Hubbert math

Jon Claerbout and Francis Muir

ABSTRACT

Hubbert fits growth and decay of petroleum production to the logistic function. Hubbert's relationship is commonly encountered in four different forms. They are all stated here, then derived from one of them, thus showing they are equivalent.

PREFACE

This article was rejected by Wikipedia. They say an encyclopedia should not contain the derivation of important equations so we prepared this article for that purpose. The many references to this paper (missing here) are web links which may be found at http://sep.stanford. edu/sep/jon/tarsand/

THE FOUR FORMS OF HUBBERT'S EQUATION

Over the long haul populations grow and decay. To describe the growth and decay of civilization's dependence on nuclear and fossil fuels, M. King Hubbert chose an equation that describes many natural processes. Introduce bacteria to food and their population will grow exponentially until there no longer is food. As we catch all the fish in the lake our daily catch will be proportional to the number of remaining fish. Hubbert's equation models both exponential growth and decay with a single equation of three parameters to be chosen from the data. He predicted 52 years ago that worldwide oil production would be peaking about now (2008). It is.

Hubbert's math has four different forms which we examine before showing they are mathematically equivalent.

Basic definitions

We define:

- t is time in years
- $Q(t)$ is cumulative production in billion barrels at year t.

- Q_∞ is the ultimate recoverable resource.
- $P(t) = dQ/dt$ is production in billion barrels/year at year t.
- τ is the year at which production peaks.
- ω is an inverse decay time (imaginary frequency).

The Hubbert's equation can be expressed in four forms. First, the differential form

$$\frac{dQ}{dt} = P = \omega\ Q \left(1 - \frac{Q}{Q_\infty}\right) \tag{1}$$

This equation is non-linear in Q but it reduces to familiar linear equations near the beginning and $Q \approx 0$ and near the end at $Q \approx Q_\infty$. As production begins and Q/Q_∞ is small, equation (1) reduces to $dQ/dt = \omega Q$ which displays exponential growth at a rate ω. As production ends near $Q \approx Q_\infty$ the non-linear equation reduces to exponential decay. To prove this fact change variables from Q to q by inserting $Q = Q_\infty - q$. Then evaluate the result at small q. The form (1) exhibits growth and decay as a dynamic process.

The second form of the Hubbert equation is found by dividing equation (1) by Q. It is sometimes called the *Hubbert Linearization*.

$$\frac{P}{Q} = \omega\ \left(1 - \frac{Q}{Q_\infty}\right) \tag{2}$$

The important thing about this equation is that it is linear in the two variables Q and P/Q. If you have historical measurements of P_i and Q_i, you can plot these points in the $(Q, P/Q)$-plane and hope for them to reasonably fit a straight line. Fitting the best line to the scattered points we can read the axis intercepts. At $Q = 0$ with equation (2) we can read off the value of the growth/decay parameter $\omega = (P/Q)_{\text{intercept}}$. For world oil, according to Deffeyes (Remember! The references are links on the web version of this paper.) it is 5.3 percent/year. At the other intercept, $P/Q = 0$ we must have $Q = Q_\infty$. According to Deffeyes, Q_∞ is two trillion barrels.

The third form of Hubbert's equation is the one best known. It looks like a Gaussian, but it isn't. (A Gaussian decays much faster.) The current production $P = dQ/dt$ is

$$P(t) = Q_\infty\ \omega\ \frac{1}{(e^{-(\omega/2)(\tau-t)} + e^{(\omega/2)(\tau-t)})^2} \tag{3}$$

This is the equation of a blob, also known as "Hubbert's pimple", symmetric about the point $t = \tau$. Asymptotically it decreases (or increases) exponentially towards its maximum value at the center at $t = \tau$. The function resembles a Gaussian but exponential decay is much weaker than Gaussian decay. Exponential growth is common in ecological systems which may also decay exponentially as resources are depleted or predator numbers grow exponentially.

All that remains is to figure out τ. The Hubbert curve is symmetrical and reaches its maximum when half the oil is gone. That happens when $Q = Q_\infty/2$. In the case of USA production which has passed its peak we can find the year that Q reached that value (about 1973). There is some debate about what year world production peaks, but general agreement is that it is about now (2008). Under Hubbert assumptions the decline curve is a mirror of the rise curve. That means we start down gently over the next decade, but about 25 years from now we hit the inflection point and see a 5 percent/year decline every year thereafter.

In real life there is no reason for the decay rate to match the growth rate. The decay could be faster because of horizontal drilling. The decay could be slower because we tax to conserve or successfully invest in technologies. As liquid oil depletes, society is switching to mining tar sands.

The Hubbert equation, in all its forms, follows as a consequence of the definition of the "logistic" function $Q(t)$. It ranges from 0 in the past to Q_∞ in the future.

$$Q(t) = \frac{Q_\infty}{1 + e^{\omega(\tau - t)}} \tag{4}$$

VERIFICATION THE FOUR FORMS ARE EQUIVALENT

If you buy the idea that your data scatter in $(Q_i, P_i/Q_i)$-space is a straight line, then you have bought equation (2). If you buy any one of equations (1),(2),(3), or (4), then you have bought them all because they are mathematically equivalent. Starting from the definition (4) using the rule from calculus that $d(1/v)/dt = -(dv/dt)/v^2$ yields equation (3).

$$\frac{dQ}{dt} = P(t) = Q_\infty \, \omega \, \frac{e^{\omega(\tau - t)}}{(1 + e^{\omega(\tau - t)})^2} \tag{5}$$

$$P(t) = Q_\infty \, \omega \, \frac{1}{(e^{-(\omega/2)(\tau - t)} + e^{(\omega/2)(\tau - t)})^2} \tag{6}$$

which is equation (3).

Equation (4) allows us to eliminate the denominator in equation (5) getting equation (2)

$$P/Q = (Q/Q_\infty)\ \omega\ e^{\omega(\tau - t)} \tag{7}$$

$$P/Q = (Q/Q_\infty)\ \omega\ ((1 + e^{\omega(\tau - t)}) - 1) \tag{8}$$

$$P/Q = (Q/Q_\infty)\ \omega\ (Q_\infty/Q - 1) \tag{9}$$

$$P/Q = \omega\ (1 - Q/Q_\infty) \tag{10}$$

which is equation (2). Multiplying both sides by Q gives equation (1).

REFERENCE

1. http://www.hubbertpeak.com/hubbert/1956/1956.pdf contains Hubbert(1956, M.King Hubbert, Nuclear energy and fossil fuels, Publication 95, Shell Oil Company.

2. Kenneth S. Deffeyes, 2006, Beyond Oil: The view from Hubbert's Peak: Hill and Wang ISBN 0-8090-2957-X.

POSTFACE

One day I learned that Firefox had a much better way of zooming web pages, zooming the pictures too. Knowing that equations are pictures I went to Wikipedia, and looked up "Fourier Analysis". I was delighted. A table of equations looked beautiful and could be zoomed up to a size suitable for public lectures! It was as if html had finally incorporated math. In reality the math had been done via LaTeX and inserted as photos. Wanting to have on-line lectures drawn exactly from my books I learned to contribute to Wikipedia including equations.

At the same time I was reading Deffeyes book "Beyond Oil" (a play on the slogan "Beyond Petroleum"). I wanted to play with Hubbert's curve fitting of worldwide oil production. Francis Muir gave me the algebraic tips I needed. I prepared my contribution in my "sandbox" and then moved it to the main encyclopedia. One of their volunteer managers soon found it and didn't like it. Rather than quote his opinions, I paraphrase saying "equation derivations do not belong in an encyclopedia."

So, I gave up and prepared this PDF file instead. It's not as seamlessly web viewable as html, but I'm much happier with it – and I am able to include it in this report!

A goal I cannot meet today is to write a single LaTeX file that becomes two things: (1) print media that is attractive, readable, and contains its web references (perhaps in footnotes), and (2) a PDF file with references as web links that work. I would like to see my books and all SEP report articles in this form. I'm not going to develop this myself. Someone else will do it; and I'll try to be an early adopter.