Appendix B

Relation between LSM model and reflectivity

The goal of this appendix is to define the relationship between the output of least-squares migration (LSM) and the reflectivity for normal incidence wave. There will be three sections. The first section focuses on deriving the acoustic wave equation and the reflectivity expression between an interface with velocity and density contrast. In the second section, I derive the LSM model parameter in terms of velocity and density perturbation. By LSM model, I am referring to the parametrization that is consistent with Chapter 2. Specifically, the relationship between the Born modeled data, $d_{\mathrm{b}}(\mathbf{x}_r, \mathbf{x}_s, t)$, and the model parameter, $m(\mathbf{x})$, as described by equatoin 2.6. Finally, the last section focuses on explaining the relationship between the LSM model parameter and the reflectivity output.

REFLECTIVITY COEFFICIENT IN 1D

I will derive the acoustic wave equation based on two fundamental principles of physics. Suppose there is an external force acting on a volume of particles, Newton's second law states that

$$\frac{\text{force}}{\text{volume}} = \frac{\text{mass X acceleration}}{\text{volume}}. \tag{B.1}$$

Consider a simple case where a volume of material resides in a tube under pressure. The tube has a cross-sectional area A and the length of the tube in consideration is Δx. The force applied to this volume is:

$$\begin{aligned} \frac{\text{force}}{\text{volume}} &= \frac{\text{force}}{A\Delta x}, \\ &= -\frac{P(x + \frac{1}{2}\Delta x) - P(x - \frac{1}{2}\Delta x)}{\Delta x}, \\ &= -\frac{\partial P(x)}{\partial x}, \end{aligned} \tag{B.2}$$

where $P(x)$ is the pressure applied along the tube. Combining equation B.1 and B.2 yields,

$$\rho \frac{\partial^2 u(x)}{\partial t^2} = -\frac{\partial P(x)}{\partial x}. \tag{B.3}$$

where $u(x)$ is the displacement of the particle at location x. The second equation needed to derive the acoustic wave equation is from the Hooke's Law. Hooke's Law states that for small displacements, the strain is proportional to the stress.

$$P = -\kappa \frac{\partial u}{\partial x}, \tag{B.4}$$

where κ is the bulk modulus. The pressure P is the pressure variations around a background hydrostatic pressure. If I take the first derivative with respect to x on equation B.3 and a second time derivative on equation B.4, I get the following,

$$
\begin{aligned}
\frac{\partial^3 u(x)}{\partial x \partial t^2} &= -\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial P(x)}{\partial x}\right), \\
\frac{\partial^2}{\partial t^2}\left(\frac{1}{\kappa}P(x)\right) &= -\frac{\partial^3 u(x)}{\partial x \partial t^2}.
\end{aligned}
$$

Equating the above two equations yields,

$$
\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial P(x)}{\partial x}\right) = \frac{1}{\kappa}\frac{\partial^2}{\partial t^2}P(x). \tag{B.5}
$$

Next, I express the bulk modulus as a function of density and velocity.

$$
\begin{aligned}
\kappa &= v^2\rho, \\
\frac{1}{v^2}\frac{\partial^2}{\partial t^2}P(x) &= \rho\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial P(x)}{\partial x}.\right)
\end{aligned} \tag{B.6}
$$

The derivation can be extended to 3D, which will produce the familiar acoustic variable-density wave equation,

$$
\frac{1}{v^2}\frac{\partial^2}{\partial t^2}P(\mathbf{x}, t) = \rho \bigtriangledown \cdot \left(\frac{1}{\rho} \bigtriangledown P(\mathbf{x}, t)\right). \tag{B.7}
$$

Next, I will derive the reflectivity expression in one dimension. Suppose a wave that travels in the z direction hits an interface with velocity and density contrast that goes from v_1 and ρ_1 to v_2 and ρ_2. The expressions of the pressure wave above (P^+)and below (P^-) the interface are,

$$
\begin{aligned}
P^+(z,t) &= e^{i(wt+kz)} + Re^{i(wt-kz)}, \\
P^-(z,t) &= Te^{i(wt+kz)},
\end{aligned} \tag{B.8}
$$

where R and T are the amplitude of the reflected and the transmitted pressure wavefield, respectively. k is the wave number. To obtain an expression for the reflectivity, we can enforce two conditions

1. Continuity of pressure across the interface.

2. Continuity of normal acceleration across the interface (equation B.3).

The first condition gives:

$$\begin{aligned} P^{+}(z=0,t) &= P^{-}(z=0,t), \\ e^{iwt} + Re^{iwt} &= Te^{iwt}, \\ 1 + R = T. & \end{aligned} \tag{B.9}$$

The second condition yields:

$$\begin{aligned} a^{+}(z=0,t) &= a^{-}(z=0,t), \\ \frac{1}{\rho_1}\frac{\partial P^{+}(z,t)}{\partial z}|_{z=0} &= \frac{1}{\rho_2}\frac{\partial P^{-}(z,t)}{\partial z}|_{z=0}, \\ \frac{1}{\rho_1}\left(ke^{iwt} - Rke^{iwt}\right) &= \frac{1}{\rho_2}Tke^{iwt}, \\ \frac{\omega}{\rho_1 v_1}(1-R) &= \frac{\omega}{\rho_2 v_2}T, \\ \frac{1}{\rho_1 v_1}(1-R) &= \frac{1}{\rho_2 v_2}T. \end{aligned} \tag{B.10}$$

To obtain an expression for the reflectivity (R), substitute equation B.9 into equation B.10.

$$\begin{aligned} \frac{1}{\rho_1 v_1}(1-R) &= \frac{1}{\rho_2 v_2}(1+R), \\ \rho_2 v_2(1-R) &= \rho_1 v_1(1+R), \\ \rho_2 v_2 - \rho_1 v_1 &= (\rho_1 v_1 + \rho_2 v_2)R, \\ R &= \frac{\rho_2 v_2 - \rho_1 v_1}{\rho_1 v_1 + \rho_2 v_2}. \end{aligned} \tag{B.11}$$

LEAST-SQUARES MIGRATION MODEL PARAMETER

When density variation is introduced into the wave equation, the least-squares migration model parameter will be a function of velocity and density. I will first derive the forward modeling equation by linearizing the acoustic isotropic variable-density wave equation using perturbation theory. I will then show the relationship between the least-squares migration output and the reflectivity model. All the derivations follow from Zhang et al. (2014). The acoustic isotropic variable density wave-equation with velocity $v(\mathbf{x})$ and density $\rho(\mathbf{x})$ is:

$$\left(\frac{1}{v^2(\mathbf{x})} \frac{\partial^2}{\partial t^2} - \rho(\mathbf{x}) \bigtriangledown \frac{1}{\rho(\mathbf{x})} \cdot \bigtriangledown \right) P(\mathbf{x}, t; \mathbf{x}_s) = f_s(t) \delta(\mathbf{x} - \mathbf{x}_s). \tag{B.12}$$

where $f_s(t)$ is the source signature, $\mathbf{x}_s$ is the source position, and $P(\mathbf{x}, t; \mathbf{x}_s)$ is the pressure wavefield as a function of subsurface position $\mathbf{x}$ and time t. To linearize the wave equation, I introduce perturbation to the velocity and density as follows,

$$\begin{aligned} v(\mathbf{x}) &= v_o(\mathbf{x}) + \delta v(\mathbf{x}), \\ \rho(\mathbf{x}) &= \rho_o(\mathbf{x}) + \delta \rho(\mathbf{x}), \end{aligned} \tag{B.13}$$

where I define $v_o(\mathbf{x})$ to be the background velocity and $\delta v(\mathbf{x})$ to be the velocity perturbation. Similarly, $\rho_o(\mathbf{x})$ is the background density and $\delta\rho(\mathbf{x})$ is the density perturbation. The wavefield can be broken down into a background component, $P_o(\mathbf{x}, t; \mathbf{x}_s)$, and a perturbed component, $\delta P(\mathbf{x}, t; \mathbf{x}_s)$,

$$P(\mathbf{x}, t; \mathbf{x}_s) = P_o(\mathbf{x}, t; \mathbf{x}_s) + \delta P(\mathbf{x}, t; \mathbf{x}_s). \tag{B.14}$$

$P_o(\mathbf{x}, t)$ is the background wavefield that satisfies the wave-equation (equation B.12) with the background velocity and density, $v_o(\mathbf{x})$ and $\rho_o(\mathbf{x})$. $\delta P(\mathbf{x}, t; \mathbf{x}_s)$ represents the deviation between the two wavefields. I assume all perturbed quantities are relatively small as compared to their respective background quantities. In a surface seismic experiment, if the background velocity is smoothly varying, the background wavefield at the receiver location, $P_o(\mathbf{x}_r, t; \mathbf{x}_s)$, would not record any reflection energy. Substituting equations B.13 and B.14 into equation B.12 yields,

$$\left(\frac{1}{(v_o + \delta v)^2} \frac{\partial^2}{\partial t^2} - (\rho_o + \delta\rho) \bigtriangledown \frac{1}{\rho_o + \delta\rho} \cdot \bigtriangledown \right) (P_o + \delta P) = f_s(t) \delta(\mathbf{x} - \mathbf{x}_s). \tag{B.15}$$

Expanding equation B.15, removing second-order perturbation terms, and cancelling the terms associated with $P_o(\mathbf{x}, t; \mathbf{x}_s)$ give,

$$\left(\frac{1}{v_o^2}\frac{\partial^2}{\partial t^2} - \rho_o \bigtriangledown \frac{1}{\rho_o} \cdot \bigtriangledown\right) \delta P = \left(\frac{2\delta v}{v_o^3}\frac{\partial^2}{\partial t^2} - \bigtriangledown \frac{\delta \rho}{\rho_o} \cdot \bigtriangledown\right)(P_o + \delta P). \tag{B.16}$$

In the Born approximation, we replace the total wavefield on the right hand side with the perturbed wavefield. Equation B.16 in the frequency domain becomes,

$$\left(\frac{-1}{v_o^2}\omega^2 - \rho_o \bigtriangledown \frac{1}{\rho_o} \cdot \bigtriangledown\right) \delta P = \left(\frac{-2\delta v}{v_o^3}\omega^2 - \bigtriangledown \frac{\delta \rho}{\rho_o} \cdot \bigtriangledown\right) P_o, \tag{B.17}$$

where ω represents the temporal frequency. The perturbed wavefield $\delta P(\mathbf{x}, \omega; \mathbf{x}_s)$ satisfying equation B.17 can be expressed using Green's function technique,

$$\delta P(\mathbf{x}, \mathbf{x}_s, \omega) = \int \left(\frac{-2\delta v(\mathbf{x}')}{v_o^3(\mathbf{x}')}\omega^2 - \bigtriangledown \frac{\delta \rho(\mathbf{x}')}{\rho_o(\mathbf{x}')} \cdot \bigtriangledown\right) P_o(\mathbf{x}, \omega; \mathbf{x}_s) G(\mathbf{x}, \mathbf{x}', \omega) d\mathbf{x}', \tag{B.18}$$

where $G(\mathbf{x}, \mathbf{x}', \omega)$ is the Green's function of the wave equation. It is the solution to equation B.12 with an impulse source. Next, I apply integration by part onto the second term of equation B.18 to get the following,

$$\delta P(\mathbf{x}, \mathbf{x}_s, \omega) = \int \left(\frac{-2\delta v(\mathbf{x}')}{v_o^3(\mathbf{x}')}\omega^2 P_o G + \frac{\delta \rho(\mathbf{x}')}{\rho_o(\mathbf{x}')} \bigtriangledown \cdot (\bigtriangledown P_o G)\right) d\mathbf{x}'. \tag{B.19}$$

To derive an expression of the model parameter for normal incidence, we can represent $P_o(\mathbf{x}, \omega)$ and $G(\mathbf{x}, \mathbf{x}', \omega)$ as plane waves,

$$\begin{aligned} P_o(\mathbf{x}, \omega) &= A_1 e^{i(k_z z - \omega t)}, \\ G(\mathbf{x}, \mathbf{x}', \omega) &= A_2 e^{i(k_z (z - z') - \omega t)}. \end{aligned} \tag{B.20}$$

where A_1 and A_2 are constants that represent the amplitude of the plane waves. Notice that the plane waves are traveling vertically, which makes equation B.20 independent from x and y. $k_z = \frac{\omega}{v_o}$ is the vertical wavenumber. Applying a gradient to the plane waves results in the following closed form,

$$\begin{aligned} \bigtriangledown P_o(\mathbf{x},\omega) &= ik_z A_1 e^{i(k_z z-\omega t)} = ik_z P_o,(\mathbf{x},\omega), \\ \bigtriangledown G(\mathbf{x},\mathbf{x}',\omega) &= ik_z A_2 e^{i(k_z(z-z')-\omega t)} = ik_z G(\mathbf{x},\mathbf{x}',\omega). \end{aligned} \tag{B.21}$$

$$\begin{aligned} \bigtriangledown \cdot (\bigtriangledown P_o G) &= (\bigtriangledown^2 P_o)G + (\bigtriangledown P_o)\cdot(\bigtriangledown G), \\ &= -k_z^2 P_o G - k_z^2 P_o G, \\ &= -2\frac{\omega^2}{v_o^2} P_o G. \end{aligned} \tag{B.22}$$

Substituting equation B.22 into equation B.19, we reach the final form of the forward modeling expression.

$$\delta P(\mathbf{x},\mathbf{x}_s,\omega) = \int \frac{2}{v_o^2(\mathbf{x}')}\left(\frac{\delta v_o(\mathbf{x}')}{v(\mathbf{x}')} + \frac{\delta\rho(\mathbf{x}')}{\rho_o(\mathbf{x}')}\right)(-\omega^2)P_o(\mathbf{x}',\omega;\mathbf{x}_s)G(\mathbf{x},\mathbf{x}',\omega)d\mathbf{x}'. \tag{B.23}$$

The Born modeled data of the acoustic isotropic variable-density wave equation, $\mathbf{d}_b$, is

$$\mathbf{d}_b(\mathbf{x}_r,\mathbf{x}_s,\omega) = \int \frac{2}{v_o^2(\mathbf{x}')}\left(\frac{\delta v(\mathbf{x}')}{v_o(\mathbf{x}')} + \frac{\delta\rho(\mathbf{x}')}{\rho_o(\mathbf{x}')}\right)(-\omega^2)P_o(\mathbf{x}',\omega;\mathbf{x}_s)G(\mathbf{x}_r,\mathbf{x}',\omega)d\mathbf{x}'. \tag{B.24}$$

Comparing this to the Born forward modeling equation in chapter 2, the model parameter for the case of acoustic isotropic variable-density wave equation would equate to,

$$m(\mathbf{x}') = \frac{2}{v_o^2(\mathbf{x}')}\left(\frac{\delta v(\mathbf{x}')}{v_o(\mathbf{x}')} + \frac{\delta\rho(\mathbf{x}')}{\rho_o(\mathbf{x}')}\right). \tag{B.25}$$

The above equation only holds for normal incidence plane wave. It is possible to derive a more generic expression for different incidence direction. I will direct the reader to Zhang et al. (2014) for further derivation.

CONNECTION

The reflectivity from equation B.11 can also be expressed in terms of impedance at the two interfaces.

$$\begin{aligned} I_1 &= \rho_1 v_1, \\ I_2 &= \rho_2 v_2, \\ R &= \frac{I_2 - I_1}{I_1 + I_2}. \end{aligned} \tag{B.26}$$

Next, I rewrite equation B.26 as a continuous function of variable x',

$$\begin{aligned} R(x') &= \frac{\delta I(x')}{2I(x')}, \\ &= \frac{\delta(\rho(x')v(x'))}{2\rho(x')v(x')}, \\ &= \frac{(\delta\rho(x'))v(x') + \rho(x')\delta v(x')}{2\rho(x')v(x')}. \\ &= \frac{1}{2}\left(\frac{\delta\rho(x')}{\rho(x')} + \frac{\delta v(x')}{v(x')}\right). \end{aligned} \tag{B.27}$$

Comparing between equations B.27 and B.25, I get the following relationship between the least-squares migration model parameter and reflectivity.

$$\begin{aligned} R(x') &= \frac{1}{4}v_o^2(x')m(x'), \\ &= \left(\frac{v_o(\mathbf{x}')}{2}\right)^2 m(\mathbf{x}'). \end{aligned} \tag{B.28}$$

Equation B.28 suggests that we can scale the LSM output by the square of half of the background velocity to obtain the reflectivity at normal incidence.