WAVE EQUATION DECONVOLVED TIME SECTIONS

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Introduction

Research efforts in wave equation multiple deconvolution can be classified according to how accurately they account for the effects of dip and of offset on seismic sections. Riley (SEP-3) developed a model valid for small offset and a range of dip. Although this approach provided a useful framework for initial investigations, it had limited success in practice because marine reflection data is not recorded about zero offset. For this reason Estevez (SEP-12) was led to consider the problem from the "slant frame" viewpoint. His simple theory, without diffractions, was able to incorporate wide offsets together with a small range of dip and allowed significant water depth differences at shots and geophones. His more general theory included diffraction terms and a wider range of dips.

Our current study was motivated by the belief that Estevez's analysis excluded an additional first order effect - the effect of vertical velocity variation. Since the velocity contrasts at the bottom of the water layer can easily be as much as 3:2, and since the lateral shifts required in coupling up- and downgoing waves turn out to be proportional to the square of velocity, this can indeed be a significant consideration.

This paper extends Estevez's theory to account for vertical velocity variations and presents two serendipitous results. The first of these is that the slant stacking process is the only way to preserve a constant reverberation period for nonzero offset data. This can be shown from purely geometrical considerations and supports Estevez's contention that slant stacks are a good starting point for the inverse problem. The second surprise is that an unexpected coordinate transformation is required to produce a time section $c(\tau)$ rather than a depth section $c(z)$. The former product is much preferred for seismic interpretation.

Uniform Reverberation Period of the Slant Stack

Correct prediction and suppression of multiples in marine prospecting requires an almost uniform reverberation time between primary events and their associated multiples. If the multiples obey these "natural" timing relationships, then construction of inverse filters to deconvolve the multiple signatures is greatly facilitated. It turns out that slant stacking the data ensures that the multiples will have these natural timing relationships in a Z-variable velocity medium. This can be seen by referring to Figure 1.

In Figure 1, the four hyperbolae represent the primary and pegleg reflections for the ray paths depicted on the left. The lines EA, FD, E'A', and F'D' have all been constructed tangent to arrivals with common slowness p_0. Now AC = A'C', since both of these lines represent the two-way traveltime of a ray of constant ray parameter p_0 through layer 1. Furthermore, CD = C'D' is the horizontal excursion of a ray with parameter p_0 which has made two trips through layer 1. Also, $\angle ACD = \angle A'C'D' = 90^o$, CD = C'D' and $\angle BDC = \angle B'D'C'$. Therefore, by construction we must have $\Delta BCD \sim \Delta B'C'D'$. This, in turn, means that AB = A'B' = EF = E'F'. We can conclude from this that the slant stack does indeed preserve the time separation of multiples from their primaries for this model. The same principles used above allow us to extend the claim to a model with any number of layers.

In addition to this most important property, the slant stack has a number of other desirable characteristics. Among these are the fact that the underlying theory can account for two-dimensional wave phenomena, and incoherent noise is attenuated. The slant stack also correctly accounts for the angular dependence of the reflection coefficients since it attenuates energy arriving with ray parameter outside a small, specified range.

It is of interest to compare the results of slant stacking with another method which focuses on events at constant p -- Taner's Radial Trace Deconvolution (1975). A radial trace in a constant velocity medium is a one-dimensional cut through a gather taken along a line of constant stepout (see Figure 2). In a constant velocity medium both the radial trace and the slant stack preserve the desired multiple timing relationships (AD = A'D' and EF = E'F'). If velocity varies as a step function in z, then the radial trace is taken along an appropriate piecewise linear trajectory (see Figure 1). In each segment the radial trace has slope $(dx/dt)_i = pv_i^2$. If we were to

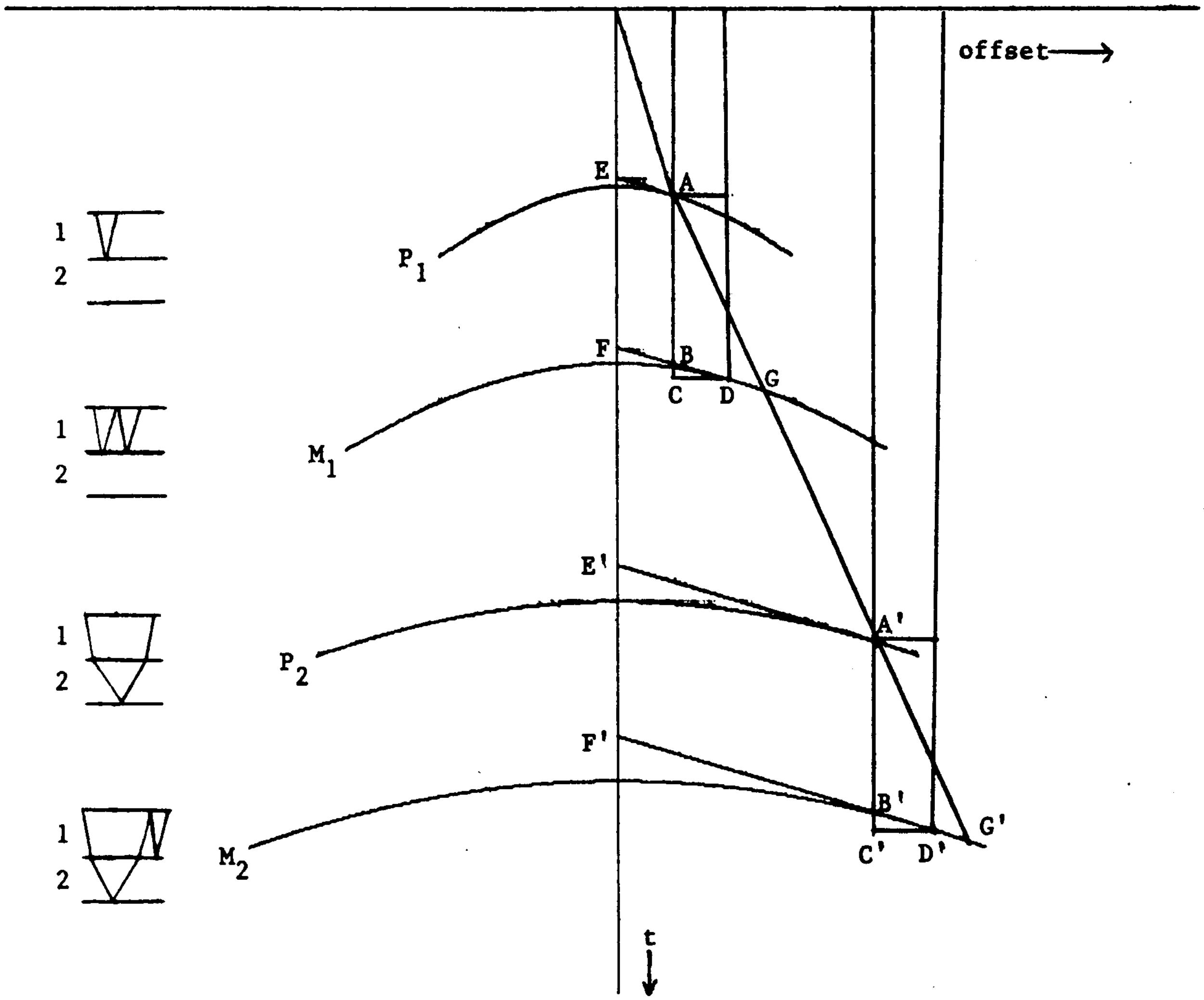

FIGURE 1.--A two-layer model with $v_2 > v_1$. Note that the time separation of P_1M_1 is not equal to the separation of P_2M_2 on the radial trace ($AG \neq A'G'$). Nevertheless, an ideal slant stack preserves a uniform reverberation period (i.e. $EF = E'F'$).

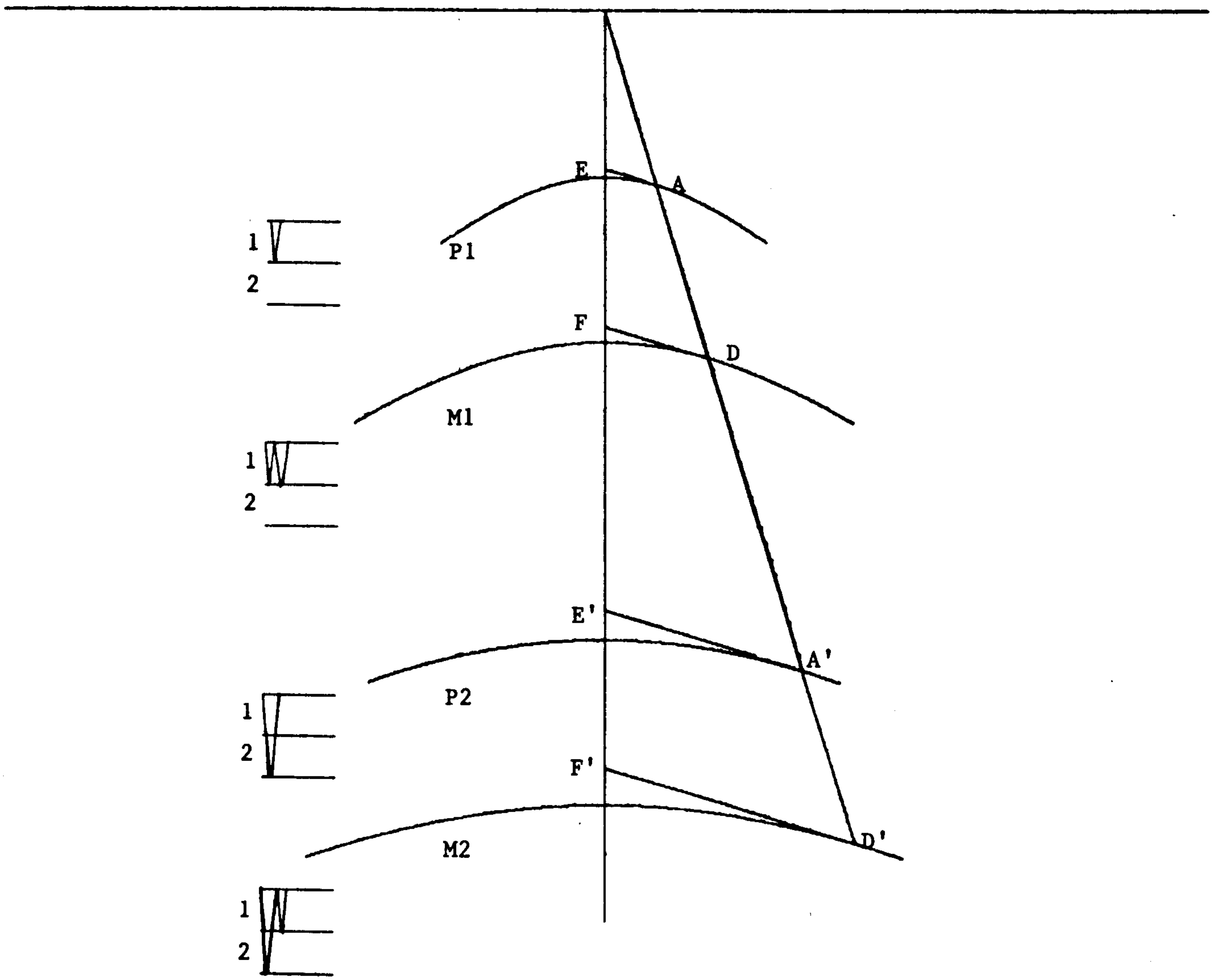

FIGURE 2.--A two-reflector model for a constant velocity medium. Note that a uniform reverberation period is maintained on both the radial trace and the ideal slant stack (AD = A'D' and EF = E'F').

imagine a continuum of horizontal reflectors, then the radial trace would be the line passing through each primary hyperbola at the point where the tangent to that event had slope p.

A radial trace defined in this manner only has the correct multiple timing relationships for constant velocity. This does not mean that the slant stack approach is always preferable since the radial trace maintains a nearly uniform reverberation period for moderate velocity contrasts. In any given situation we face a tradeoff between imperfect multiple timing relationships of the radial trace and waveform distortion on the slant stack caused by spatial aliasing.

Coupled Vertical Incidence Equations

Before considering the effect of z-variable velocity on the coupling of up- and downgoing waves in slant frames, it will be useful to address the problem for vertically incident plane waves. This is not a difficult problem by itself but it will illustrate one of the basic principles involved in our selection of slant frame coordinates. An obvious pair of coordinate frames to remove the stretch effect of variable velocity are:

$$
\begin{aligned}
x'' &= x' = x \\
\tau'' &= \tau' = 2 \int^{z} \frac{dz}{v} \\
t'' &= t + \int^{z} \frac{dz}{v} \\
t' &= t - \int^{z} \frac{dz}{v}
\end{aligned}
\qquad (1)
$$

τ represents the two-way traveltime of a ray traveling from the surface to depth z.

We will begin by considering the (coupled) 15-degree equation in unprimed coordinates (Claerbout, 1976, Equation 10-5-16b).

$$U_{zt} = \frac{1}{v} U_{tt} - \frac{v}{2} U_{xx} - c(x,z)\, D_t \qquad (2)$$

Using the operator identities:

$$\partial_x = \partial_{x'} = \partial_{x''}$$
$$\partial_t = \partial_{t'} = \partial_{t''} \tag{3}$$
$$\partial_z = \frac{2}{v}\frac{\partial}{\partial\tau'} - \frac{1}{v}\frac{\partial}{\partial t'} = \frac{2}{v}\frac{\partial}{\partial\tau''} + \frac{1}{v}\frac{\partial}{\partial t''}$$

and the invariance relations

$$U(x,z,t) = U'(x',\tau',t') = U''(x'',\tau'',t'')$$
$$D(x,z,t) = D'(x',\tau',t') = D''(x'',\tau'',t'')$$
$$c(x,z) = c'(x',\tau') = c''(x'',\tau'') \tag{4}$$
$$v(z) = v'(\tau') = v''(\tau'')$$

(2) becomes

$$\frac{2}{v''}U''_{\tau''t''} + \frac{1}{v''}U''_{t''t''} = \frac{1}{v''}U''_{t''t''} - \frac{v''}{2}U''_{x''x''} - c''(x'',\tau'')\,D''_{t''}(x'',\tau'',t'') \tag{5}$$

or

$$\frac{2}{v''}U''_{\tau''} = -\frac{v''}{2}U''^{\,t''}_{x''x''} - c''(x'',\tau'')\,D'(x'{=}x'',\tau'{=}\tau'',t'{=}t''-\tau'') \tag{6}$$

The accompanying downward-continuation equation is

$$D'_{\tau'} = \left(\frac{v'}{2}\right)^2 D^{t'}_{x'x'} \tag{7}$$

We can conclude that algorithmically, the stratified velocity/vertical incidence problem is identical to the constant velocity problem, but the reflection coefficients are scaled by $v''(\tau'')/2$. If we ignore the diffraction term and absorb the scale factor into c'', the solution to (6) is given by

$$U'' = \int c''(x'',\tau'')\,D'(x'',\tau'',t''-\tau'')\,d\tau'' \tag{8}$$

Now, since $D'_{\tau'} = 0$, we have the result that

$$\dot{U}'' = c'' *_{t''} D' \tag{9}$$

where $*_{t''}$ represents convolution over the t'' variable.

Apart from the coefficient of $U''_{\tau''}$ and $D'_{\tau'}$, Equations (6) and (7) are of the same form as the constant velocity/vertical incidence equations derived by Claerbout (1976, p. 258). Although the transformations outlined in *Fundamentals of Geophysical Data Processing* were successful in eliminating translation terms, the t-coupling was only of convolutional form for constant velocity. Equation (6) is of convolutional form for any general $v(z)$.

Slanted Coordinate Coupled Equations

In two adjoining SEP-7 papers, Claerbout (p. 30 and p. 33) derived a pair of slanted wave equations valid for constant velocity. The final result of the second paper was:

$$\partial_{z''} U''(t'',x'',z'') = -\frac{v}{2\cos^3\theta}\,\partial^{t''}_{x''x''}\,u''(t'',x'',z'')$$

$$- c''(x'',z'')\,D'(t'' - \frac{2z''}{v}\cos\theta, x''-2z''\tan\theta, z'') \tag{10a}$$

$$\partial_{z'} D'(t',x',z') = \frac{v}{2\cos^3\theta}\,\partial^{t'}_{x'x'}\,D'(t',x',z') \tag{10b}$$

The coordinate transformations used in this derivation were

$$\begin{aligned} x'' &= x + z\tan\theta \\ x' &= x - z\tan\theta \end{aligned} \tag{11a}$$

$$z'' = z' = z \tag{11b}$$

$$\begin{aligned} t'' &= t + \frac{z\cos\theta}{v} - \frac{x\sin\theta}{v} \\ t' &= t - \frac{z\cos\theta}{v} - \frac{x\sin\theta}{v} \end{aligned} \tag{11c}$$

The double prime transformations were constructed so that a plane wavefield, incident at angle θ to the horizontal in a medium of constant velocity v, remained fixed in the coordinate frame. The single prime coordinates define a corresponding downgoing frame.

In seeking a similar transformation valid for z-variable velocity we are guided by the following two basic principles:

1) The downward continuation operators for U'' must not contain any first order terms in $\partial_{x''}$ or $\partial_{t''}$. Such terms only apply uniform translations in U'' with increasing z'' and invalidate our neglect of the Fresnel $\partial_{z''z''}$ terms.

2) The differential equation coupling U'' and D' must be expressible in the form

$$\mathrm{Op}(U'') = c''(x'',z'')\, D'[x'' - \chi''(z''),z'',t''-z''] \tag{12}$$

where χ'' is (for now) some undetermined function of z''.

As shown in the previous section, if $\mathrm{Op}(\;) \propto \partial/\partial z''$, the t'' dependence of U'' is obtained by convolving c'' with D' over z''. For this reason we say that (12) represents the "convolutional" property of the coordinate frame. A coordinate frame having this property will have a computationally efficient solution and, more important, will ensure that our final product is a time section. In practice this will allow us to see if our attempts at multiple suppression have succeeded or failed since residual multiple reflections will have their familiar timing relationships.

A set of coordinate transformations which meets the above two criteria is

$$\begin{aligned} x'' &= x + \int^{z} \tan\theta \, dz \\ x' &= x - \int^{z} \tan\theta \, dz \end{aligned} \tag{13a}$$

$$\begin{aligned} \tau'' &= 2 \int_0^{z} \frac{\cos\theta}{v} \, dz \\ \tau' &= 2 \int_0^{z} \frac{\cos\theta}{v} \, dz \end{aligned} \tag{13b}$$

$$t'' = t - px + \int^{z} \frac{\cos\theta}{v}\, dz$$

$$t' = t - px - \int^{z} \frac{\cos\theta}{v}\, dz \qquad (13c)$$

The ray parameter is $p = \sin\theta/v$ — a constant along any raypath. We should also note here that $\cos\theta = (1-p^2v^2)^{1/2}$ and $\tan\theta = pv(1-p^2v^2)^{-1/2}$. The transformations (13a) and (13c) have been recommended previously by Estevez (SEP-5, p. 34). They can be seen by inspection to be the natural extension of Equations (11a) and (11c).

The physical significance of (13b) is not as apparent. Our first guess at choosing this coordinate transformation was to put $\cos\theta$ in the denominator of (13b). This would represent the two-way traveltime along a raypath. This guess, however, was inconsistent with the previously mentioned convolution criterion. It turns out that τ'' is the product of depth with the mean projection of the vertical slowness of a wavefront onto its raypath (see Figure 3).

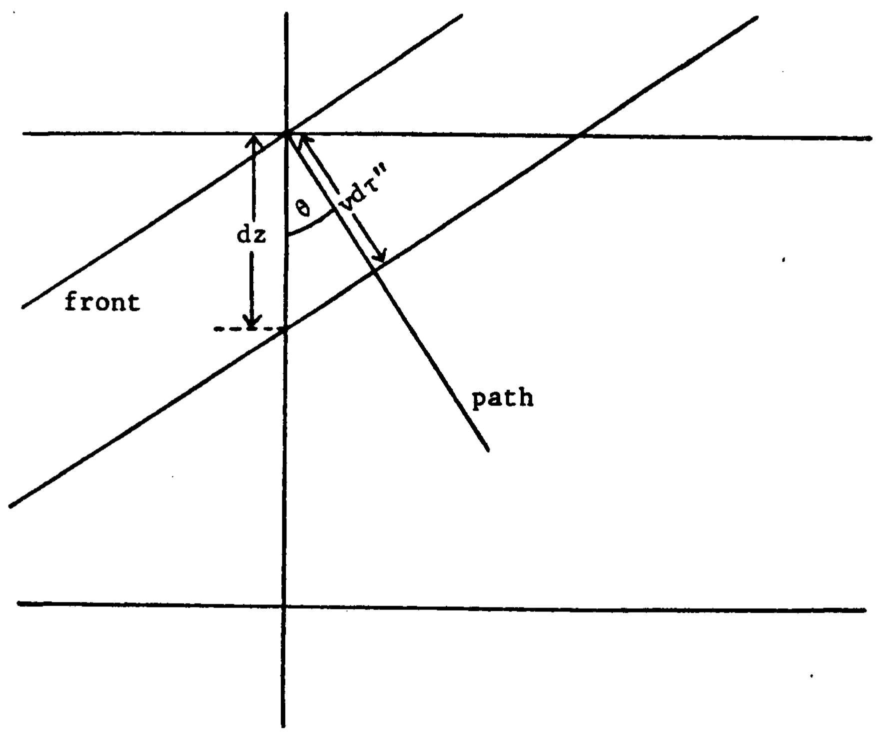

FIGURE 3.-- $d\tau''$ = [incremental depth, dz][projection of vertical slowness of the front onto the path, $(\cos\theta/v)$]

The wave equation for U in (x,z,t) space is

$$(\partial_{xx} + \partial_{zz} - \frac{1}{v^2}\partial_{tt})\, U = 0 \tag{14}$$

Using the chain rule for differentiation,

$$\partial_t = \partial_{t''} \tag{15a}$$

$$\partial_z = \tan\theta\, \partial_{x''} + \frac{2\cos\theta}{v}\partial_{\tau''} + \frac{\cos\theta}{v}\partial_{t''} \tag{15b}$$

$$\partial_x = \partial_{x''} - p\,\partial_{t''} \tag{15c}$$

Equation (14) becomes

$$\Big\{\partial_{x''x''} - 2p\partial_{x''t''} + p^2\partial_{t''t''} + \tan^2\theta\,\partial_{x''x''} + \frac{4\cos^2\theta}{v^2}\partial_{\tau''\tau''}$$

$$+ \frac{\cos^2\theta}{v^2}\partial_{t''t''} + \frac{2\sin\theta}{v}\partial_{x''t''} + \frac{4\cos^2\theta}{v^2}\partial_{\tau''t''} + \frac{4\sin\theta}{v}\partial_{x''\tau''}$$

$$+ \left[\left\langle\frac{2\cos\theta}{v}\partial_{\tau''}(\tan\theta)\right\rangle\partial_{x''} + \left\langle\frac{2\cos\theta}{v}\partial_{\tau''}(\frac{\cos\theta}{v})\right\rangle (2\partial_{\tau''} + \partial_{t''})\right]$$

$$- \frac{1}{v^2}\partial_{t''t''}\Big\}\, U'' = 0 \tag{16}$$

The $\partial_{\tau''}$ operators within the "⟨ ⟩" brackets only operate on $\tan\theta$ and $\cos\theta/v$. The transmission terms in the square brackets may now be dropped provided that $v''_{\tau''}$ is sufficiently small. Using the paraxial approximation, we also choose to neglect the term in $\partial_{\tau''\tau''}$.

A further simplification is possible if we restrict our attention to small dip. This justifies dropping the term in $\partial_{x''\tau''}$. Using all these approximations, recalling that $p = \sin\theta/v$, and adding the coupling term, gives

$$(\sec^2\theta\,\partial^{t''}_{x''x''} + \frac{4\cos^2\theta}{v''^2}\partial_{\tau''})\, U'' = -c''(x'',\tau'')\, D''(x'',\tau'',t'') \tag{17}$$

The corresponding equation for downward continuation is

$$\left(\sec^2\theta\, \partial^{t'}_{x'x'} - \frac{4\cos^2\theta}{v'^2}\, \partial_{\tau'}\right) D' = 0 \tag{18}$$

Recalling the definition of τ'' [Equation (13b)] and the statements of invariance, we can rewrite the right-hand side of (17) as

$$-c''(x'',\tau'')\; D'(x'=x'' - 2\int^{z} \tan\theta\, dz, \tau'=\tau'', t'=t''-\tau'') \tag{19}$$

We now have our desired "convolutional form" in t'.

Using the fact that

$$\frac{dz}{dz''} = \frac{v}{2\cos\theta} \tag{20}$$

together with the identities for $\tan\theta$ and $\cos\theta$, we can rewrite the integral-shifting term in (19) as

$$p \int^{\tau''} \frac{v''^2}{1-p^2 v''^2}\, d\tau'' \tag{21}$$

This is the "χ-shift" term anticipated by Equation (12).

Equations (17) to (19) have the same general structure as Estevez's (SEP-12, p. 34) coupled slant equations but have one notable difference: the "χ-shift" is now a non-linear function of τ''. At small angles, it approximates the mean square velocity, and it becomes infinite as the critical angle is approached.

Summary

In this paper we have extended Estevez's results to account for the effects of z-variable velocity on coupled up- and downgoing wave equations in slant coordinate frames. In doing this we have been able to retain the convolutional form over the t'' coordinate. This was useful since it guaranteed that our final product was a time section rather than a depth section. Finally, we have shown that the slant stack theoretically preserves a uniform reverberation period and is therefore a good starting point for the inverse problem.

REFERENCES

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