One-way Wave Equations for Variable Velocity Media

by David Brown

In this paper, the physical interpretation of various one-way wave equations is discussed in terms of the WKBJ theory for asymptotic expansion solutions to the acoustic wave equation.

Begin by reviewing the derivation of the 15-degree equation for constant velocity media. The acoustic wave equation may be written as

$$P_{zz} + P_{xx} = (1/v^2)\ P_{tt}, \qquad (1)$$

where for the moment, v is a constant. This equation may then be factored into two parts: one for upgoing waves and one for downgoing waves:

$$(\partial_z - [(1/v^2)\partial_{tt} - \partial_{xx}]^{1/2})(\partial_z + [(1/v^2)\partial_{tt} - \partial_{xx}]^{1/2})P = 0. \qquad (2)$$

To get the 15-degree equation, a simple approximation to the square-root can be made:

$$\partial_z P = [(1/v^2)\partial_{tt} - \partial_{xx}]^{1/2} = (1/v)\partial_t[1 - v^2\partial^{tt}_{xx}]^{1/2}\ P$$

$$\tilde{=}\ (1/v)\partial_t(\ 1 - [v^2/2)\partial^{tt}_{xx}\ ]\ P$$

$$==> \qquad P_{zt} = (1/v)P_{tt} - (v/2)P_{xx}. \qquad (3)$$

When the problem is generalized to the case where the velocity is a function of both space variables, x and z, the factoriztion given by equation (2) is not valid, because the velocity, v, and the space derivative operators, ∂_z and ∂_x , do not commute. Clearly, another approach must be taken.

One possibility is to develop one-way wave equations in terms of the WKBJ asymptotic expansion solutions to the wave equation. WKBJ solutions are valid when the wavelengths of the acoustic waves of interest are short in comparison to the inhomogeneities of the medium. In reflection seismology the appropriate description is that the velocity is a "slowly varying function of x and z". This method of solution is reviewed in the next paragraph.

To simplify the discussion, consider the acoustic wave equation in one space-variable:

$$P_{zz} = (1/v^2)\, P_{tt}. \qquad (5)$$

To solve this equation, assume that a solution exists which is of the form

$$P(z,t) = \exp(\theta(z)\partial_t - t\partial_t). \qquad (6)$$

To find $\theta(z)$, assume that it may be expressed in terms of an asymptotic expansion in ∂_t:

$$\theta(z) = A(z) + B(z)\partial^t + C(z)\partial^{tt} + \ldots \qquad (7)$$

Once the functions A,B,C and so on have been determined, they can be substituted back into equation (6) to obtain an approximation to the solution of (5) or alternatively, the differential equation which will give the asymptotic expansion solution exactly can be derived. The latter approach is taken here.

Begin by including only the first term in the expansion for $\theta(z)$, i.e. let $\theta(z) = A(z)$. Substituting into (6) and differentiating twice with respect to z,

$$P_{zz} = (A_{zz}\partial_t + A_z^2\partial_{tt})P.$$

Substituting this into (5), an ordinary differential equation for A results:

$$([A_z^2 - (1/v^2)]\partial_{tt} + A_{zz}\partial_t)P = 0. \qquad (8)$$

The coefficients of the different orders of ∂_t must individually be equal to zero. This gives that

$$A_z = \pm(1/v) \qquad (9a)$$

and

$$A_{zz} = \mp(v_z/v) = 0. \qquad (9b)$$

The second relation indicates that this approximation is exact if the velocity doesn't depend on the spatial variables; the first equation can be integrated to get A, which gives the first order WKBJ solution to (5):

$$P(z,t) = e^{i\omega[\int(dz/v) - t]}. \qquad (10)$$

It is simple, as well, to get the one-way equation which will give (10) as its exact solution. Taking one derivative with respect to z of (6),

$$P_z = A_z \partial_t P,$$

or substituting from (9),

$$P_z = \pm(1/v)P_t. \qquad (11)$$

Comparison of (11) with (2) shows that this is just the one-dimensional equivalent of the one-way equations from which the more familiar migration equations are derived.

Higher-order asymptotic solutions and the corresponding one-way differential equations can be obtained by including more of the terms in the expansion for θ (equation 7). The general form of the one-way equations will be

$$P_z = (A_z\partial_t + B_z + C_z\partial^t + D_z\partial^{tt} + \ldots)P, \quad (12)$$

where the coefficients A,B,C,D,... are determined by substituting the expression

$$P_{zz} = [A_{zz}\partial_t + B_{zz} + C_{zz}\partial^t + D_{zz}\partial^{tt} + \ldots + (A_z\partial_t + B_z + C_z\partial^t + D_z\partial^{tt} + \ldots)^2]P$$

into (5) and setting the coefficients of different orders of ∂_t equal to zero. The second-order asymptotic one-way equations and solutions are given by

$$P_z = \pm(1/v)P_t + (v_z/2v)P \qquad (13a)$$

and

$$P(z,t) = (v_o/v)e^{i\omega[\int(dz/v) - t]} . \qquad (13b)$$

Inspection of equations (10) and (13b) and solutions for higher-order expansions indicates what the new terms in the one-way differential equations mean. The first-order solution (10) expresses a simple phase delay as a function of time and depth. The second order solution includes first-order amplitude effects due to the changes in the velocity of the medium. If terms with coefficients which depend on C in the asymptotic expansion are included, the corresponding correction term in the equation will amount to a second-order phase correction in the solution, due also to the variation of the velocity with z. The differential equation which results will be

$$P_z = \pm(1/v)P_t + (v_z/2v)P \pm[(v_z^2/8v) - (v_{zz}/4)]\partial^t P. \qquad (14)$$

The extension of this method of development of one-way equations to two space dimensions is quite difficult. An alternate method for deriving equations for the two-dimensional case is given by Björn Engquist in SEP-13. In that paper, the first-order amplitude terms and the second order phase correction terms are derived for the 15-degree equation. The result is repeated below:

$$\partial_t P = [-(1/v)\partial_t P + (v/2)\partial^{t}_{xx} P] + [(v_z/2v)P + (v_x/2)\partial^{t}_{x} P] + \{[(v_{zz}/4) - (v_z^2/8v)] + [(v_{xx}/4) - (v_x^2/4v)]\}\partial^{t} P \qquad (15)$$

Comparison with the one-dimensional case above allows identification of the significance of the terms in this equation: The first group of terms on the right-hand side of the equation represent the first-order propagation terms. The second group is the first-order amplitude terms, and the third group represent the second-order phase correction terms. Engquist's method is not limited in it's application to the 15-degree equation. It appears that at the moment, it is also the more tractable of the two methods for deriving higher-order one-way equations. The WKBJ approach can then be applied to the one-dimensional problem in order to identify the resulting terms.