IMAGE-SPACE WAVE-EQUATION TOMOGRAPHY IN THE SHOT-PROFILE DOMAIN ACCORDING TO THE ADJOINT-STATE METHOD

Image-space wave-equation tomography

Image-space wave-equation tomography aims to solve for the slowness model, $s = s(\mathbf{x})$, that minimizes the linearized objective function

$$J(\mathbf{s}) = \frac{1}{2} \left|\left|\Delta r\right|\right|^2 , \tag{1}$$

where $\Delta r = \Delta r(\mathbf{x}, \mathbf{h})$ is the image perturbation which measures the goodness of the slowness model. Δr is computed either by applying wave-equation migration-velocity analysis (WEMVA) (Sava and Biondi, 2004a,b) or differential-semblance optimization (DSO) (Shen and Symes, 2008) operators to the image $r = r(\mathbf{x}, \mathbf{h})$. Here, these operators are called indistinctively $\mathbf{M}$. Therefore, the objective function reads

$$J(s) = \frac{1}{2} \left|\left|\mathbf{M}r\right|\right|^2 . \tag{2}$$

If $\mathbf{M}$ is independent on the slowness, the gradient of this objective function, evaluated at the current slowness, $\widehat{s} = \widehat{s}(\mathbf{x})$, is

$$\nabla J(s) = \left(\frac{\partial \widehat{r}}{\partial s}\right)' \Bigg|_{s=\widehat{s}} \mathbf{M}'\mathbf{M}\widehat{r}. \tag{3}$$

where ‘$'$’ denotes the adjoint and $\widehat{r} = \widehat{r}(\mathbf{x}, \mathbf{h})$ is the image obtained with the current slowness model. The linear operator $\frac{\partial \widehat{r}}{\partial s}$ defines a mapping between the slowness perturbation Δs to the image perturbation Δr, and it is called image-space wave-equation tomographic operator.

The image-space wave-equation tomographic operator is composed of different operators. However, this is not clear from the representation of equation 3. Therefore, for a clear explanation of the operators involved, I use the adjoint-state method to derive the gradient of the objective function in equation 2.

In shot profile migration, the source and receiver wavefields are propagated independently and the image, $r_z = r_z(\mathbf{x}, \mathbf{h})$, at a depth level z, considering just one shot, is computed by the crosscorrelation

$$r_z(\mathbf{x}, \mathbf{h}) = \sum_{\omega} p_z^*(\mathbf{x} - \mathbf{h}, \omega) u_z(\mathbf{x} + \mathbf{h}, \omega), \tag{4}$$

where $p_z(\mathbf{x}, \omega)$ is the source wavefield for a single frequency ω at horizontal coordinates $\mathbf{x} = (x, y)$; $u_z(\mathbf{x}, \omega)$ is the receiver wavefield and $\mathbf{h} = (h_x, h_y)$ is the subsurface half-offset, and ‘$*$’ stands for the complex-conjugate. In a more compact notation, not explicitly writing the dependencies on $\mathbf{x}$ and $\mathbf{h}$, equation 4 can be written as:

$$r_z = \mathbf{S}\mathbf{P}'_z(\omega) u_z(\omega) = \mathbf{S}\mathbf{U}_z(\omega) p_z^*(\omega), \tag{5}$$

where $\mathbf{P}$ and $\mathbf{U}$ are convolutional matrices composed of (h_x, h_y)-shifted versions of $p_z(\mathbf{x}, \omega)$ and $u_z(\mathbf{x}, \omega)$, respectively. Operator $\mathbf{S}$ corresponds to the summation over frequency.

For subsequent depth levels, $p(\mathbf{x}, \omega)$ is computed by means of the recursive downward propagation

$$\begin{cases} p_{z+1}(\omega) = T_z^{\downarrow}(\omega, s) p_z(\omega) \\ p_1(\omega) = f_s(\omega) \delta(\mathbf{x} - \mathbf{x_s}), \end{cases} \tag{6}$$

where $T_z^{\downarrow}$ is the downward continuation operator, which is function of the slowness, s, and $f_s(\omega)$ is the source signature located at $\mathbf{x}_s = (x_s, y_s, 0)$.

The downward continuation of the receiver wavefield is performed by

$$\begin{cases} u_{z+1}(\omega) = T_z^{\downarrow}(\omega, s) u_z(\omega) \\ u_1(\omega) = d(\omega), \end{cases} \tag{7}$$

where $d(\omega)$ is the data at the surface. In equations 6 and 7, I omitted the dependencies of the wavefield with respect to $\mathbf{x}$.

In the image-space wave-equation tomography problem, the perturbed source and receiver wavefields and image perturbations are used to compute the slowness perturbation to update the current slowness model. From the perturbation theory, we have that $p + \Delta p$, $u + \Delta u$ and, consequently, $r + \Delta r$ are physical realizations with $s + \Delta s$. To the first order, these perturbed fields are given by

$$\Delta p_{z+1}(\omega) = T_z^{\downarrow}(\omega, s) \Delta p_z(\omega) + \Delta T_z^{\downarrow}(\omega, s) p_z(\omega), \tag{8}$$

$$\Delta u_{z+1}(\omega) = T_z^{\downarrow}(\omega, s) \Delta u_z(\omega) + \Delta T_z^{\downarrow}(\omega, s) u_z(\omega), \tag{9}$$

In equations 8 and 9, $\Delta T_z^{\downarrow}$ is the scattering operator

$$\Delta T_z^{\downarrow}(\omega, s) = i \frac{\omega^2 s}{\sqrt{\omega^2 s^2 - |\mathbf{k}|^2}} dz \Delta s. \tag{10}$$

Its derivation is provided in the Apendix A.

The perturbed image is

$$\Delta r_z = \mathbf{S} \left(\Delta \mathbf{P}'_z(\omega) u_z(\omega) + \mathbf{P}'_z(\omega) \Delta u_z(\omega) \right) = \mathbf{S} \left(\mathbf{U}_z(\omega) \Delta p_z^*(\omega) + \mathbf{P}'_z(\omega) \Delta U_z(\omega) \right). \tag{11}$$

Notice that the perturbations are evaluated around the current slowness, $\widehat{s}$. Therefore, equations 8, 9 and 11 read

$$\Delta p_{z+1}(\omega) = T_z^{\downarrow}(\omega, \widehat{s}) \Delta p_z(\omega) + \widetilde{p}_z(\omega) \Delta s_z, \tag{12}$$

$$\Delta u_{z+1}(\omega) = T_z^{\downarrow}(\omega, \widehat{s}) \Delta u_z(\omega) + \widetilde{u}_z(\omega) \Delta s_z, \tag{13}$$

$$\Delta r_z = \mathbf{S}\left(\widehat{\mathbf{U}}_z(\omega)\Delta p_z^*(\omega) + \widehat{\mathbf{P}}'_z(\omega)\Delta u_z(\omega)\right). \tag{14}$$

The scattered wavefields $\widetilde{p}_z$ and $\widetilde{u}_z$ are given by

$$\widetilde{p}_z(\omega) = i\frac{\omega^2\widehat{s}}{\sqrt{\omega^2\widehat{s}^2 - |\mathbf{k}|^2}}dz\ \widehat{p}_z(\omega), \tag{15}$$

and

$$\widetilde{u}_z(\omega) = -i\frac{\omega^2\widehat{s}}{\sqrt{\omega^2\widehat{s}^2 - |\mathbf{k}|^2}}dz\ \widehat{u}_z(\omega). \tag{16}$$

The matrix representation for equations 12, 13, 14 is

$$\Delta\underline{\mathbf{p}} = \mathbf{T}^{\downarrow}\Delta\underline{\mathbf{p}} + \widetilde{\mathbf{P}}\mathbf{S}'\Delta\mathbf{s}, \tag{17}$$

$$\Delta\underline{\mathbf{u}} = \mathbf{T}^{\downarrow}\Delta\underline{\mathbf{u}} + \widetilde{\mathbf{U}}\mathbf{S}'\Delta\mathbf{s}, \tag{18}$$

$$\Delta\underline{\mathbf{r}} = \mathbf{S}\left(\widehat{\mathbf{U}}\Delta\underline{\mathbf{p}}^* + \widehat{\mathbf{P}}'\Delta\underline{\mathbf{u}}\right). \tag{19}$$

where $\widetilde{\mathbf{P}}$ and $\widetilde{\mathbf{U}}$ are diagonal operators containing the scattered source and receiver wavefields, respectively.

Equations 17, 18 and 19 are the forward equations of the image-space wave-equation tomography problem using a shot profile scheme. They depend on the state variables $\Delta\underline{\mathbf{p}}$, $\Delta\underline{\mathbf{u}}$ and $\Delta\underline{\mathbf{r}}$. The augmented functional reads

$$\begin{aligned}\mathcal{L}(\Delta\underline{\mathbf{p}}, \Delta\underline{\mathbf{u}}, \Delta\underline{\mathbf{r}}, \underline{\lambda}_p, \underline{\lambda}_u, \underline{\lambda}_r; \Delta\mathbf{s}) = \mathcal{R}\Big[\tfrac{1}{2}\,||\Delta\underline{\mathbf{r}}||^2 - &\\ \left\langle \underline{\lambda}_p, \left(\mathbf{I} - \mathbf{T}^{\downarrow}\right)\Delta\underline{\mathbf{p}} - \widetilde{\mathbf{P}}\mathbf{S}'\Delta\mathbf{s}\right\rangle - &\\ \left\langle \underline{\lambda}_u, \left(\mathbf{I} - \mathbf{T}^{\downarrow}\right)\Delta\underline{\mathbf{u}} - \widetilde{\mathbf{U}}\mathbf{S}'\Delta\mathbf{s}\right\rangle - &\\ \left\langle \underline{\lambda}_r, \Delta\underline{\mathbf{r}} - \mathbf{S}\left(\widehat{\mathbf{U}}\Delta\underline{\mathbf{p}}^* + \widehat{\mathbf{P}}'\Delta\underline{\mathbf{u}}\right)\right\rangle &\end{aligned}$$

The adjoint state variables are computed by taking the derivative of $\mathcal{L}$ with respect to the state variables and equal to zero, which gives

$$\left(\mathbf{I} - \mathbf{T}^{\downarrow}\right)'\underline{\lambda}_p = \widehat{\mathbf{U}}\underline{\lambda}_r, \tag{20a}$$
$$\left(\mathbf{I} - \mathbf{T}^{\downarrow}\right)'\underline{\lambda}_u = \widehat{\mathbf{P}}\underline{\lambda}_r, \tag{20b}$$
$$\underline{\lambda}_r = \Delta\underline{\mathbf{r}}. \tag{20c}$$

Notice that

$$\left(\mathbf{I} - \mathbf{T}^{\downarrow}\right)' = \left(\mathbf{I} - \mathbf{T}^{\downarrow'}\right) = \left(\mathbf{I} - \mathbf{T}^{\uparrow}\right) \tag{21}$$

corresponds to the upward propagation operator. Therefore, equations 20a and 20b, can be written as

$$\underline{\lambda}_p = \mathbf{T}^{\uparrow}\underline{\lambda}_p + \widehat{\mathbf{U}}\underline{\lambda}_r, \tag{22a}$$

$$\underline{\lambda}_u = \mathbf{T}^{\uparrow}\underline{\lambda}_u + \widehat{\mathbf{P}}\underline{\lambda}_r, \tag{22b}$$

which correspond to the recursive upward propagation of the perturbed wavefields resulting from the convolution of the wavefields computed with the current slowness and the perturbed image.

The gradient of J is

$$\nabla_s J(\mathbf{s}) = \mathbf{S}\left(\widetilde{\mathbf{P}}'\underline{\lambda}_p + \widetilde{\mathbf{U}}'\underline{\lambda}_u\right). \tag{23}$$

To compute the gradient, the upward propagated perturbed wavefields, $\underline{\lambda}_p$ and $\underline{\lambda}_u$, are crosscorrelated in time with the scattered wavefields.

REFERENCES

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APPENDIX A

The perturbed wavefields satisfy the one-way wave equation linearized with respect to the slowness. Take, for instance, the recursive solution of the one-way wave equation for the source wavefield

$$\mathbf{d}^{z+\Delta z} = e^{ik_z\Delta z}\mathbf{d}^z \tag{A-1}$$

to be linearized with respect to the slowness evaluated around the background slowness. The vertical wavenumber, k_z, can be defined as the sum of the background wavenumber, $\widehat{k}_z$, and the wavenumber perturbation, Δk_z. By substituting this definition for k_z into equation A-1 we get

$$\begin{aligned}\mathbf{d}^{z+\Delta z} &= e^{i\widehat{k}_z\Delta z}e^{i\Delta k_z\Delta z}\mathbf{d}^z \\ &= e^{i\Delta k_z\Delta z}\widehat{\mathbf{d}}^{z+\Delta z},\end{aligned} \tag{A-2}$$

where

$$\Delta k_z = \left.\frac{dk_z}{d\mathbf{s}}\right|_{\mathbf{s}=\widehat{\mathbf{s}}} \Delta\mathbf{s}, \tag{A-3}$$

and

$$\left.\frac{dk_z}{d\mathbf{s}}\right|_{\mathbf{s}=\widehat{\mathbf{s}}} = \frac{\omega^2 \widehat{s}}{\sqrt{\omega^2 \widehat{s}^2 - |\mathbf{k}|^2}},$$

where $\mathbf{k} = (k_x, k_y)$ is the spatial wavenumber vector.

The exponential in equation A-2 can be linearized using $e^{i\theta} \approx 1 + \theta$. With this linearization and equation A-3, the linearized version of equation A-2 reads

$$\mathbf{d}^{z+\Delta z} = (1 + i\frac{dk_z}{d\mathbf{s}}\Delta\mathbf{s}\Delta z)\widehat{\mathbf{d}}^{z+\Delta z}. \tag{A-4}$$

By rearranging the terms we finally get

$$\Delta\mathbf{d}^{z+\Delta z} = i\frac{dk_z}{d\mathbf{s}}\Delta\mathbf{s}\Delta z\widehat{\mathbf{d}}^{z+\Delta z} = e^{i\widehat{k}_z\Delta z}\{i\frac{dk_z}{d\mathbf{s}}\Delta\mathbf{s}\Delta z\widehat{\mathbf{d}}^z\}. \tag{A-5}$$

The perturbed source wavefield, $\Delta\mathbf{d}^{z+\Delta z}$, in equation A-5 is computed by applying the scattering operator, $i\frac{dk_z}{d\mathbf{s}}\Delta\mathbf{s}\Delta z$, to the background source wavefield at the previous depth level generating the scattered source wavefield. Then, the scattered source wavefield is propagated to the next depth level using the background slowness. The same reasoning can be applied to compute the perturbed receiver wavefield, which gives

$$\Delta\mathbf{u}^{z+\Delta z} = e^{-i\widehat{k}_z\Delta z}\{-i\frac{dk_z}{d\mathbf{s}}\Delta\mathbf{s}\Delta z\widehat{\mathbf{u}}^z\}. \tag{A-6}$$

APPENDIX B

In matrix form, for every depth level, the terms in the equations 12, 13, 14, 15 and 16 can be written as

$$\Delta\mathbf{p}(\omega) = \begin{bmatrix} 0 \\ \Delta p_2(\omega) \\ \vdots \\ \Delta p_{n_z-1}(\omega) \\ \Delta p_{n_z}(\omega) \end{bmatrix}; \Delta\mathbf{u}(\omega) = \begin{bmatrix} 0 \\ \Delta u_2(\omega) \\ \vdots \\ \Delta u_{n_z-1}(\omega) \\ \Delta u_{n_z}(\omega) \end{bmatrix}; \Delta\mathbf{r} = \begin{bmatrix} 0 \\ \Delta r_2 \\ \vdots \\ \Delta r_{n_z-1} \\ \Delta r_{n_z} \end{bmatrix};$$

$$\widetilde{\mathbf{p}}(\omega) = \begin{bmatrix} 0 \\ \widetilde{p}_2(\omega) \\ \vdots \\ \widetilde{p}_{n_z-1}(\omega) \\ \widetilde{p}_{n_z}(\omega) \end{bmatrix}; \widetilde{\mathbf{u}}(\omega) = \begin{bmatrix} 0 \\ \widetilde{u}_2(\omega) \\ \vdots \\ \widetilde{u}_{n_z-1}(\omega) \\ \widetilde{u}_{n_z}(\omega) \end{bmatrix};$$

$$\widehat{\mathbf{P}}(\omega) = \begin{bmatrix} \widehat{P}_1(\omega) & 0 & \dots & 0 & 0 \\ 0 & \widehat{P}_2(\omega) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \widehat{P}_{n_z-1}(\omega) & 0 \\ 0 & 0 & \dots & 0 & \widehat{P}_{n_z}(\omega) \end{bmatrix};$$

$$\Delta\mathbf{P}(\omega) = \begin{bmatrix} 0 & \Delta P_2(\omega) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \Delta P_{n_z-1}(\omega) & 0 \\ 0 & 0 & \dots & 0 & \Delta P_{n_z}(\omega) \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix};$$

$$\Delta\mathbf{U}(\omega) = \begin{bmatrix} 0 & \Delta U_2(\omega) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \Delta U_{n_z-1}(\omega) & 0 \\ 0 & 0 & \dots & 0 & \Delta U_{n_z}(\omega) \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix};$$

$$\mathbf{T}^{\downarrow}(\omega) = \begin{bmatrix} 0 & T_2^{\downarrow}(\omega) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & T_{n_z-1}^{\downarrow}(\omega) & 0 \\ 0 & 0 & \dots & 0 & T_{n_z}^{\downarrow}(\omega) \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix};$$

$$\Delta\mathbf{T}^{\downarrow}(\omega) = \begin{bmatrix} 0 & \Delta T_2^{\downarrow}(\omega) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \Delta T_{n_z-1}^{\downarrow}(\omega) & 0 \\ 0 & 0 & \dots & 0 & \Delta T_{n_z}^{\downarrow}(\omega) \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix};$$

In a furthermore compact notation, now for every frequency, we can write

$$\Delta\underline{\mathbf{p}} = \begin{bmatrix} \Delta\mathbf{p}(\omega_{\mathbf{1}}) \\ \Delta\mathbf{p}(\omega_{\mathbf{2}}) \\ \vdots \\ \Delta\mathbf{p}(\omega_{\mathbf{n}_\omega-\mathbf{1}}) \\ \Delta\mathbf{p}(\omega_{\mathbf{n}_\omega})) \end{bmatrix}; \Delta\underline{\mathbf{u}} = \begin{bmatrix} \Delta\mathbf{u}(\omega_{\mathbf{1}}) \\ \Delta\mathbf{u}(\omega_{\mathbf{2}}) \\ \vdots \\ \Delta\mathbf{u}(\omega_{\mathbf{n}_\omega-\mathbf{1}}) \\ \Delta\mathbf{u}(\omega_{\mathbf{n}_\omega})) \end{bmatrix};$$

$$\widetilde{\underline{\mathbf{p}}} = \begin{bmatrix} \widetilde{\mathbf{p}}(\omega_1) \\ \widetilde{\mathbf{p}}(\omega_2) \\ \vdots \\ \widetilde{\mathbf{p}}(\omega_{n_\omega-1}) \\ \widetilde{\mathbf{p}}(\omega_{n_\omega}) \end{bmatrix}; \widetilde{\underline{\mathbf{u}}} = \begin{bmatrix} \widetilde{\mathbf{u}}(\omega_1) \\ \widetilde{\mathbf{u}}(\omega_2) \\ \vdots \\ \widetilde{\mathbf{u}}(\omega_{n_\omega-1}) \\ \widetilde{\mathbf{u}}(\omega_{n_\omega}) \end{bmatrix};$$

$$\widehat{\mathbf{P}} = \begin{bmatrix} \widehat{\mathbf{P}}(\omega_1) & 0 & \dots & 0 & 0 \\ 0 & \widehat{\mathbf{P}}(\omega_2) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \widehat{\mathbf{P}}(\omega_{n_\omega-1}) & 0 \\ 0 & 0 & \dots & 0 & \widehat{\mathbf{P}}(\omega_{n_\omega}) \end{bmatrix};$$

$$\Delta\mathbf{P} = \begin{bmatrix} \Delta\mathbf{P}(\omega_1) & 0 & \dots & 0 & 0 \\ 0 & \Delta\mathbf{P}(\omega_2) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \Delta\mathbf{P}(\omega_{n_\omega-1}) & 0 \\ 0 & 0 & \dots & 0 & \Delta\mathbf{P}(\omega_{n_\omega}) \end{bmatrix};$$

$$\Delta\mathbf{U} = \begin{bmatrix} \Delta\mathbf{U}(\omega_1) & 0 & \dots & 0 & 0 \\ 0 & \Delta\mathbf{U}(\omega_2) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \Delta\mathbf{U}(\omega_{n_\omega-1}) & 0 \\ 0 & 0 & \dots & 0 & \Delta\mathbf{U}(\omega_{n_\omega}) \end{bmatrix};$$

$$\mathbf{T}^{\downarrow} = \begin{bmatrix} \mathbf{T}^{\downarrow}(\omega_1) & 0 & \dots & 0 & 0 \\ 0 & \mathbf{T}^{\downarrow}(\omega_2) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathbf{T}^{\downarrow}(\omega_{n_\omega-1}) & 0 \\ 0 & 0 & \dots & 0 & \mathbf{T}^{\downarrow}(\omega_{n_\omega}) \end{bmatrix};$$

$$\Delta\mathbf{T}^{\downarrow} = \begin{bmatrix} \Delta\mathbf{T}^{\downarrow}(\omega_1) & 0 & \dots & 0 & 0 \\ 0 & \Delta\mathbf{T}^{\downarrow}(\omega_2) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \Delta\mathbf{T}^{\downarrow}(\omega_{n_\omega-1}) & 0 \\ 0 & 0 & \dots & 0 & \Delta\mathbf{T}^{\downarrow}(\omega_{n_\omega}) \end{bmatrix};$$

The contribution of every frequency to the image perturbation is achieved by the summation matrix, $\mathbf{S}$,

$$\mathbf{S} = \begin{bmatrix} \mathbf{I}(n_\mathbf{x} n_z) & \mathbf{I}(n_\mathbf{x} n_z) & \dots & \mathbf{I}(n_\mathbf{x} n_z) & \mathbf{I}(n_\mathbf{x} n_z) \end{bmatrix}, \tag{B-1}$$

composed of n_ω block-identity matrices of $n_\mathbf{x} n_z$-dimension.