Appendix A

Linearizing the acoustic wave equation

In this appendix, I derive the linear forward modeling and adjoint operators for the least-squares reverse-time migration method. Just like in migration, least-squares migration requires a velocity or slowness model as input. In this derivation, I use the acoustic isotropic constant-density wave equation,

$$\left(s^2(\mathbf{x})\frac{\partial^2}{\partial t^2} - \bigtriangledown^2\right) P(\mathbf{x}, t; \mathbf{x}_s) = f_s(t)\delta(\mathbf{x} - \mathbf{x}_s). \tag{A.1}$$

where $f_s(t)$ is the source signature, $\mathbf{x}_s$ is the source position, and $P(\mathbf{x}, t; \mathbf{x}_s)$ is the pressure wavefield as a function of subsurface position $\mathbf{x}$ and time t. Let $s(\mathbf{x})$ represents the true slowness, and $s_o(\mathbf{x})$ the migration slowness. If the migration slowness deviates from the true slowness, we can introduce a perturbation term, $\Delta s(\mathbf{x})$, that is defined as,

$$\begin{aligned} s(\mathbf{x}) &= s_o(\mathbf{x}) + \Delta s(\mathbf{x}), \\ s^2(\mathbf{x}) &\approx s_o^2(\mathbf{x}) + 2\Delta s(\mathbf{x}) s_o(\mathbf{x}) = s_o^2(\mathbf{x}) + m(\mathbf{x}), \end{aligned} \tag{A.2}$$

where I define our model, $m(\mathbf{x})$, as a product between the migration slowness and the perturbation slowness. Similarily, the wavefield can be broken down into a background component, $P_o(\mathbf{x}, t; \mathbf{x}_s)$, and a perturbed component, $\delta P(\mathbf{x}, t; \mathbf{x}_s)$,

$$P(\mathbf{x}, t; \mathbf{x}_s) = P_o(\mathbf{x}, t; \mathbf{x}_s) + \delta P(\mathbf{x}, t; \mathbf{x}_s). \tag{A.3}$$

$P_o(\mathbf{x}, t)$ is the wavefield that satisfies the acoustic wave-equation with slowness, $s_o(\mathbf{x})$, as described in equation A.4. $\delta P(\mathbf{x}, t; \mathbf{x}_s)$ represents the deviation between the two wavefields.

$$\left(s_o^2(\mathbf{x}) \frac{\partial^2}{\partial t^2} - \bigtriangledown^2 \right) P_o(\mathbf{x}, t; \mathbf{x}_s) = f_s(t) \delta(\mathbf{x} - \mathbf{x}_s). \tag{A.4}$$

Substituting equations A.2 and A.3 into equation A.1 yields,

$$\left((s_o^2(\mathbf{x}) + m(\mathbf{x})) \frac{\partial^2}{\partial t^2} - \bigtriangledown^2 \right) (P_o(\mathbf{x}, t; \mathbf{x}_s) + \delta P(\mathbf{x}, t; \mathbf{x}_s)) = f_s(t) \delta(\mathbf{x} - \mathbf{x}_s) \tag{A.5}$$

Expanding equation A.1 and cancelling the terms associated with equation A.4 gives,

$$\left(s_o^2(\mathbf{x}) \frac{\partial^2}{\partial t^2} - \bigtriangledown^2 \right) \delta P(\mathbf{x}, t; \mathbf{x}_s) = -m(\mathbf{x}) \frac{\partial^2}{\partial t^2} (P_o(\mathbf{x}, t; \mathbf{x}_s) + \delta P(\mathbf{x}, t; \mathbf{x}_s)) \tag{A.6}$$

The forcing term also contains the perturbed wavefields. In the Born approximation, the perturbed terms are ignored to produce equation A.7.

$$\left(s_o^2(\mathbf{x}) \frac{\partial^2}{\partial t^2} - \bigtriangledown^2 \right) \delta P(\mathbf{x}, \mathbf{x}_s, t) = -m(\mathbf{x}) \frac{\partial^2}{\partial t^2} P_o(\mathbf{x}, \mathbf{x}_s, t). \tag{A.7}$$

Notice that the force term in equation A.7, $(-m(\mathbf{x}) \frac{\partial^2}{\partial t^2} P_o(\mathbf{x}, \mathbf{x}_s, t))$, is a product of the source-side wavefield and the image model defined earlier. The Born approximation of the linearized acoustic wave equation, $\mathbf{d}_{born}$, is

$$d_{born}(\mathbf{x}_r, \mathbf{x}_s, t) = \delta P(\mathbf{x}, t; \mathbf{x}_s) \delta(\mathbf{x} - \mathbf{x}_r), \tag{A.8}$$

From equation A.8 and A.7, we can see that d_{born} scales linearly with $m(\mathbf{x})$.

The adjoint of the Born modeling operation (equation A.8) can be derived by calculating the gradient of the least-squares reverse-time migration objective function.

$$
\begin{aligned}
S(m) &= \frac{1}{2}\|\vec{d}_o + \vec{d}_{\mathrm{born}} - \vec{d}_{\mathrm{obs}}\|^2 && \text{(A.9)} \\
&= \frac{1}{2}\|\Delta\vec{d}\|^2 && \text{(A.10)}
\end{aligned}
$$

where $\vec{d}_{\mathrm{obs}}$ is the observed data, $\vec{d}_{\mathrm{born}}$ is the linearized Born modeled data, and $\Delta\vec{d}$ represents the data residual. $\vec{d}_o$ is the background data, which is calculated by using full wave modeling based on the background slowness, s_o. Next, I rewrite equation A.1 in matrix and vector notations.

$$
\mathbf{F}\vec{u} = \vec{f} \qquad \text{(A.11)}
$$

where $\mathbf{F}$ and $\vec{f}$ are

$$
\begin{aligned}
\mathbf{F} &= (s_o^2\frac{d^2}{dt^2} + \bigtriangledown^2) && \text{(A.12)} \\
\vec{u} &= \mathbf{F}^{-1}\vec{f} && \text{(A.13)}
\end{aligned}
$$

The equivalent of equation A.8 in matrix-vector notations is,

$$
\vec{d}_{\mathrm{born}} = \mathbf{S}\vec{u}, \qquad \text{(A.14)}
$$

where $\mathbf{S}$ is the selection operator that extracts the wavefield $\vec{u}$ at the receiver location.

Taking the gradient of the objecive function, equation A.9, gives:

$$\begin{aligned}
\nabla_m S &= \langle \frac{\partial \vec{d}_{\text{born}}}{\partial m}, \Delta \vec{d} \rangle, &\text{(A.15)}\\
&= \left(\frac{\partial \vec{d}_{\text{born}}}{\partial m} \right)^* \Delta \vec{d}, &\text{(A.16)}\\
&= -\left(\mathbf{S}\mathbf{F}^{-1} \frac{\partial \mathbf{F}}{\partial m} \mathbf{F}^{-1} \vec{f} \right)^* \Delta \vec{d}, &\text{(A.17)}\\
&= -\left(\mathbf{F}^{-1} \vec{f} \right)^* \left(\frac{\partial \mathbf{F}}{\partial m} \right)^* \left(\mathbf{S}\mathbf{F}^{-1} \right)^* \Delta \vec{d}, &\text{(A.18)}\\
&= -\left(\mathbf{F}^{-1} \vec{f} \right)^* \left(\frac{d^2}{dt^2} \right)^* \left(\mathbf{S}\mathbf{F}^{-1} \right)^* \Delta \vec{d}. &\text{(A.19)}
\end{aligned}$$

Notice that the gradient of the objective function is exactly the reverse-time migration operation.

$$m(\mathbf{x}) = -\sum_{\mathbf{x}_s, t} P_o^*(\mathbf{x}, \mathbf{x}_s, t) \left(\frac{d^2}{dt^2} \right)^* P_r(\mathbf{x}, \mathbf{x}_r, t; \mathbf{x}_s) \qquad \text{(A.20)}$$

where t is time, $m(\mathbf{x})$ represents the structural image at subsurface location $\mathbf{x}$, $\mathbf{x}_s$ represents the source location and $\mathbf{x}_r$ represents the receiver location. $P_o(\mathbf{x}, \mathbf{x}_s, t)$ and $P_r(\mathbf{x}, \mathbf{x}_r, t; \mathbf{x}_s)$ are solutions to the two-way acoustic constant density equation.