Appendix B

Crosscorrelations and spectra

Fourier transformations are powerful tools to analyze passive seismic recordings, primarily because of their ability to characterize an infinitely long time series by a compact spectrum. We employ the following definition for the continuous-time Fourier transformation:

$$G(f) = \mathcal{F}_t\left\{g(t)\right\} = \int_{-\infty}^{\infty} \mathrm{e}^{-i2\pi ft} g(t)dt, \tag{B.1}$$

and its inverse

$$g(t) = \mathcal{F}_t^{-1}\left\{G(f)\right\} = \int_{-\infty}^{\infty} \mathrm{e}^{i2\pi ft} G(f)df, \tag{B.2}$$

where $G(f)$ denotes the temporal Fourier domain counterpart of the function $g(t)$, t and f denote time and temporal frequency respectively. The temporal frequency f is regularly substituted by angular frequency $w = 2\pi f$. In practice, we do not record continuous and infinitely long time series, but instead record a limited number of discrete samples. Correspondingly, the Fourier spectrum is not continuous; the discrete time-series is decomposed into a discrete number of Fourier coefficients. For a time series containing N samples with sampling Δt, the time and frequency axes are discretised as:

$$t \rightarrow \mathbf{f} = t_n = n\Delta f \quad \text{with } \; n = -N/2, -N/2+1, ..., N/2-1, \tag{B.3}$$

$$f \rightarrow \mathbf{f} = f_m = m\Delta f \quad \text{with } \; m = -N/2, -N/2+1, ..., N/2-1, \tag{B.4}$$

where the frequency sampling, Δf, is related to N and Δt as shown below. The continuous time and frequency functions $g(t)$ and $G(f)$ are consequently discretised as

$$g(t) \rightarrow \mathbf{g}(t_n) \quad \text{with} \;\; n = -N/2, -N/2+1, ..., N/2-1, \tag{B.5}$$

$$G(f) \rightarrow \mathbf{G}(f_m) \quad \text{with} \;\; m = -N/2, -N/2+1, ..., N/2-1. \tag{B.6}$$

Substituting these into the definition of the forward and inverse Fourier transformations, Equations B.1 and B.2, we find

$$\mathbf{G}(f_m) = \sum_{n=-N/2}^{N/2-1} \mathbf{g}(t_n) e^{-i2\pi(\Delta t \Delta f \; n \; m)} \Delta t, \tag{B.7}$$

$$\mathbf{g}(t_n) = \sum_{m=-N/2}^{N/2-1} \mathbf{G}(f_m) e^{i2\pi(\Delta t \Delta f \; n \; m)} \Delta f. \tag{B.8}$$

Notice that the periodicity of the Fourier kernel relates Δt, Δt and N as:

$$N \Delta f \Delta t = 1 \tag{B.9}$$

Although Equations B.7 and B.8 are the discrete versions of Equations B.1 and B.2, more modifications are needed to find the Discrete Fourier Transformations (DFTs) as commonly applied by packages such as the Fastest Fourier Transform in the West (FFTW) (Frigo, 1999). First, replace the negative indices with $n \rightarrow n + N$ and $m \rightarrow m+N$ such that the summations run over positive indices only. The summations now integrate from 0 to 2π instead of from $-\pi$ to π. As a consequence of this, the DFT algorithms expect the positive time-lags first followed by the negative time-lags organized in the time-array, $\mathbf{g}(t_n)$. A similar rearrangement in the frequency-array, $\mathbf{G}(f_m)$. Secondly, scale the variable G_m by Δt and substitute the factor $\Delta f \Delta t$ with $1/N$. Now the DFT algorithm operates independently of Δt and Δf while maintaining invertability, we have:

$$\mathbf{G}(f_m) = \sum_{n=0}^{N-1} \mathbf{g}(t_n) e^{-i2\pi \frac{n\ m}{N}}, \tag{B.10}$$

$$\mathbf{g}(t_n) = \frac{1}{N} \sum_{m=0}^{N-1} \mathbf{G}(f_m) e^{-i2\pi \frac{n\ m}{N}}. \tag{B.11}$$

If we wish to find the discrete version of the continuous Fourier transformations as defined in Equations B.1 and B.2 using the FFTW package, we need to scale the outcome of the forward transformation by Δt and that of the inverse transformation by $\frac{1}{\Delta t}$.

A time-series can be characterized after Fourier transformation by its amplitude and phase spectra, and the energy in a time-series can be characterized by its power spectrum. In this thesis I employ the definitions for amplitude and power spectra as used in ocean acoustics to analyze continuous time series (Erbe, 2011). If a time series of length $N\Delta t = T$ (in $[s]$), measures an electrical voltage (in $[V]$), the application of the DFT transforms the unit into $[Vs]$. For time-series that are purely real, the symmetry of the resulting Fourier domain spectrum can be exploited by multiplying the amplitude spectrum by two and analyzing the spectrum for positive frequencies only. Thus the amplitude spectrum is defined by:

$$\mathbf{S}(f_m) = 2\,|\mathbf{G}(f_m)| \quad \text{with} \quad m = 0, 1, ..., N/2 - 1 \tag{B.12}$$

The power spectrum is defined as the square of the Fourier spectrum:

$$\mathbf{P}(f_m) = 2\,(\mathbf{G}(f_m)\mathbf{G}^*(f_m)) \, \text{with} \quad m = 0, 1, ..., N/2 - 1 \tag{B.13}$$

It represents the energy in signal normalized per $T(s)$ used for the DFT. The units of the power spectrum are $[V^2 s]$. Finally Parceval's theorem requires that

$$\Delta t \sum_{n=-N/2}^{N/2-1} \mathbf{g}(t_n) = \Delta f \sum_{n=-N/2}^{N/2-1} \mathbf{G}(f_m) \text{with} \quad n = 0, 1, ..., N/2 - 1 \tag{B.14}$$

which validates if the scaling is performed correctly. I prefer to additionally normalize the amplitude spectrum by the recording-time, T, used for the DFT (i.e. multiplication by Δf). This results in the average rate in which energy is being transferred. This rate must still be integrated over a finite time-period to represent energy. In this thesis I integrate over 1 s. The unit for power levels is $[V^2 s]$ and equivalently the unit for amplitude is $[V\sqrt{s}]$. The amplitude spectrum then corresponds to the amplitudes of the time domain sinusoidal basis functions that are used to decompose the time series.

Amplitude and power spectra have a large dynamic range and are therefore often studied on a logarithmic scale with a reference variable. For a power spectrum we define a decibel as:

$$\mathbf{P}(f_m)[dB] = 10\mathrm{Log}_{10}\left(\frac{\mathbf{P}(f_m)[nm/s]}{\alpha_{ref}}\right). \tag{B.15}$$

For an amplitude spectrum we define a decibel as

$$\mathbf{S}(f_m)[dB] = 10\mathrm{Log}_{10}\left(\frac{\mathbf{S}^2(f_m)[nm/s]}{\alpha_{ref}^2}\right) = 20\mathrm{Log}_{10}\left(\frac{\mathbf{S}(f_m)[nm/s]}{\alpha_{ref}}\right). \tag{B.16}$$

The factor 20 stems from the square and keeps the decibel levels similar for amplitude and power spectra. The unit is typically denoted as ($[dB$ re $1\alpha_{ref}]$), where α_{ref} is the reference level. Standard reference levels for pressure are $p_{ref} = 20[\mu Pa\sqrt{s}]$ (in air) and $p_{ref} = 1[\mu Pa\sqrt{s}]$ (under water) and for particle velocity $v_{ref} = 50[nm/s\sqrt{s}]$.

Instead of looking at the power spectrum of one signal we could study the cross spectrum between two signals, **G** and **H**:

$$\mathbf{C}(f_m) = 2\left(\mathbf{G}(f_m)\mathbf{H}^*(f_m)\right) \tag{B.17}$$

After inverse Fourier transformation this yields a crosscorrelation function, which is best normalized by time to be insensitive to the amount of time crosscorrelated. The unit of this crosscorrelation signal is now $[V^2 s]$.

Figure B.1 shows the discrete amplitude and power spectra computed for a sinusoid. The input signal $\sin(2\pi f_0 t)$ with $f_0 = 2.5$ Hz in Figure B.1a, the absolute squared in Figure B.1b, the coefficients of the discrete amplitude spectrum in Figure B.1c, and the coefficients of the discrete amplitude spectrum in Figure B.1d. Evaluating Parceval's theorem in the time-domain (averaging Figure B.1b and integrating over 1 s) results in 0.5 V^2s, evaluating Parceval's theorem in the frequency-domain (summing the coefficients in Figure B.1d) results in 0.5 V^2s.

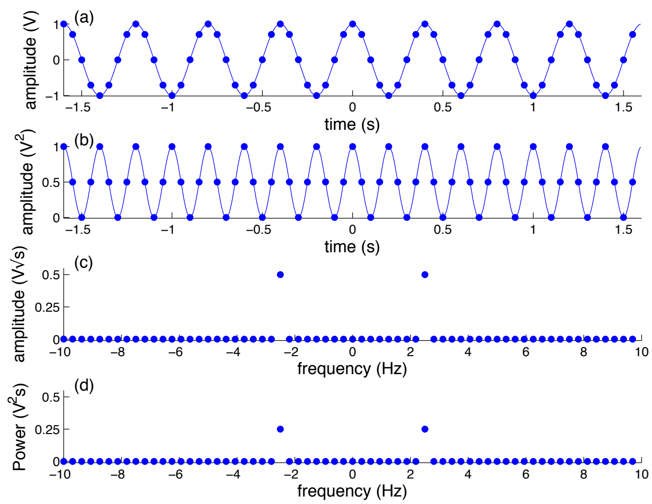

Figure B.1: a) Recorded signal b) It's absolute value squared, c) the coefficients of the discrete amplitude spectrum, and d) the coefficients of the discrete amplitude spectrum. [**ER**] spectra

Finally we employ the following definition for the spatial Fourier transformation:

$$G(\xi) = \mathcal{F}_x\left\{g(x)\right\} = \int_{-\infty}^{\infty} \mathrm{e}^{i2\pi\xi x} g(x)dx, \tag{B.18}$$

and its inverse

$$g(x) = \mathcal{F}_t^{-1}\left\{G(x)\right\} = \int_{-\infty}^{\infty} \mathrm{e}^{-i2\pi\xi x} G(x)d\xi. \tag{B.19}$$

The spatial frequency ξ is regularly substituted by the wave number $k = 2\pi\xi$. Equations B.18 and B.19 are discretized similarly as Equations B.1 and B.2.

Further reading suggestions are Erbe (2011) and Bracewell (2000).