An algorithm for interpolation using Ronen's pyramid

Jon Claerbout and Antoine Guitton

ABSTRACT

Shuki Ronen has shown that a dip spectrum in 2-D may be characterized by a 1-D Prediction Error Filter (PEF) after his "pyramid transform" where (t, x)-space is transformed to $(\omega, u = \omega x)$-space. The transform to (ω, u)-space creates empty locations (missing data). Thus we have the question of unknown PEF along with missing data. Here we propose to find both simultaneously by iterative linear least squares.

INTRODUCTION

A Prediction Error Filter (PEF) is good at characterizing a 1-D spectrum. A short simple PEF will easily model a narrow bandwidth, a broad one, or a combination — whatever fits. Unfortunately, the spectrum characterized by a PEF is a temporal spectrum or a spatial spectrum, when in practice it is more generally the dip spectrum that we seek a good representation for.

Although the helix transform brings us the power of PEFs to 2-D and 3-D data it does not directly offer us exactly what often seek, a dip spectrum. Shuki Ronen's pyramid transform does. It brings PEF power from time and space axes to the dip axis.

First we review Ronen's principles noticing that raw data traces are moved to radial traces on a new grid. The 1-D PEF must be found on this new grid. But the new grid has empty spaces (missing data) between the known radial traces. Thus we approach the problem as one of nonlinear least squares which we propose to solve by iterative linear least squares.

FIRST PRINCIPLES

Let us review Ronen's pyramid transform (Hung et al., 2005). Fourier transform data from (t, x) to (ω, x). Ronen then shrinks and stretches the space axis to transform from x to $u = \omega x$. All the information on a trace (constant x_0), in the (t, x) plane, lands on the line $u = \omega x_0$ in the the $(\omega, u = \omega x)$-plane. This is a line through the origin. Ordinary traces have become radial traces.

Look ahead to Figure 1 and you will see an example of the basic property of Ronen's space, the property we will prove next, namely: Whereas dips in (t, x) space are curves in (ω, x) space, they return to straight lines in (ω, u) space. Dips in physical (t, x) space become frequencies on Ronen's u-axis, the same u-space frequency for all ω. To track the three events in Figure 1 into Ronen's space, it's helpful to notice that each has a different ω spectrum.

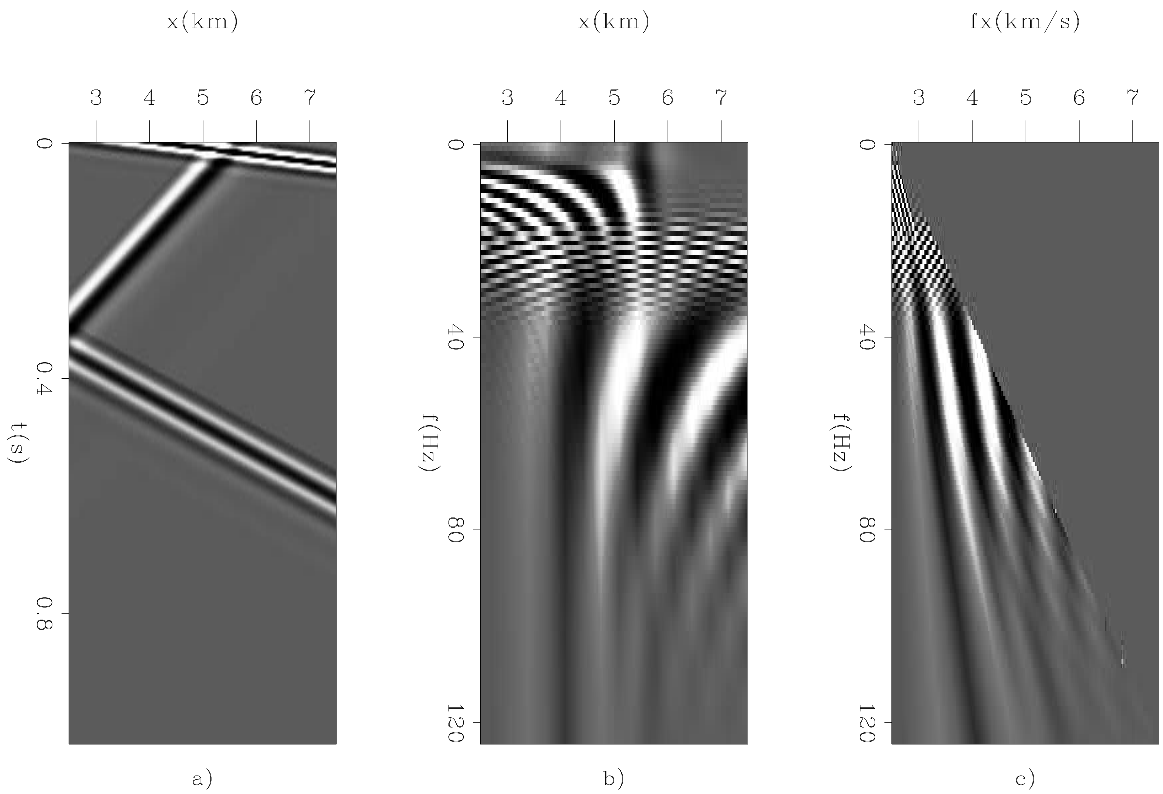

Figure 1: LEFT: Input data (t, x) with three events, each with different frequency content (and spatial extent). MIDDLE: Real part of (ω, x)-space of the panel on the left. RIGHT: Same in $(\omega, u = \omega x)$-space, Ronen's pyramid space in which we again see straight lines as we did in (t, x)-space. Observe three frequencies on the u-axis, each corresponding to a different slope in the (t, x)-plane. To help identify which is which, notice each of the three events in the (t, x)-plane has a different ω spectrum. [**ER**]

To establish the basic property of pyramid space we consider a dipping signal in (t, x)-space, say $g(t, x) = g(t - px)$. Transform time t to frequency ω. Now we have a plane of complex valued information say $G(\omega, x)$. What is it in Ronen's (ω, u) space? Say $g(t, x = 0)$ fourier transforms to $G(\omega)$. Then $g(t - t_0)$ transforms to $G(\omega)e^{i\omega t_0}$. Take the time shift t_0 to be distance x times slope p. Then $g(t - xp)$ transforms to $G(\omega)e^{i\omega xp} = G(\omega)e^{ipu}$. Add some extra plane waves.

$$G(\omega, u) \quad = \quad G_1(\omega)e^{ip_1 u} + G_2(\omega)e^{ip_2 u} + \cdots \tag{1}$$

Examine any ω. You see sinusoids on the u-axis. The frequencies of these sinusoids are $(p_1, p_2, \cdots)$. We see those same frequencies for each ω. Thus the PEF on the u-axis informs us of a spectrum on the p-axis. To a seismologist, p is a dip axis though it would be more correct to call it an axis of stepout.

What happens in 3-D? Instead of (x, y)-axes we have $(u, v) = (\omega x, \omega y)$ axes. It is not immediately clear what applications will bring us to do in the (u, v)-plane. Some applications may call for a 2-D PEF there, while others call for two 1-D PEFs. What are we expecting to describe in (p_x, p_y)-space? Globally, there may be a bounding range in a slowness circle corresponding to a slowest material velocity. Locally things will look more like plane waves.

APPLICATION: AN ALGORITHM FOR INTERPOLATION

In seismology we often deal with instruments spaced more widely than they should be, more widely than they should be for typical data processing such as Fourier transform, more widely than is suitable for data display. Fundamentally something is lost, but that does not detract from our goal of regular spaced data on a dense enough mesh. Given regular data, finding a PEF is a linear problem. Given a PEF, interpolating data is a linear problem (demonstrated by many examples in Claerbout's free on-line book, "Image Estimation by Example"). In both cases we are minimizing the energy of filtered data. A more generalized approach minimizes energy in the filtered data where some unknowns are in the PEF while others are missing data values among the knowns. This minimization is nonlinear (because the PEF multiplies missing data).

The main difficulty of trying to utilize Ronen's pyramid in practice is the issue of bringing the x-space to the u-space. On first glance it seems to require interpolation. On trial, this interpolation seems to need to be done extremely carefully. An alternate approach, which we take here, is to sample the u-space very densely. This, of course, introduces many locations not touched directly by the data. We have traded the interpolation problem for a missing data problem, nonlinear because we must estimate this new missing data at the same time we estimate the PEF. We'd like to come up with a reliable pyramid method of interpolating aliased data that is devoid of low frequency information. An attractive feature is that the pyramid concept does not require the original data on a regular mesh in x-space. Our method will build a dense regular mesh in model u-space.

The problem is non-linear because of the product of unknowns, the PEF multiplying the missing data. Here we approach the problem by multistage linear least squares which can be iterated to solve the nonlinear problem. In any nonlinear problem the initial guess must be "near enough". Hopefully the proposed method will not demand unaliased low frequency information.

Imagine five unevenly spaced traces on the x-axis. The data space is defined as $d(\omega, x)$ at five known values of the coordinate x. Define a model space $m(\omega, u = \omega x)$ that is dense (many uniformly spaced points) on u-space. We are interested in fitting

$$\mathbf{0} \;\approx\; \boldsymbol{L}\boldsymbol{m} - \boldsymbol{d}$$

The operator $\boldsymbol{L}$ linearly interpolates from the dense u-axis to the sparse x-space (which need not be regularly sampled on x). In the limit of an extremely dense u-space we might choose $\boldsymbol{L}$ to be "extraction" basically "nearest-neighbor inverse binning". A zeroth order model space is $\boldsymbol{m_0} = \boldsymbol{L}'\boldsymbol{d}$. This model is simply dropping the several data traces into radial traces in pyramid space. Because u-space has very many points, $m_0(\omega, u)$ has many empty regions (triangularly shaped). Thus we use preconditioning. Let us precondition with the convolutional roughener $A0 = (1, -1)$ on the u-axis. This particular $A0$ leads to a solution $m(u)$ that linearly interpolates the given data. Such preconditioning should be good at large ω where the radial traces are far apart. Thus $\boldsymbol{p} = \boldsymbol{A_0}\boldsymbol{m}$ or $\boldsymbol{m} = \boldsymbol{A_0}^{-1}\boldsymbol{p}$. To find $\boldsymbol{p}$ and $\boldsymbol{m}$ (independently for each ω) we iterate on the regression

$$\mathbf{0} \;\approx\; \boldsymbol{L}\boldsymbol{A_0}^{-1}\boldsymbol{p} - \boldsymbol{d}$$

The resulting $\boldsymbol{m}$ we will call $\boldsymbol{m_1}$ (because we will eventually improve on it getting an $\boldsymbol{m_2}$).

Next let us upgrade $\boldsymbol{A_0}$. At each ω from the model space $\boldsymbol{m_1}$, make an operator $\boldsymbol{M_1}$ for convolution over the $\boldsymbol{u}$ axis. Simultaneously for all ω we find the regression for an upgraded PEF $\boldsymbol{a}$ (which is constant over ω).

$$\mathbf{0} \;\approx\; \boldsymbol{W}\boldsymbol{M_1}\boldsymbol{a}$$

Here $\boldsymbol{W}$ is a diagonal weighting matrix defined next. As mentioned earlier, there are large empty spaces in the zeroth order model space $\boldsymbol{m_0} = \boldsymbol{L}'\boldsymbol{d}$. Although our improved model space $m_1(\omega, u)$ has filled the holes with artificial data, we don't want to use regression equations except where the model space points directly to real data, namely where $m_0(\omega, u)$ is non-zero. Thus we define $\boldsymbol{W}$ to be a diagonal matrix of ones and zeros, zeros where $m_0(\omega, u)$ is zero.

Although we are planning to iterate, we will never change $\boldsymbol{W}$. From solution of the regression above we have the vector $\boldsymbol{a}$ which we use to make the filter operator $\boldsymbol{A_1}$. Use it in place of $\boldsymbol{A_0}$ in the regression for $\boldsymbol{p}$ above. Iterate to get an $\boldsymbol{m}$ that is improved over $\boldsymbol{m_1}$. Call it $\boldsymbol{m_2}$. Iterate.

CONCLUSIONS

We have an algorithm. We should code it and test it.

REFERENCES

Hung, B., C. Notfors, and S. Ronen, 2005, Seismic data reconstruction using the pyramid transform: 67th Meeting, EAGE, Expanded Abstracts, A039.