# Stable simulations of illumination patterns caused by focusing of sunlight by water waves

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## ABSTRACT

Illumination patterns of underwater sunlight have fascinated various researchers in the past. I derive a set of equations that models these patterns at arbitrary depths from arbitrary surface topology functions. The rays are approximated by a statistical distribution function that is shifted to different lateral positions depending on the ray's refraction angle at the water surface. I perform simulations using either delta or Gaussian distributions. Using delta distributions proves unstable because of the Gibbs phenomenon, which can be suppressed using a low-pass filter. The simulations using a Gaussian distribution are stable and, apart from minor smoothing, do not suffer from additional artifacts.

## INTRODUCTION

The topography of the water surface bends sunlight to form sharp stripes and exotic patterns under the surface. Claerbout (2007) relates these light patterns under water to seismology, bright spots in seismic sections usually associated with hydrocarbon reservoirs. Some of these bright spots do not reflect rock or reservoir characteristics, but are instead manifestations of wavefield focusing by lateral velocity variations in the overburden. This paper addresses accurate modeling illumination patterns caused by focusing of sunlight by water waves.

Much work has been done to understand and predict the amplitude spectra of light-field fluctuations with water depth. Schenck (1957) was the first to derive a model of underwater sunlight focusing, he derived the direction of light rays under the water surface in 2D using Snell's law and a function describing the water surface. His research and later work aimed at satisfying the needs of marine biologists to predict the photosynthetic production in the photic zone (Dera and Gordon, 1968). Snyder and Dera (1970) did a more careful analysis in 3D, deriving a set of equations that describe the focusing to first order but are valid only near to the water surface. This model could be used as a starting point for inversions or adjoint modeling for illumination patterns close to the water surface. Nikolayev and Khulapov (1976) derived a relation describing how the statistical properties of illumination fluctuations depend on average wave slopes. Stramski and Dera (1988) extended Schenck's gravity wave model to include shorter wavelength capillary waves. They also claimed that for practical purposes, only the statistical properties of the water surface can be derived from measured illumination fluctuation profiles. More recently, Zaneveld et al. (2001) estimated the diffuse attenuation coefficient of light intensity with depth, from real data. A study by Sabbah and Shashar (2006) considers how diffuse light sources, as opposed to direct sunlight, contribute to underwater light polarization and radiance fluctuations.

All previous simulations of underwater illumination patterns were performed by counting the ray density after ray tracing. This method suffers from instabilities caused by the finite number of rays used to calculate the densities. In this paper I formalize the ray-tracing method by approximating rays by delta distributions of illumination. These are extrapolated to depth and shifted, as a function of the angle of the ray, using the shift property of the delta function. I derive a close form expression of the illumination pattern in the spatial Fourier domain. I propose Gaussian distributions as an alternative to delta distributions for modeling incident light rays at the surface. They yield a more stable and smoother calculation in the space domain.

## THEORY

First, I describe my theory for simulating underwater illumination patterns of sunlight. I define the Fourier transformation of a space-dependent function, $g = g(x)$ to be

$$\hat{g}(k) = \mathcal{F}\left\{g(x)\right\} = \int_{-\infty}^{\infty} g(x)\, \exp\left\{ikx\right\}\, \mathrm{d}x, \tag{1}$$

where $k$ is the wavenumber. The illumination as a function of distance $x$ at a depth $z$ is denoted as $f = f(x, z)$, with z positive downwards. I treat illumination at a point as being caused by the concentration of rays arriving from different directions.

Consider the special depth level, $z_0$, at which the illumination at each point, $f_0(x) = f(x, z_0)$, is created by one light ray with an unique angular direction with respect to the vertical, $\theta = \theta(x)$. A straightforward derivation of the angle $\theta$ is given by Schenck (1957). Equation 2 gives the angle of a ray just under the water surface as a function of incidence angle $\theta^i = \theta^i(x)$, the function describing the water surface $S = S(x)$, and the indices of refraction of air and water, respectively $\eta_a$ and $\eta_w$:

$$\theta(x) = \mathrm{atan}\left\{\partial_x S(x)\right\} - \mathrm{asin}\left[\eta_a \eta_w^{-1} \sin\left(\mathrm{atan}\left\{\partial_x S(x)\right\} - \theta^i\right)\right]. \tag{2}$$

All angles are defined as positive clockwise. The intensity of sunlight, $I_s$, distributes over the surface as $f_0(x) = \sin\left\{\theta^i(x)\right\}\, I_s$. The illumination at depth $z_0$ is written as an integral of delta functions,

$$f_0(x) = f(x, z_0) = \int_{-\infty}^{\infty} f_0(r)\, \delta\left(x - r\right)\, \mathrm{d}r. \tag{3}$$

Consider these delta functions as the contribution from each ray. Each ray is shifted to a different lateral position at greater depths; the shift is given by $\Delta z\, \tan\left\{\theta\right\}$, where

$\Delta z = z - z_0$. In this way I write the illumination at a depth $z$ as

$$f(x,z) = \int_{-\infty}^{\infty} f_0(r)\ \delta\left(x - [r + \Delta z\ \tan\theta(r)]\right)\ \mathrm{d}r, \tag{4}$$

or, using equation 1,

$$\hat{f}(k,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_0(r)\ \delta\left(x - [r + \Delta z\ \tan\theta(r)]\right)\ \exp\{ikx\}\ \mathrm{d}x\mathrm{d}r \tag{5}$$

in the Fourier domain. Evaluating the integration over $x$ modifies the argument of the exponent:

$$\hat{f}(k,z) = \int_{-\infty}^{\infty} f_0(r)\ \exp\{ik\,[r + \Delta z\ \tan\theta(r)]\}\ \mathrm{d}r. \tag{6}$$

After inverse Fourier transformation, Equation 6 becomes

$$f(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_0(r)\ \exp\{ik\,[r + \Delta z\ \tan\theta(r)]\}\ \exp\{-ikx\}\ \mathrm{d}r\mathrm{d}k. \tag{7}$$

Calculation of the delta distribution in the wavenumber domain is unstable, because it would require infinite bandwidth. The result would suffer from the Gibbs phenomenon, as discussed in the next section. One can apply a smoothing filter in the wavenumber domain to suppress the worst of the ringing. An alternative approach is to replace the delta distribution with a different distribution. I propose a Gaussian distribution centered at $x = b$ of the following form:

$$\mathcal{G}(x,b) = \frac{1}{c\sqrt{2\pi}}\ \exp\left\{-\frac{(x-b)^2}{2c^2}\right\}. \tag{8}$$

Note that the integral $\int_{-\infty}^{\infty}\mathcal{G}(x,b)\ \mathrm{d}x = 1$. Smoothing the illumination at $z = z_0$ with a Gaussian distribution yields

$$f(x,z_0) = \int_{-\infty}^{\infty} f_0(r)\ \mathcal{G}(x,r)\ \mathrm{d}r, \tag{9}$$

which projects to a depth $z$ in a fashion similar to that in Equation 4,

$$f(x,z) = \int_{-\infty}^{\infty} f_0(r)\ \mathcal{G}(x,\ r + \Delta z\ \tan\{\theta(r)\}\ )\ \mathrm{d}r. \tag{10}$$

Using Equation 10, this expands to

$$f(x,z) = \frac{1}{c\sqrt{2\pi}}\ \int_{-\infty}^{\infty} f_0(r)\ \exp\left\{-\frac{(\ x - [r + \Delta z\ \tan\theta(r)]\ )^2}{2c^2}\right\}\ \mathrm{d}r. \tag{11}$$

In the next section I show examples of illumination patterns calculated using these formulas.

# EXAMPLES

The shape of the water surface is modeled as a Gaussian function, similar to Equation 8 with $a = 5$, $b = 0$ and $c = 3$. This wavelet exhibits various focusing and bending aspects and is well suited to assess the accuracy to which the equations in the previous section can be computed. In addition, the sun is modeled as a point source at infinity, at an inclination of 85° to the horizon. Thus, the sunlight forms an incident plane wave at a 5° angle with the vertical. Let the intensity of the sun be $I_s = 1$, which yields an intensity of $f_0(x) = \cos 5°$. Figure 1 shows a ray tracing example of the configuration, where the thickness of the water surface lens is not neglected. The illumination pattern at a depth of 75 m is calculated using Equation 7

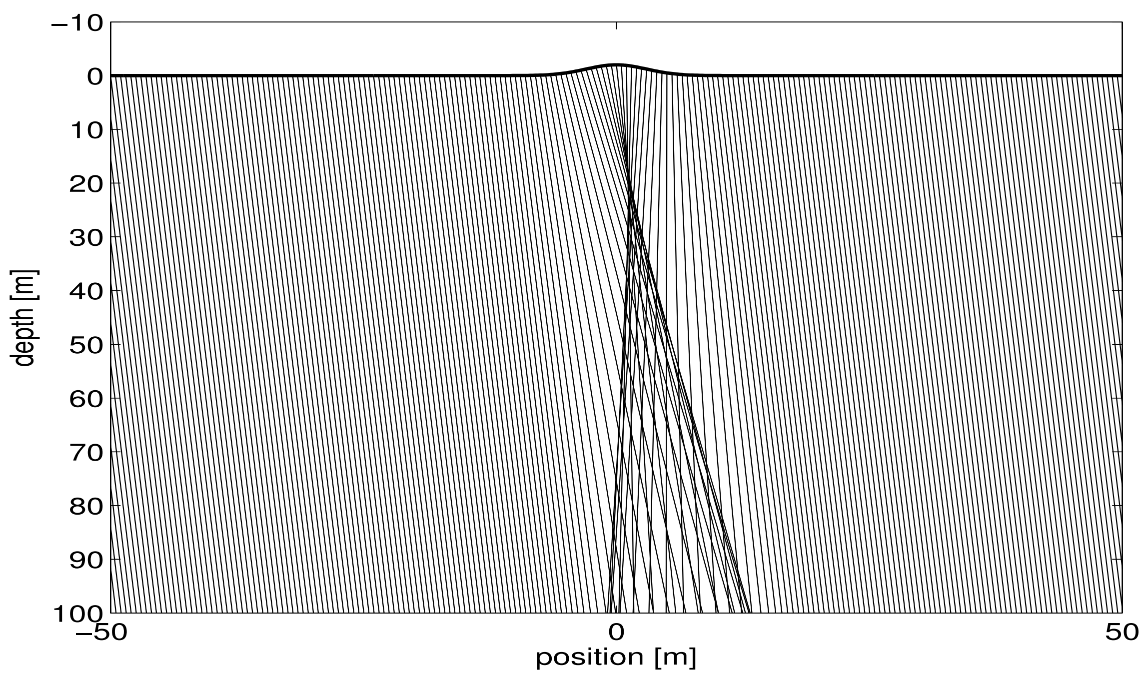


Figure 1: Ray tracing for sunlight passing through a Gaussian-shaped water wave. [**ER**]

and shown in the top panel of Figure 2. The Gibbs phenomenon is clearly visible and obscures the computed pattern. The real part of the spatial Fourier domain spectrum is shown in the top panel of Figure 3. The Gibbs phenomenon is suppressed in the Fourier domain using a Hanning low-pass filter penalizing the spectrum above and under $k = \pm 18$ m$^{-1}$. The filtered spectrum is shown in the middle panel of Figure 3. The obtained illumination pattern, seen in the middle panel of Figure 2, is now free of much of the ringing but still over- and under-shoots at the edges of the horn shape.

I use the Gaussian distributions to approximate the rays at the surface, and equation 11 to simulate the illumination pattern at a depth of 75 m. The resulting pattern is shown in the bottom panel of Figure 2. The real part of its spectrum is shown in Figure 3.

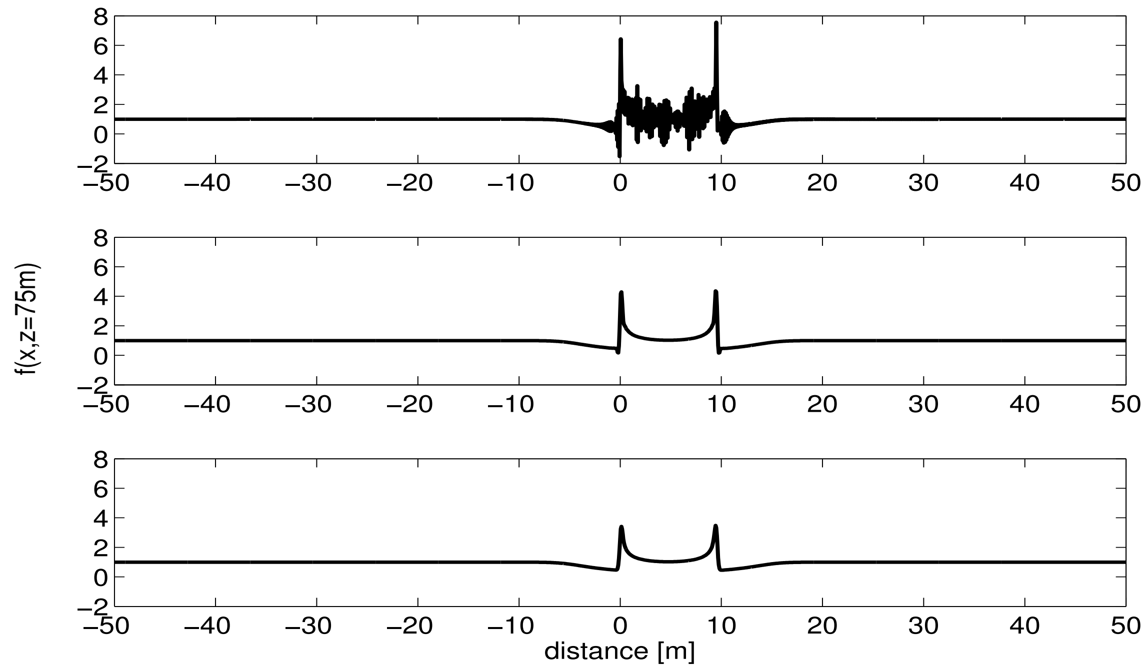


Figure 2: Various profiles at 75 meters deep. Top: rays modeled as delta distributions. Middle: Fourier-domain Hanning low-pass filtered version of the top panel. Bottom: rays approximated as Gaussian distributions. [**ER**]

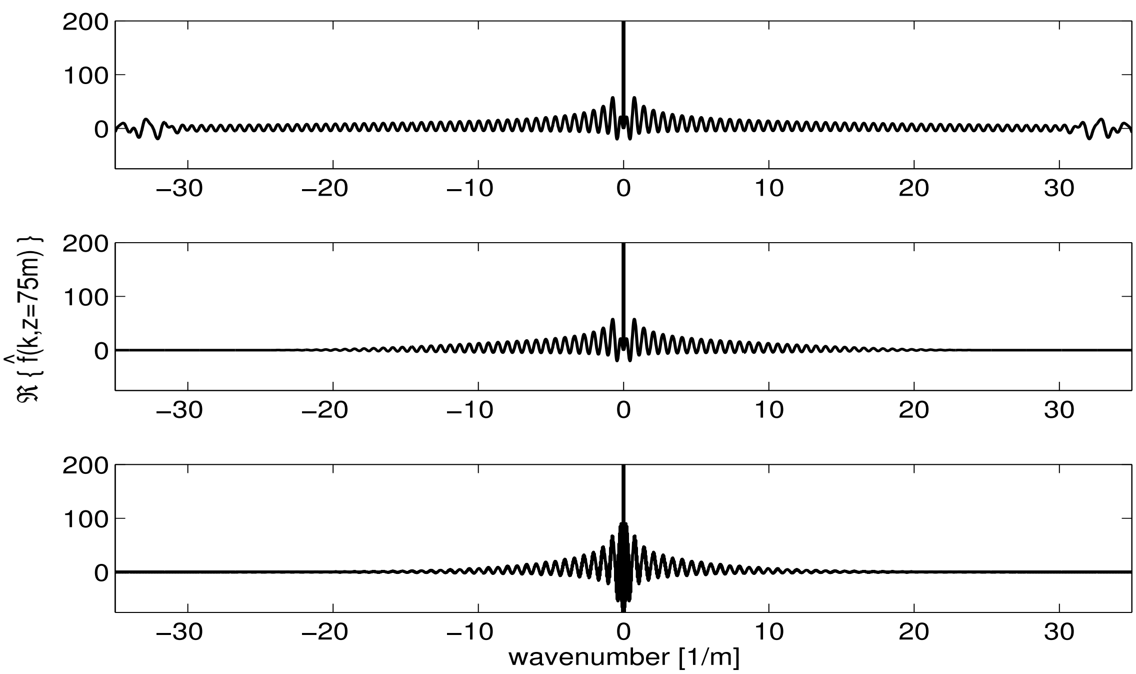


Figure 3: Real part of spatial Fourier domain spectrum for various profiles at 75 meters deep. Top: rays modeled as delta distributions. Middle: Fourier-domain Hanning low-pass filtered version of the top panel. Bottom: rays approximated as Gaussian distributions. [**ER**]

## DISCUSSION AND CONCLUSIONS

The equations in this paper describe a new approach for modeling underwater illumination patterns of focused sunlight. The shift approach creates remarkably detailed results that are accurate in areas with low or high illumination intensity. Ray counting methods suffer in low coverage areas because of their discrete nature. Simulations using delta functions are unstable due to the Gibbs phenomenon, although applying a filter in the Fourier domain successfully improves the computed pattern. When we use a Gaussian distribution to approximate rays, there is no need to enter the Fourier domain. The standard deviation that modulates the width of the Gaussian should be matched to the sampling of the water surface function. I find that a standard deviation of $c = 2\Delta x$ generally produces good results. However, there is a limit on the second derivative of the function $\theta(x)$, for the ratio of $c = 2\Delta x$ to remain valid.

Filtering is a processing step that needs adjusting, but so is the parameter $c$ in the Gaussian distribution. My simulations become more stable but less accurate when a broader Gaussian is used to approximate rays. The mapping of $\theta(x)$ onto $f(x, z)$ is unique but non-linear. The map does not seem invertible, making processing by adjoint methods very difficult. Non-uniqueness is introduced by discretization of the presented integral equations. Posing a smoothness restriction on the function $\theta(x)$ will perhaps linearize the forward problem and reduce the number of non-unique solutions.

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