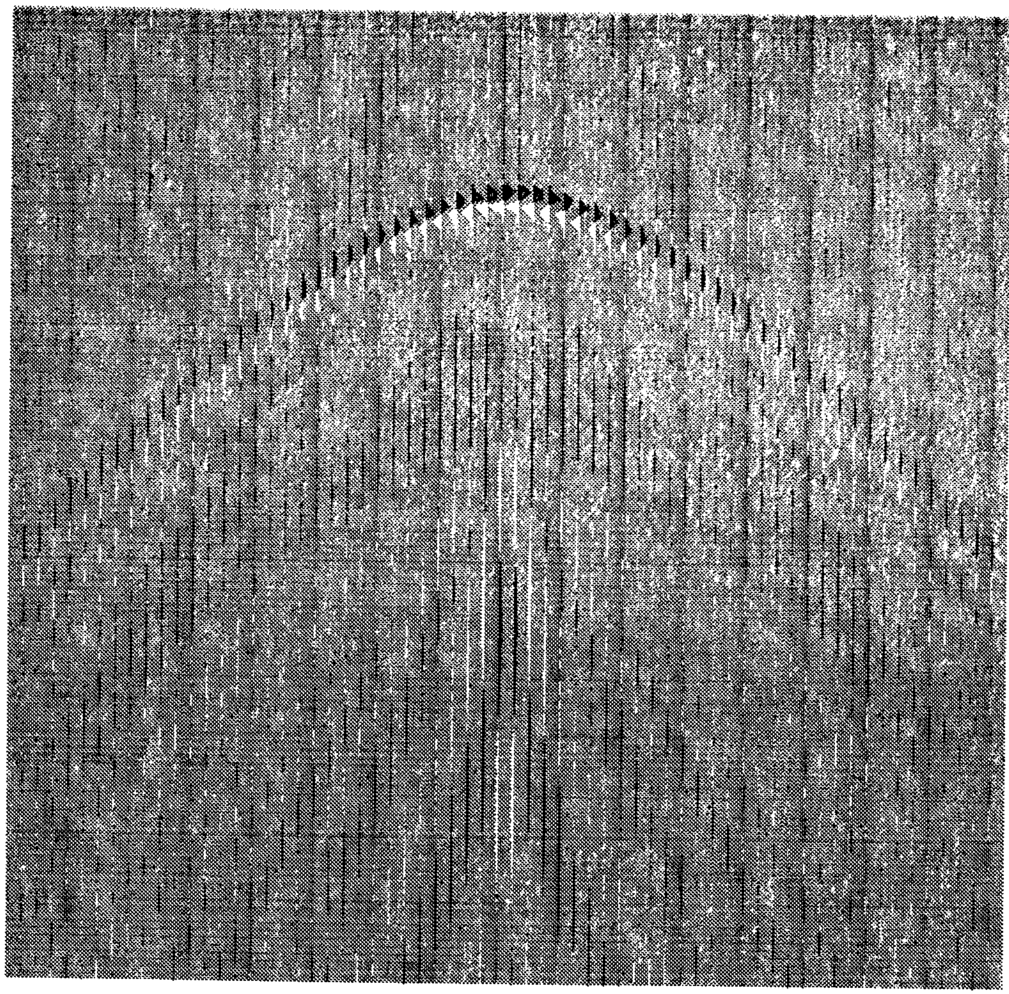

FIGURE 1.--Impulse response of monochromatic 45° program without viscosity. Prominent noise streaks are due to evanescent energy at low ω and high k_x, which the all-pass filter propagates.

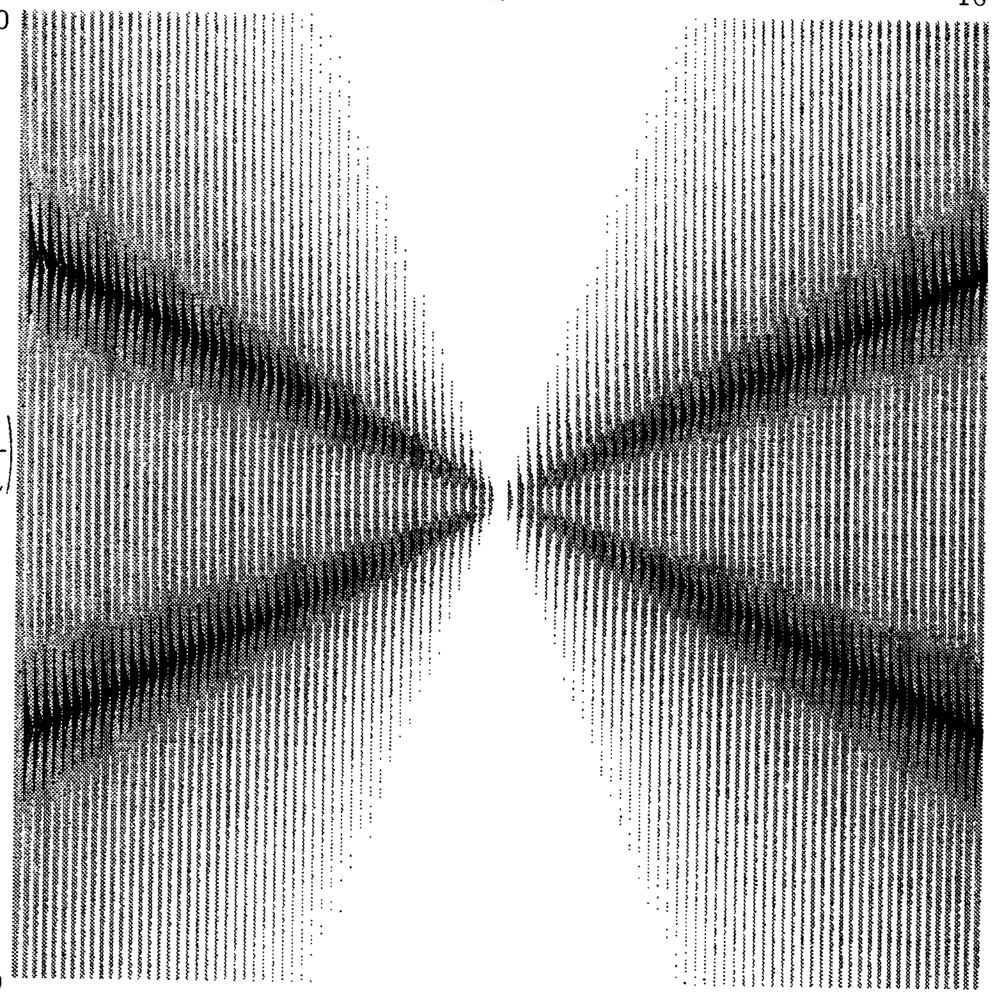

FIGURE 2.--Plot of ln(attenuation) as function of k_x/m_ℓ and m/m_ℓ for $\varepsilon_1 = \varepsilon_2 = 0.1$ ml. The plot is scaled to a maximum of 5.5. Black shading about $|m/k_x| = 1/2$ delineates a zone of strong attenuation. The trend towards white denotes zones of progressively weaker attenuation.

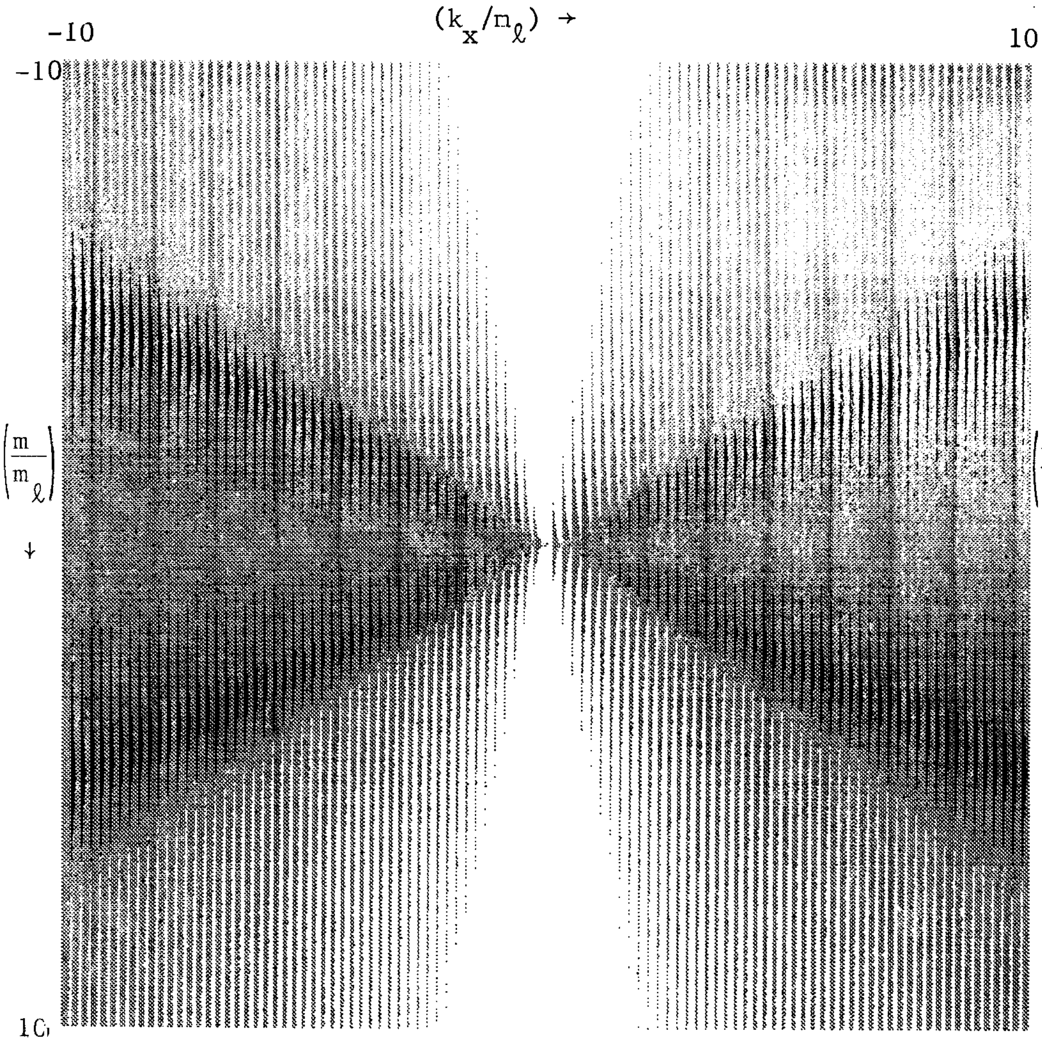

FIGURE 3.--Same as Figure 2 but with $\varepsilon_1 = 0$, $\varepsilon_2 = .5\ m_\ell$. Maximum value is 4.5.

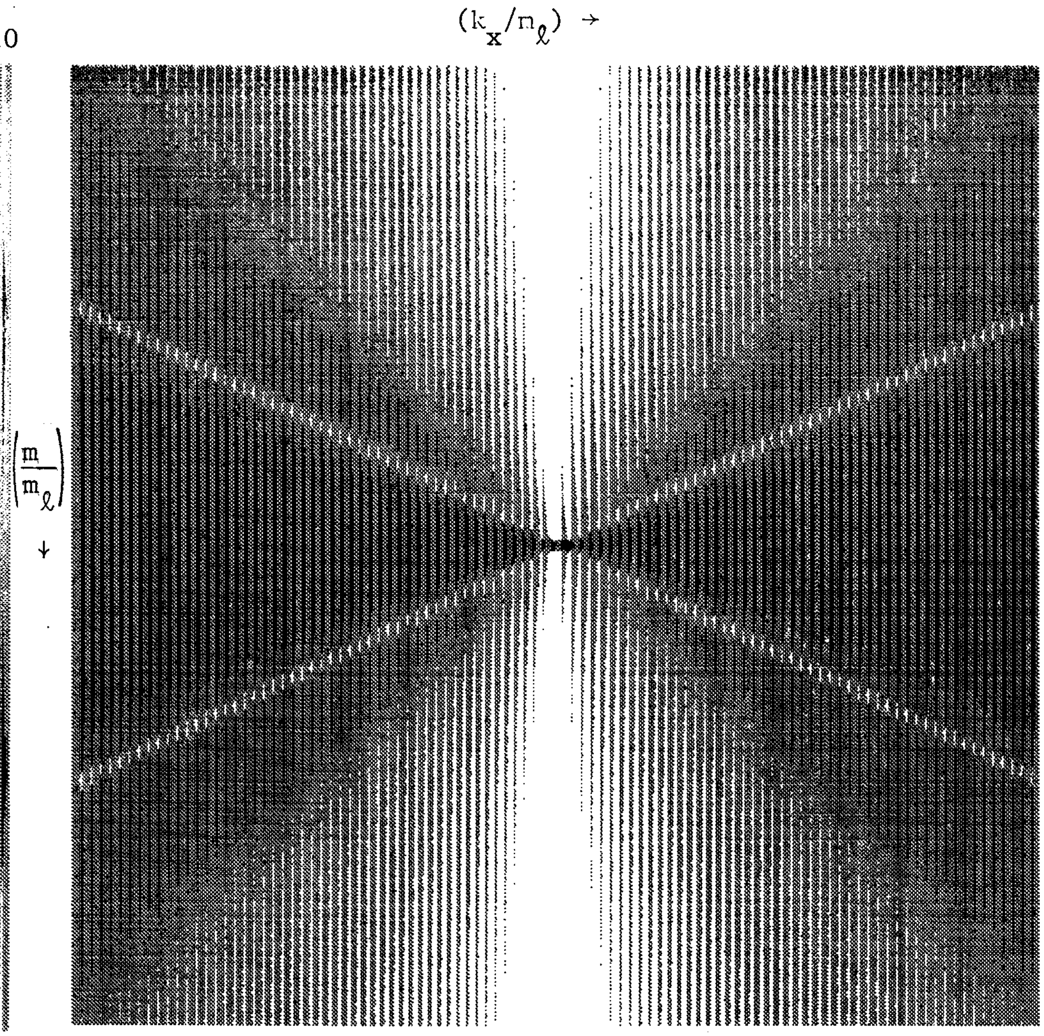

FIGURE 4.--Plot of $\ln[\text{real } k_z(2)/\text{real } k_z(1)]$ as function of m/m_ℓ and k_x/m_ℓ. [Recall that $\varepsilon_1 = \varepsilon_2$ for $k_z(1)$ but $\varepsilon_1 = 0$ for $k_z(2)$.] Since this quantity is negative for all $|m| > |k_x|$ and positive for $|m| < |k_x|$ except in the narrow zone of maximum attenuation, $k_z(2)$ is clearly the more desirable operator.

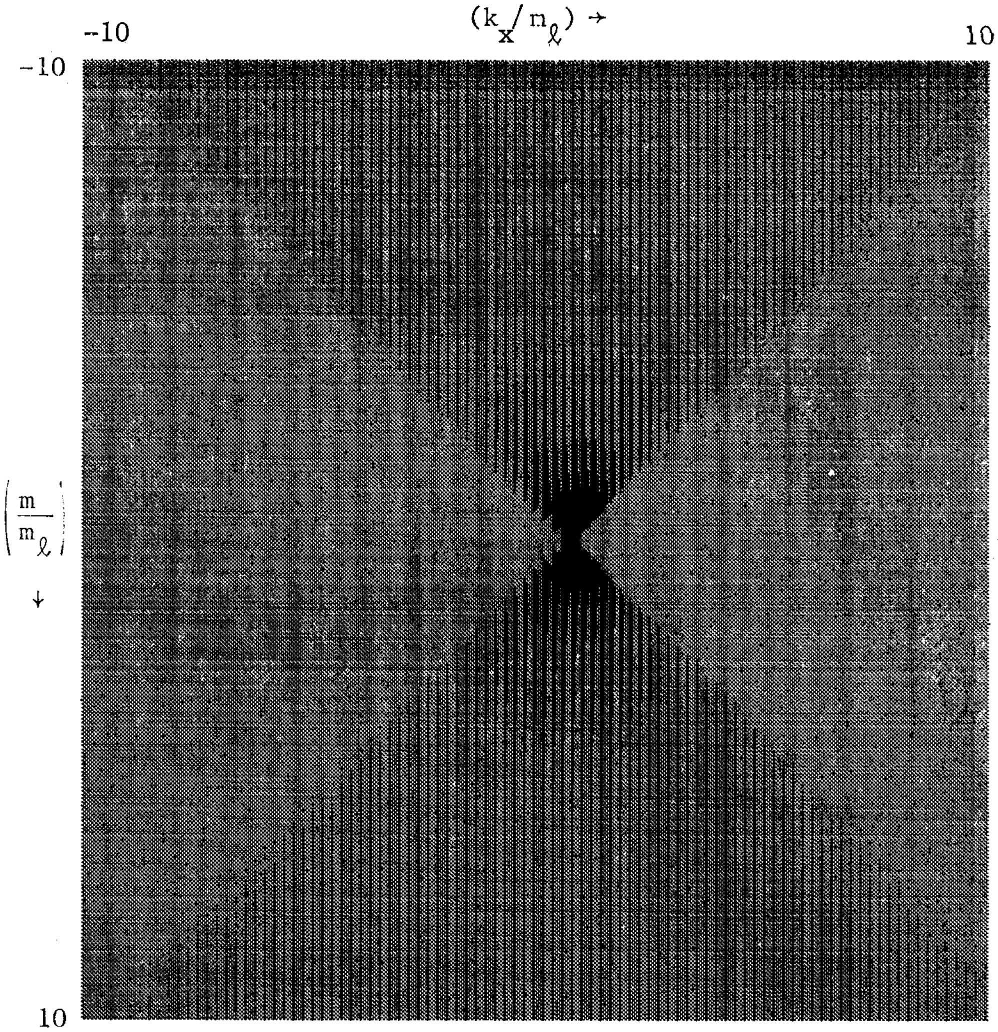

FIGURE 5.--Plot of differences in absolute phase errors between $k_z(1)$ and $k_z(2)$ in the zone of propagation. The positivity of this quantity means that the $k_z(2)$ operator has less phase error everywhere in the propagating region. The apparent improvement is unimpressive when it is noted that the plot is scaled to ten millidegrees. Nevertheless, it is significant that the phase error is no worse for $k_z(2)$ than $k_z(1)$. That is, we do not have to introduce any phase distortion in the all-pass region in order to improve the attenuation performance of the continuation filter in the evanescent zone.

"Phase error" is defined as absolute deviation from the phase of the 45^{o} equation.

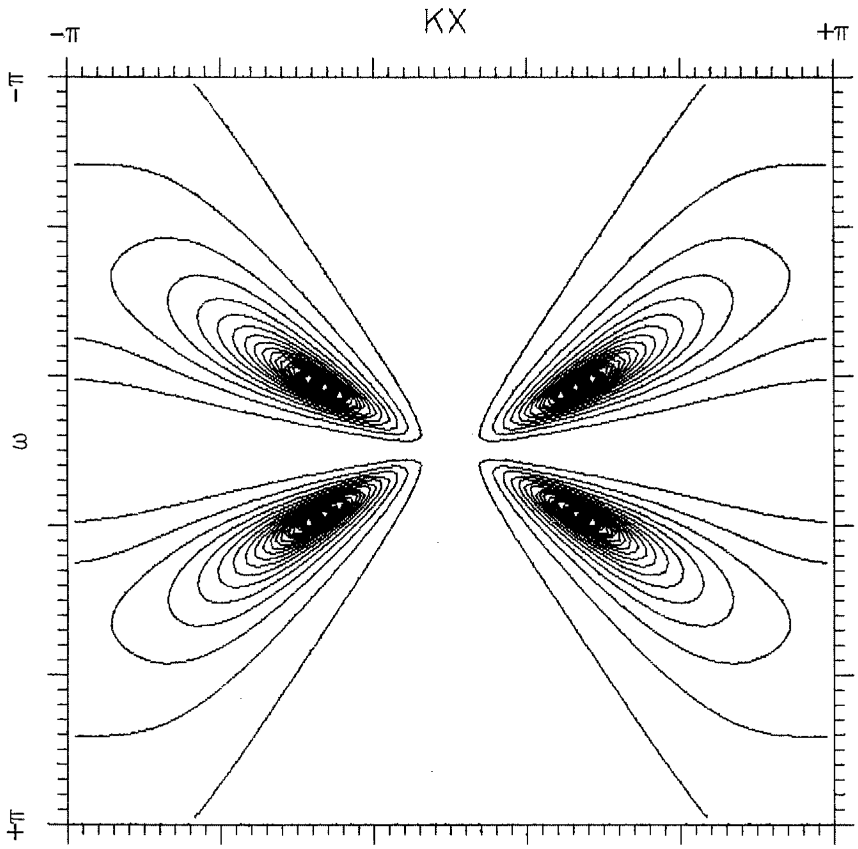

FIGURE 6.--One db. contours of attenuation per unit z-step, with $\varepsilon = .1\pi/\Delta t$, $\varepsilon = 0$, $\Delta x = v = \Delta t = 1$, for the 45-degree equation. The approximation $\partial_{xx} = \delta_{xx}/(1 + \gamma\delta_{xx})$ with $\gamma = 0.14$ was used. The transfer function to go from one z-level to the next is given by:

$$H(\omega,k_x) = \frac{aae^{ik_x\Delta x} + bb + aae^{-ik_x\Delta x}}{ae^{ik_x\Delta x} + b + ae^{-ik_x\Delta x}}$$

where

$$a = 4\Delta x^2\gamma m_1 m_2 - i\Delta z m_2 + 1$$

$$b = 4\Delta x^2 m_1 m_2 - 2a$$

$$aa = a + 2i\Delta z m_2$$

$$bb = b - 4i\Delta z m_2$$

and

$$m_j = \frac{\omega + i\varepsilon_j}{v}; \qquad j = 1,2.$$

$$\text{attenuation} = -\text{real}(\ln H)$$

Note that the region of low $|\omega|$ and high $|k_x|$ is essentially all pass for this choice of ε's.

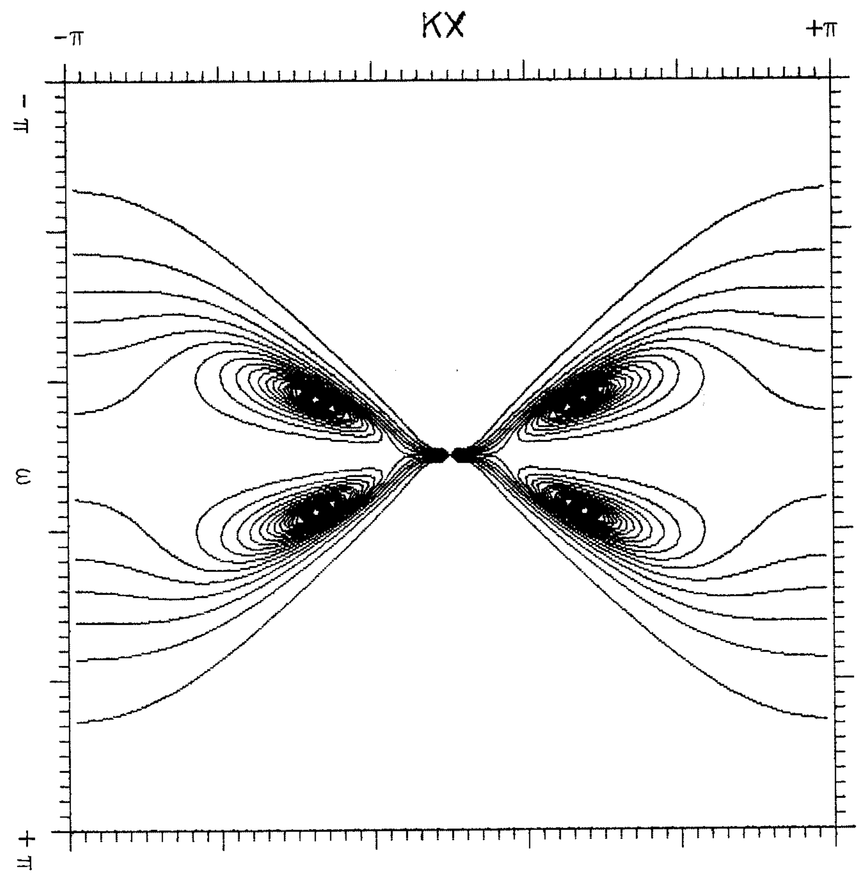

FIGURE 7.--db of attenuation/unit z-step for $\varepsilon_1 = 0$, $\varepsilon_2 = 0.1\ \pi/\Delta t$. All other parameters are the same as in Figure 6. Energy in the region of low $|\omega|$ and high $|k_x|$ is now well attenuated.

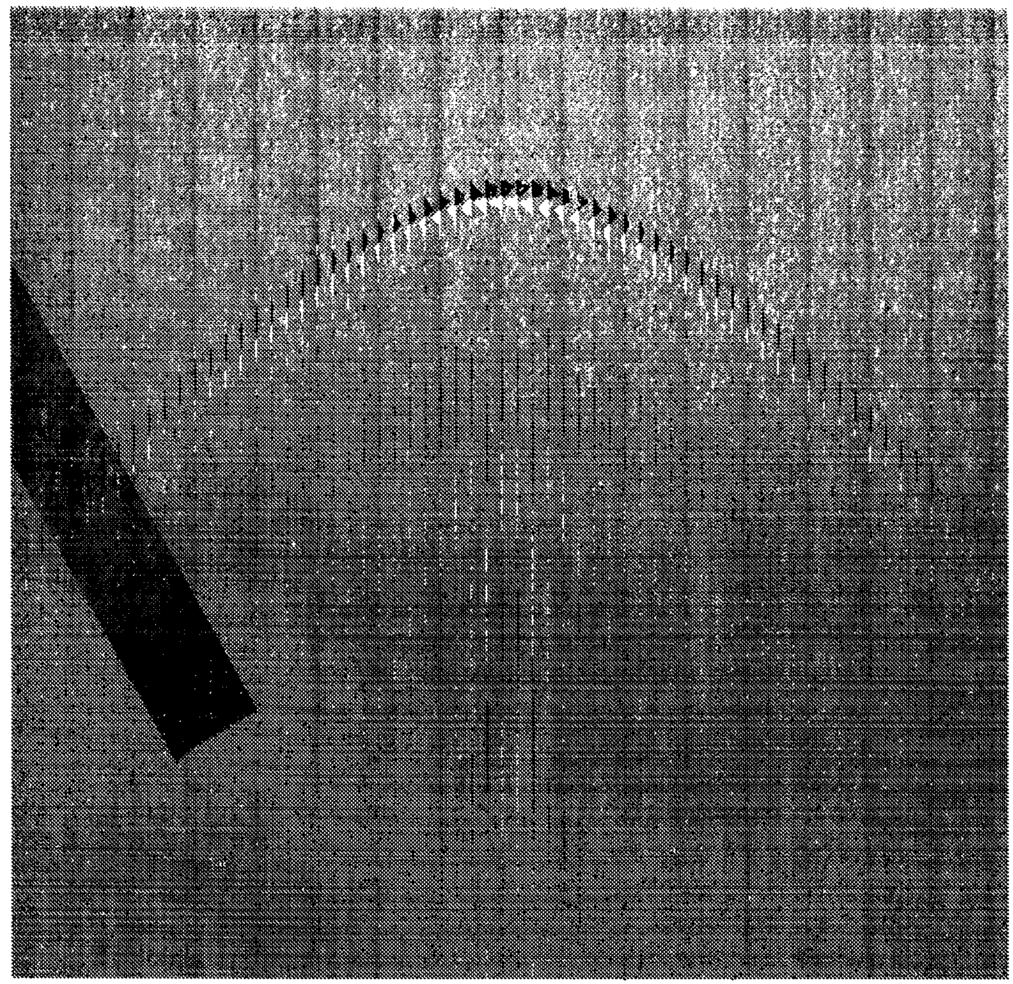

FIGURE 8.--45-degree impulse response of Figure 1 but with $\varepsilon_1 = \varepsilon_2 = .005\ \pi/\Delta t$ instead of 0. This is an improvement over Figure 1 but noise streaks are still visible.

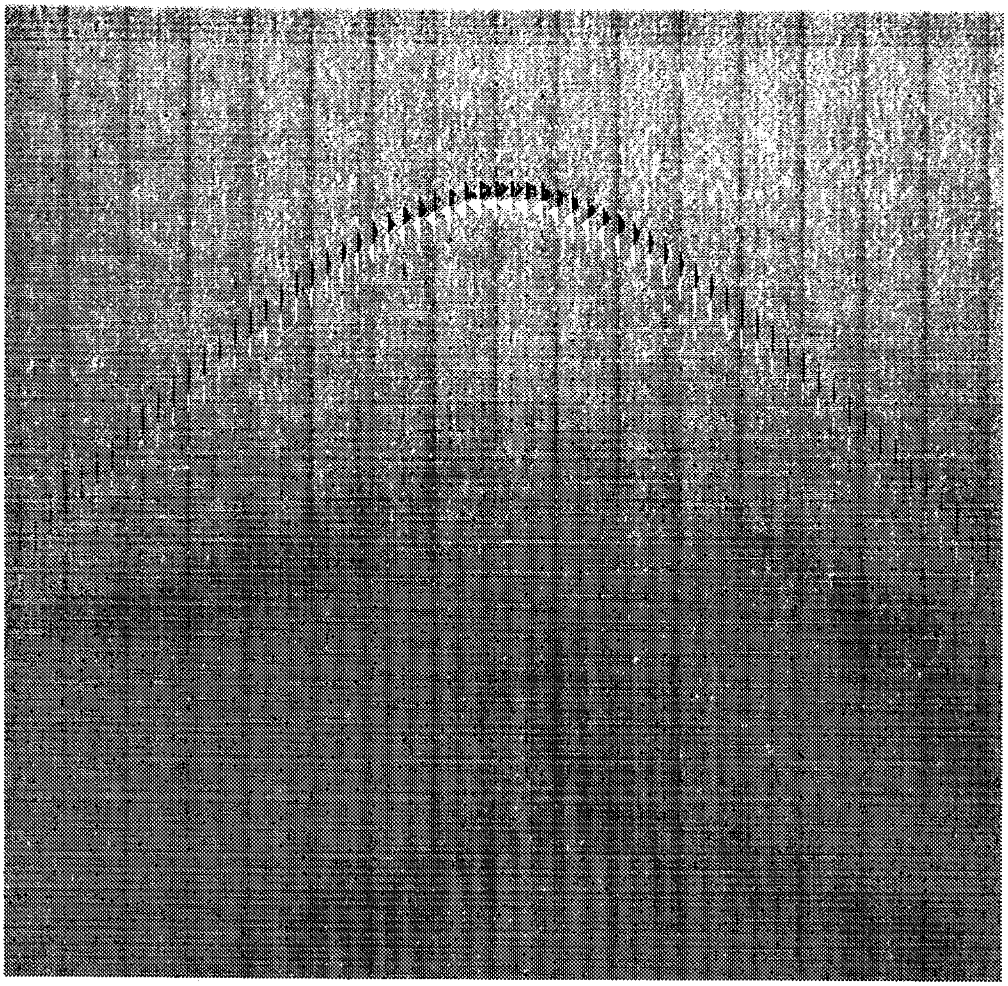

FIGURE 9.--Same as Figure 8 except $\varepsilon_1 = 0$, $\varepsilon_2 = .015\ \pi/\Delta t$. Noise streaks are completely removed *without* any degradation of signal.