Effective medium theory for elastic composites

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ABSTRACT

The theoretical foundation of a variant on effective medium theories for elastic constants of composites is presented and discussed. The connection between this approach and the methods of Zeller and Dederichs, Korringa, and Gubernatis and Krumhansl is elucidated. A review of the known relationships between the various effective medium theories and rigorous bounding methods for elastic constants is also provided.

INTRODUCTION

In a series of papers [Berryman (1979, 1980a,b)], a variant on effective medium theories for elastic composites was developed by the author. In this paper, I will review the derivation of the effective medium formulas for the elastic constants of composites while elucidating the relationships between my results and the results from effective medium theories proposed by others. These results are then compared to known rigorous bounds on the effective elastic constants.

The general background for theories of elastic composites with special emphasis on earth sciences applications is provided in the review articles by Watt et al. (1976) and Berryman (1995). Another review of effective medium theories with an emphasis on connections to general applied physics applications is given by Elliott et al. (1974). Related work by Willis (1977, 1981) is also especially useful for some of the cases not considered here, including anisotropic media and polycrystalline composites.

EFFECTIVE ELASTIC CONSTANTS

Mal and Knopoff (1967) derived an integral equation for the scattered displacement field from a single elastic scatterer. Let Ω_i symbolize the volume of the region occupied by a single inclusion i. Let the incident field be $\vec{u}^0(\vec{x})\exp(-i\omega t)$ and let $\vec{u}(\vec{x})\exp(-i\omega t)$ and $\vec{v}(\vec{x})\exp(-i\omega t)$ be the total field outside and inside the inclusion volume such that

$$\begin{aligned} \vec{u}(\vec{x}) &= \vec{u}^0(\vec{x}) + \vec{u}^s(\vec{x}) \qquad \text{for} \qquad \vec{x} \notin \Omega_i, \\ \vec{v}(\vec{x}) &= \vec{u}^0(\vec{x}) + \vec{v}^s(\vec{x}) \qquad \text{for} \qquad \vec{x} \in \Omega_i. \end{aligned} \tag{1}$$

The scattered fields are $\vec{u}^s$ and $\vec{v}^s$. Both $\vec{u}(\vec{x})$ and $\vec{u}^0(\vec{x})$ satisfy the same equation:

$$c^m_{\ell npq}\frac{\partial^2 u_p}{\partial x_n \partial x_q} + \rho_m \omega^2 u_\ell = 0 \tag{2}$$

outside the inclusion, while $\vec{v}(\vec{x})$ satisfies

$$c^i_{\ell npq}\frac{\partial^2 v_p}{\partial x_n \partial x_q} + \rho_i\omega^2 v_\ell = 0 \tag{3}$$

inside the inclusion. The indices ℓ,n,p,q take the values $1, 2, 3$ for the three spatial dimensions, and the Einstein summation convention applies in Equations (2) and (3), and also throughout this paper. The elastic tensor for the matrix and inclusion are respectively:

$$c^m_{\ell npq} = \lambda_m\delta_{\ell n}\delta_{pq} + \mu_m\left(\delta_{\ell p}\delta_{nq} + \delta_{np}\delta_{\ell q}\right), \tag{4}$$

$$c^i_{\ell npq} = \lambda_i\delta_{\ell n}\delta_{pq} + \mu_i\left(\delta_{\ell p}\delta_{nq} + \delta_{np}\delta_{\ell q}\right) \equiv c^m_{\ell npq} + \Delta c^i_{\ell npq}, \tag{5}$$

and ρ_m and $\rho_i(\equiv \rho_m + \Delta\rho_i)$ are the respective densities.

Green's function for a point source in an infinite, isotropic, homogeneous elastic medium of the matrix material is given by

$$g_{pq}(\vec{x},\vec{\zeta}) = \frac{1}{4\pi\rho_m\omega^2}\left[s^2\frac{\exp{(isr)}}{r}\delta_{pq} - \frac{\partial^2}{\partial x_p \partial x_q}\left(\frac{\exp{(ikr)}}{r} - \frac{\exp{(isr)}}{r}\right)\right], \tag{6}$$

where $r = |\vec{x} - \vec{\zeta}|$, $k = \omega[\rho_m/(\lambda_m + 2\mu_m)]^{1/2}$, and $s = \omega[\rho_m/\mu_m]^{1/2}$ — with k and s being, respectively, the magnitudes of the wavevectors for compressional and shear waves in the matrix. Given the form of g_{pq}, Mal and Knopoff (1967) then derive an integral equation for $\vec{u}(\vec{x})$. Since the derivation follows standard lines of argument, I will not repeat it here. The result is

$$u_\ell(\vec{x}) = u^0_\ell(\vec{x}) + \int_{\Omega_i} d\vec{\zeta}\left[\Delta\rho_i\omega^2 v_n(\vec{\zeta}) - \Delta c^i_{njpq}\epsilon_{pq}\frac{\partial}{\partial\zeta_j}\right]g_{\ell n}(\vec{x},\vec{\zeta}). \tag{7}$$

Equation (7) is an exact integral equation for the displacement field in the region exterior to the scatterer in terms of the displacement and strain fields inside the inclusion volume Ω_i.

To evaluate the integral (7), estimates of the interior displacement and strain fields are required. Considering the first Born approximation from quantum scattering theory suggests the estimates for wave speed and strain at $\vec{\zeta} \in \Omega_i$:

$$\vec{v}(\vec{\zeta}) \simeq \vec{u}^0(\vec{\zeta}), \tag{8}$$

and

$$\epsilon_{pq}(\vec{\zeta}) \simeq \epsilon^0_{pq}(\vec{\zeta}). \tag{9}$$

By Equations (8) and (9), I mean to approximate $\vec{v}$ and ϵ by the values $\vec{u}^0$ and ϵ^0 would have achieved at position $\vec{\zeta}$ if the matrix contained no scatterers. For scatterers with small volumes, it follows from (7) that $\vec{u}^s(\vec{x})$ and its derivatives are small quantities for $\vec{x}$ outside of Ω_i. Since the displacement is continuous across the boundary, it follows that Equation (8) will be a good approximation to $\vec{v}(\vec{\zeta})$. However, this argument fails for Equation (9), because the strains are not continuous across the boundary. Equation (9) should therefore be replaced by the formula:

$$\epsilon_{pq} = T_{pqrs}\epsilon^0_{rs}, \tag{10}$$

where T is Wu's tensor [Wu (1966)], relating ϵ_{pq} for an arbitrary ellipsoidal inclusion to the uniform strain at infinity ϵ^0_{pq}. Now, if the wavelength of the incident waves is large compared to the size of the ellipsoid (*i.e.*, $a/\bar{\lambda} << 1$, where $\bar{\lambda}$ is the wavelength), then the fields both near the ellipsoid and inside scatterer volume Ω_i will be essentially static and uniform [Eshelby (1957)]. Thus, to the lowest order of approximation, it is valid to make the substitutions (8) and (10). When the ellipsoid is centered at ζ_i, it follows easily that

$$u^s_\ell(\vec{x}) = \Omega_i\left[\Delta\rho_i\omega^2 u^0_n(\vec{\zeta}_i)g_{\ell n}(\vec{x},\vec{\zeta}_i) - \left(\Delta\lambda^i T_{pprs}\delta_{nj} + 2\Delta u^i T_{njrs}\right)\epsilon^0_{rs}g_{\ell n,j}(\vec{x},\vec{\zeta}_i)\right], \tag{11}$$

where the symmetry properties of T have been used in simplifying the expression. A comma preceding a subscript indicates a derivative with respect to the as-labelled component.

Equation (11) gives the first order estimate of the scattered wave from an ellipsoidal inclusion whose principal axes are aligned with the coordinate axes. When the ellipsoid is oriented arbitrarily with respect to the coordinate axes, Equation (11) must be changed by replacing T_{pqrs} everywhere with

$$U_{pqrs} = \ell_{p\alpha}\ell_{q\beta}\ell_{r\gamma}\ell_{s\delta}T_{\alpha\beta\gamma\delta}, \tag{12}$$

where $\ell_{\alpha\beta}$ are the appropriate direction cosines. For homogeneous, isotropic composites with randomly oriented ellipsoidal inclusions, the general form of the average tensor as given by Wu (1966) is

$$\bar{U}_{pqrs} = \frac{1}{3}(P-Q)\delta_{pq}\delta_{rs} + \frac{1}{2}Q(\delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr}), \tag{13}$$

where

$$P = \frac{1}{3}T_{ppqq} \qquad \text{and} \qquad Q = \frac{1}{5}\left(T_{pqpq} - T_{ppqq}\right). \tag{14}$$

Finally, suppose N inclusions are contained in a small volume of radius a centered at $\vec{\zeta}_0$. Assume that the effects of multiple scattering may be neglected at sufficiently low frequencies (*i.e.*, long wavelengths appropriate for seismology) to the lowest order. Then, to the same degree of approximation used in Equation (11) (*i.e.*, $a/\bar{\lambda} << 1$), the scattered wave has the form:

$$\langle u^s_\ell(\vec{x})\rangle^m \simeq \sum_{i=1}^N \Omega_i\left[\Delta\rho_i\omega^2 u^0_n(\vec{\zeta}_0)g_{\ell n}(\vec{x},\vec{\zeta}_0) - \left(\Delta\lambda^i\bar{U}^{mi}_{pprs}\delta_{nj} + 2\Delta\mu^i\bar{U}^{mi}_{njrs}\right)\epsilon^0_{rs}g_{\ell n,j}(\vec{x},\vec{\zeta}_0)\right], \tag{15}$$

where the superscripts m and i again refer to matrix and inclusion properties, respectively. Note especially that distinct superscripts i must be used in Equation (15) to specify both the inclusion *material* itself, and also the *shape* of each distinct type of inclusion.

To apply this thought experiment to the analytical problem of estimating elastic constants, consider replacing the true composite sphere with a sphere composed of matrix material identical to the imbedding material and of ellipsoidal inclusions of the same materials as those in the true composite, and also in the same proportions. Then, if multiple scattering effects may be (and are) neglected, the theoretical expression which determines the elastic constants is

$$\langle u_\ell^s(\vec{x})\rangle^* = 0, \tag{16}$$

where the left hand side is given by Equation (15) with matrix-type $m = *$. Equation (16) states simply that the net (overall) scattering — due to many scatterers – in the self-consistently determined medium vanishes to lowest order.

If the volume fraction of the i-th component is defined by $f_i = \Omega_i / \sum_{j=1}^N \Omega_j$, then Equation (16) implies the following formulas:

$$\sum_{i=1}^N f_i(\rho_i - \rho^*) = 0, \tag{17}$$

$$\sum_{i=1}^N f_i(K_i - K^*)P^{*i} = 0, \tag{18}$$

and

$$\sum_{i=1}^N f_i(\mu_i - \mu^*)Q^{*i} = 0. \tag{19}$$

Equation (17) states that the effective density ρ^* is just the volume average density (which is what one might reasonably expect, but nevertheless is not always true for effective medium theories). Equations (18) and (19) provide implicit formulas for K^* and μ^*. Such implicit formulas are typically solved numerically by iteration [Berryman (1980b)]. This step is usually necessary because the factors P^{*i} and Q^{*i} are themselves both typically functions of both the unknown quantities K^* and μ^*. Experience has shown that such iterative methods often converge in a stable fashion, and usually after a small number of iterations (typically 10 or less).

The derivation given and final results attained here are very similar to methods discussed by Elliott et al. (1974) and Gubernatis and Krumhansl (1975). I will therefore refer to the resulting effective medium method as the "coherent potential approximation" (or CPA), as is typically done in the physics literature, since the early work of Soven (1967). Equations (18) and (19) were also obtained independently by Korringa et al. (1979), while using an entirely different method. In the following sections, I will compare the results obtained from this effective medium theory to the known rigorous bounds on elastic constants and also to the results of other effective medium theories.

RIGOROUS BOUNDS ON EFFECTIVE MODULI

In their review article, Watt et al. (1976) discuss various rigorous bounds on the effective moduli of composites. For example, the well-known Voigt (arthimetic) and Reuss (harmonic) averages are, respectively, rigorous [Hill (1952)] upper and lower bounds for both K^* and μ^*. Generally tighter bounds have also been given by Hashin and Shtrikman (1961, 1962, 1963).

Still tighter bounds have been obtained in principle by Beran and Molyneux (1966) for the bulk modulus and by McCoy (1970) for the shear modulus. However, the resulting formulas depend on three-point spatial correlation functions for the composite and are therefore considerably more difficult to evaluate than the expressions for the Hashin-Shtrikman [Hashin and Shtrikman (1961, 1962, 1963)] bounds, which depend only on the material constants and volume fractions. Miller (1969b,a) evaluated the bounds of Beran and Molyneux (1966) by treating an isotropic homogeneous distribution of statistically independent cells. Silnutzer (1972) used the same approach to simplify the bounds of McCoy (1970) for cell materials. Furthermore, Milton (1981) has shown that the bounds of Beran and Molyneux (1966) and McCoy (1970) can be simplified somewhat even if the composite is not a cell material. Nevertheless, the bounds which are most easily evaluated are still the Hashin-Shtrikman [Hashin and Shtrikman (1961, 1962, 1963)] (HS) bounds, the Beran-Molyneux-Miller (BMM) bounds, and the McCoy-Silnutzer (MS) bounds. I will compare these bounds to the estimates obtained from the coherent potential approximation (CPA), the specific effective medium theory being stressed here.

To aid in the following comparisons, it is convenient to introduce two functions:

$$\Lambda(x) = \left(\sum_{i=1}^{N} \frac{f_i}{K_i + 4x/3} \right)^{-1} - \frac{4}{3}x, \tag{20}$$

$$\Gamma(y) = \left(\sum_{i=1}^{N} \frac{f_i}{\mu_i + y} \right)^{-1} - y, \tag{21}$$

together with a third function that is needed in conjunction with Γ:

$$F(x,z) = \frac{x}{6}\left(\frac{9z + 8x}{z + 2x} \right). \tag{22}$$

It has been shown previously [Berryman (1980b, 1995)] that $\Lambda(x)$ and $\Gamma(y)$ are monotonically increasing functions of their real arguments. Similarly, I find that

$$\frac{\partial F}{\partial z} = \frac{5x^2}{3(z+2x)^2} \geq 0 \tag{23}$$

and

$$\frac{\partial F}{\partial x} = \frac{9z^2 + 16xz + 16x^2}{6(z+2x)^2} \geq 0 \quad \text{if} \quad x \geq 0, \quad \& \quad z \geq 0. \tag{24}$$

So, when both arguments of $F(x,z)$ are non-negative (which will soon be shown to be the case in these applications), it follows that F is a monotonically increasing function of both arguments.

Now, if I define the minimum and maximum moduli among all the constitutents by

$$\begin{aligned} K_+ &= \max(K_1,\ldots,K_N), & K_- &= \min(K_1,\ldots,K_N), \\ \mu_+ &= \max(\mu_1,\ldots,\mu_N), & \mu_- &= \min(\mu_1,\ldots,\mu_N), \end{aligned} \tag{25}$$

then the Hashin-Shtrikman bounds are also given in general by

$$K^{\pm}_{HS} = \Lambda(\mu_\pm) \tag{26}$$

and

$$\mu^{\pm}_{HS} = \Gamma\left[F(\mu_\pm, K_\pm)\right]. \tag{27}$$

[Note that the only combinations considered on the right-hand side of (27) are those having both pluses or both minuses – no mixing of the subscripts.]

The Beran-Molyneux-Miller bounds and the McCoy-Silnutzer bounds are known for two-phase composites (*i.e.*, $N=2$). These bounds can be written in succinct form using the notation of Milton (1981). By defining two geometric parameters $\zeta_1 = 1-\zeta_2$ and $\eta_1 = 1-\eta_2$, and two related averages [analogous to the volume fraction weighted average $\langle M\rangle = f_1M_1 + f_2M_2$] of any modulus M by $\langle M\rangle_\zeta = \zeta_1M_1+\zeta_2M_2$, and $\langle M\rangle_\eta = \eta_1M_1+\eta_2M_2$, then the bounds can be written very concisely as:

$$K^+_{BMM} = \Lambda\left(\langle\mu\rangle_\zeta\right), \tag{28}$$

$$K^-_{BMM} = \Lambda\left(\langle 1/\mu\rangle_\zeta^{-1}\right), \tag{29}$$

$$\mu^+_{MS} = \Gamma(\Theta/6), \tag{30}$$

and

$$\mu^-_{MS} = \Gamma(\Xi^{-1}/6), \tag{31}$$

where

$$\Theta = \left[10\langle\mu\rangle^2\langle K\rangle_\zeta + 5\langle\mu\rangle\langle 2K+3\mu\rangle\langle\mu\rangle_\zeta + \langle 3K+\mu\rangle^2\langle\mu\rangle_\eta\right]/\langle K+2\mu\rangle^2 \tag{32}$$

and

$$\Xi = \left[10\langle K\rangle^2\left\langle\frac{1}{K}\right\rangle_\zeta + 5\langle\mu\rangle\langle 2K+3\mu\rangle\left\langle\frac{1}{\mu}\right\rangle_\zeta + \langle 3K+\mu\rangle^2\left\langle\frac{1}{\mu}\right\rangle_\eta\right]/\langle 9K+8\mu\rangle^2. \tag{33}$$

For symmetric cell materials, it is known that $\zeta_1 = \eta_1 = f_1$ for spherical cells, $\zeta_1 = \eta_1 = f_2$ for disks, while $\zeta_1 = (3f_1+f_2)/4$, and $\eta_1 = (5f_1+f_2)/6$ for needles.

It is particularly simple to compare these bounds with the results of effective medium theory when the inclusions are assumed to be spherical in shape. Then, the estimates of the moduli are given by the self-consistent formulas (which are mutually interdependent):

$$K^* = \Lambda(\mu^*) \tag{34}$$

and

$$\mu^* = \Gamma\left[F\left(\mu^*, K^*\right)\right]. \tag{35}$$

Furthermore, the bounds (28)–(30) simplify in this case and are given by

$$K^+_{BMM} = \Lambda\left(\langle\mu\rangle\right), \tag{36}$$

$$K^+_{BMM} = \Lambda\left(\langle 1/\mu\rangle^{-1}\right), \tag{37}$$

and

$$\mu^+_{MS} = \Gamma\left[F\left(<\mu>, <K>\right)\right]. \tag{38}$$

From the monotonicity properties of the functions (20)–(22), from elementary arguments relating the estimates to the Voigt and Reuss averages, and also from the fact that all the arguments of these functions depend on quantities composed of elastic constants averaged using positive measures such as volume fractions and the related quantities for various cell-material shapes, I find for the bulk modulus that

$$\Lambda(\mu_-) \le \Lambda\left(\langle 1/\mu\rangle^{-1}\right) \le \Lambda(\mu^*) = K^* \le \Lambda\left(\langle\mu\rangle\right) \le \Lambda(\mu_+), \tag{39}$$

or equivalently that

$$K^-_{HS} \le K^-_{BMM} \le K^* \le K^+_{BMM} \le K^+_{HS}. \tag{40}$$

Similarly, by making use of $\Gamma(y)$ from (21), it follows for the shear modulus that

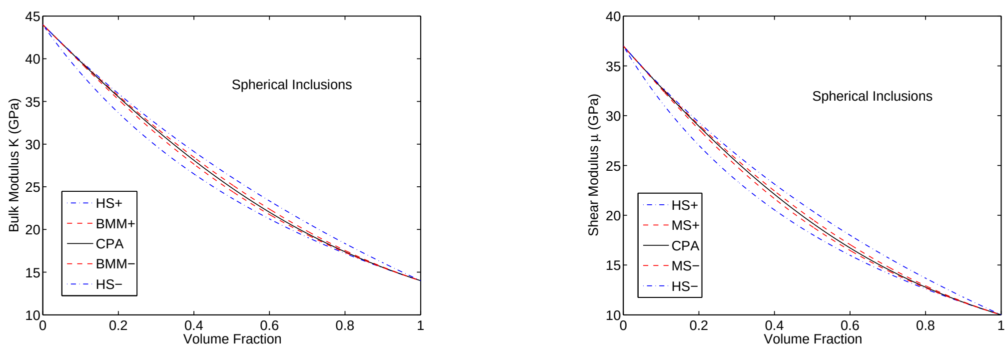
$$\mu^-_{HS} \le \mu^* \le \mu^+_{MS} \le \mu^+_{HS}. \tag{41}$$

The detailed argument leading to Equation (39) is a little involved: First, I must show that K^*, μ^* are bounded by the Hashin-Shtrikman bounds [Berryman (1980a)]. Then, since the Hashin-Shtrikman bounds are themselves bounded by the Voigt and Reuss bounds, Equation (39) follows from

$$\Lambda(\langle 1/\mu\rangle^{-1}) \le \Lambda(\mu^-_{HS}) \le \Lambda(\mu^*) \le \Lambda(\mu^+_{HS}) \le \Lambda(\langle\mu\rangle). \tag{42}$$

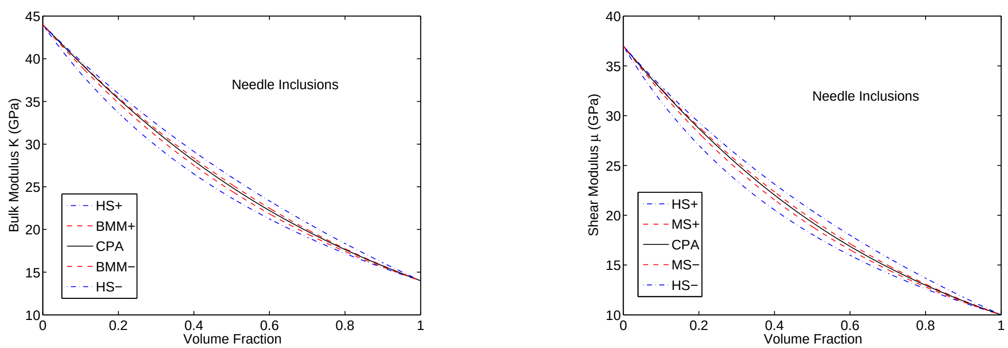
The arguments just given are valid *only* for the case of spherical inclusions. The author knows of no general argument relating the effective medium results to the rigorous bounds for arbitrary inclusion shapes. However, as will be observed in the following Figures, numerical examples illustrate the effective medium estimates always lying between the bounds.

Typical results are presented in Figures 1–3. The values of the constituents' moduli were chosen to be: $K_1 = 44.0$ GPa, $\mu_1 = 37.0$ GPa, $K_2 = 14.0$ GPa, and $\mu_2 = 10.0$ GPa. The values of K_2 and μ_2 were chosen as a compromise between two extremes: (a) If K_2 and μ_2 are too close to K_1 and μ_1, then the bounds are too close together to be distinguishable on the plots. (b) If K_2 and μ_2 are both chosen to be zero, the iteration to the effective medium theory results does not converge for the case of disk-like inclusions [Berryman (1980b)], although all the other cases converge without difficulties. I find in all cases considered that the effective medium theory results lie between the rigorous bounds, as stated above.



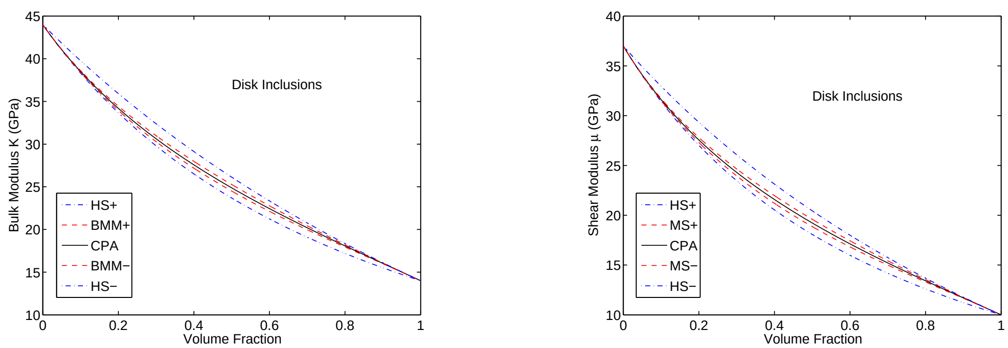
(a) (b)

Figure 1: Estimates of the effective bulk (a) and shear (b) moduli of elastic composites with constituents $K_1 = 44.0$ GPa, $\mu_1 = 37.0$ GPa, $K_2 = 14.0$ GPa, and $\mu_2 = 10.0$ GPa as the volume fraction of type-2 increases. The curves are respectively the CPA (or coherent potential approximation: a self-consistent estimator) — which is the black solid line, the Beran-Molyneux-Miller bounds for the bulk modulus and the McCoy-Silnutzer bounds for the shear modulus — which are the red dashed lines, and the Hashin-Shtrikman bounds — which are the blue dot-dashed lines. Inclusions are treated as having spherical shape. **NR**



(a) (b)

Figure 2: Estimates of the effective bulk (a) and shear (b) moduli of elastic composites with constituents $K_1 = 44.0$ GPa, $\mu_1 = 37.0$ GPa, $K_2 = 14.0$ GPa, and $\mu_2 = 10.0$ GPa as the volume fraction of type-2 increases. The curves are respectively the CPA (or coherent potential approximation: a self-consistent estimator) — which is the black solid line, the Beran-Molyneux-Miller bounds for the bulk modulus and the McCoy-Silnutzer bounds for the shear modulus — which are the red dashed lines, and the Hashin-Shtrikman bounds — which are the blue dot-dashed lines. Inclusions are treated here as having needle-like shape. **NR**



(a) (b)

Figure 3: Estimates of the effective bulk (a) and shear (b) moduli of elastic composites with constituents $K_1 = 44.0$ GPa, $\mu_1 = 37.0$ GPa, $K_2 = 14.0$ GPa, and $\mu_2 = 10.0$ GPa as the volume fraction of type-2 increases. The curves are respectively the CPA (or coherent potential approximation: a self-consistent estimator) — which is the black solid line, the Beran-Molyneux-Miller bounds for the bulk modulus and the McCoy-Silnutzer bounds for the shear modulus — which are the red dashed lines, and the Hashin-Shtrikman bounds — which are the blue dot-dashed lines. Inclusions are treated here as having disk-like shape. **NR**

OTHER EFFECTIVE MEDIUM THEORIES

A great variety of effective medium theories exist for studies of the elastic properties of composites. Of these theories, the scattering theory presented by Zeller and Dederichs (1973), Korringa (1973), and Gubernatis and Krumhansl (1975) have the most in common with the scattering-theory approach presented here. However, the present approach appears to be unique among the self-consistent scattering-theory variety, being dynamic (*i.e.*, frequency dependent), while all the others are based on static or quasi-static derivations. This difference becomes a very useful advantage if we want to generalize the approach to finite (nonzero) frequencies, as is required for viscoelastic media. The bounding arguments presented here do not carry over directly to the frequency dependent case, but they actually can be generalized — as shown by Gibianksy and Milton (1993), Milton and Berryman (1997), and Gibiansky et al. (1999).

Another class of effective medium theories studied by Hill (1965), Budiansky (1965), Wu (1966), Walpole (1969), and Boucher (1974) does *not* yield the same results as the present one, except for the case of spherical inclusions. It has been shown elsewhere [Berryman (1980b)] how the derivation of the approach of Hill, Budiansky, and others can be kinds of symmetrized to yield the symmetrical results as presented here that I prefer. Since the CPA class of effective medium theories gives results equivalent to the Hashin-Shtrikman [Hashin and Shtrikman (1961, 1962, 1963)] bounds when the inclusions are disk-shaped, I conclude that these results are preferred – since they do satisfy these bounding constraints, while the alternatives do not. The numerical results show general satisfaction of the bounds.

To elucidate somewhat further the relationship between the static and dynamic derivations of the effective medium results, I will outline the static derivation next. The integral equations for the static strain field are given by

$$\epsilon_{ij}(\vec{x}) = \epsilon^0_{ij}(\vec{x}) + \int d^3x' G_{ijkl}(\vec{x},\vec{x}')\Delta c_{klmn}(\vec{x}')\epsilon_{mn}(\vec{x}'), \tag{43}$$

where Green's function is

$$G_{ijkl}(\vec{x},\vec{x}') = \frac{1}{2}\left(g^0_{ik,jl} + g^0_{jk,il}\right), \tag{44}$$

with the Kelvin solution given by

$$g_{pq}(\vec{x},\vec{x}') = \frac{1}{4\pi\mu_m}\left[\frac{\delta_{pq}}{r} - \frac{1}{4(1-\nu_m)}\frac{\partial^2 r}{\partial x_p \partial x_q}\right], \tag{45}$$

where $r = |\vec{x} - \vec{x}'|$ and μ_m and ν_m are, respectively, the shear modulus and Poisson's ratio of the matrix material. Equation (43) may be rewritten formally as

$$\epsilon = \epsilon^0 + G\Delta c\epsilon, \tag{46}$$

where G is now an integral operator defined by

$$Gf = \int d^3x' G(\vec{x},\vec{x}')f(\vec{x}'). \tag{47}$$

Iterating Equation (46), I obtain the well-known Born series

$$\epsilon = \epsilon^0 + G\Delta c\epsilon^0 + G\Delta cG\Delta c\epsilon^0 + \dots, \tag{48}$$

and then summing the Born series formally yields

$$\epsilon = (I + Gt)\,\epsilon^0 = (I - G\Delta c)^{-1}\,\epsilon^0, \tag{49}$$

where the so-called t-matrix is defined by

$$t = \Delta c\,(I - G\Delta c)^{-1} = \Delta c\,(I + Gt)\,. \tag{50}$$

Taking the ensemble average of Equation (49), I have

$$\langle\epsilon\rangle = (I + G\,\langle t\rangle)\,\epsilon^0 = \left\langle (I - G\Delta c)^{-1}\right\rangle \epsilon^0. \tag{51}$$

For a single scatterer, Equation (49) is equivalent to Equation (10). Therefore, it is worth noting that Wu's (1966) tensor T is formally related to the t-matrix by

$$T = I + Gt = (I - G\Delta c)^{-1}\,. \tag{52}$$

Equation (51) is now in a convenient form for use in determining the effective elastic tensor c^* of a composite defined by

$$\langle\sigma\rangle = \langle c\,\epsilon\rangle \equiv c^*\,\langle\epsilon\rangle\,, \tag{53}$$

where the averages in Equation (53) are again ensemble averages over possible composites having similar physical and statistical properties. Using the standard definition $c = c^m + \Delta c$, I find that

$$\langle c\,\epsilon\rangle = c^m\,\langle\epsilon\rangle + \langle\Delta c\,\epsilon\rangle = c^m\,\langle\epsilon\rangle + \langle t\rangle\,\epsilon^0. \tag{54}$$

From Equation (54), it follows easily that the effective elastic tensor is given by

$$c^* = c^m + \langle t\rangle\,(I + G\,\langle t\rangle)^{-1}\,. \tag{55}$$

The choice of matrix elastic tensor c^m is still completely free since the decomposition $c = c^m + \Delta c$ is not unique. Thus, I am free to choose, for example, $c^m = c^*$ (*i.e.*, the matrix material has now exactly the properties of the equivalent composite material), which implies:

$$\langle t\rangle \equiv 0. \tag{56}$$

Equation (56) is an implicit formula determining the effective elastic tensor c^*, and says that the effective scattering t-matrix averages to zero.

In principle, Equation (56) provides an exact solution for the effective moduli. However, the total t-matrix itself is generally too difficult to calculate. It turns out to be more reasonable and more effective [Velicky et al. (1968)] to rearrange the terms of the total t-matrix into a series of terms with repeated scattering from individual scatterers (t_i). Then, by setting the ensemble average of the individual t matrices to zero

$$\langle t_i \rangle = \sum_{i=1}^{N} f_i \Delta c_i \left(I - G\Delta c_i \right)^{-1} = 0, \tag{57}$$

and neglecting terms corresponding to fluctuations in the scattered wave [Velicky et al. (1968)], a tractable approximation for the estimate of the elastic moduli is obtained.

When the constituents and the composite as a whole are all relatively homogeneous and isotropic, the tensor Equation (57) reduces to two coupled equations:

$$\sum_{i=1}^{N} f_i (K_i - K^*) P^{*i} = 0, \tag{58}$$

and

$$\sum_{i=1}^{N} f_i (\mu_i - \mu^*) Q^{*i} = 0, \tag{59}$$

where Equations (13), (14), and (52) were used to simplify Equation (57). Note that Equations (58) and (59) are identical to Equations (18) and (19), thereby establishing the equivalence of these two approaches in the isotropic case.

SUMMARY AND CONCLUSIONS

I conclude that my preferred choice of effective medium theory (the CPA) satisfies all the known constraints on a viable theory: (a) it gives correct values and slopes for both large and small volume fractions of inclusions; (b) numerical evidence indicates that the results always satisfy the Hashin-Shtrikman bounds, the Beran-Molyneux-Miller bounds, and the McCoy-Silnutzer bounds; (c) the theory is also known [Berryman (1980b)] to reproduce Hill's exact result [Hill (1963)] for composites with uniform shear modulus — which fact is a fairly simple exercise to check, so that the reader might find it instructive to carry this through.

The single-scatterer theory is designed to minimize multiple scattering effects while yielding formulas that are relatively easy to use. Nevertheless, the theory is not exact, and some potentially significant effects have been neglected. The neglected terms become more important for propagation of higher frequency elastic waves. But it is important to note that bounding methods and formulas are also much harder to implement rigorously for the frequency dependent (viscoelastic) case. This fact is surely one reason that the theory is seldom applied at significantly higher frequencies than typical seismic frequencies, or in regions of very much higher viscosity, and wave dissipation and dispersion. So, it is expected that, for small ranges of frequency — and especially those that are pertinent to exploration seismology — will naturally be included in the range of useful applications since the seismic band is fairly narrow. Then the viscoelastic effects can typically be treated without great additional difficulty. Some future efforts should nevertheless be directed towards extending this effective medium theory to scattering from clusters of inclusions at finite frequency — thereby including within the expanded theory more of the important scattering effects discussed (but then specifically neglected, and therefore not treated in any detail) here.

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