Seismic anisotropy for polar media and an extended Thomsen formulation for longer offsets

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ABSTRACT

Crack-influence parameters of Sayers and Kachanov (1991) have been shown to be directly related to Thomsen weak-anisotropy parameters for seismic wave speeds. These results are applied to the problem of seismic propagation in reservoirs having polar (HTI) symmetry due to aligned vertical fractures. To take full advantage of these relationships, it is also helpful to obtain more accurate expressions for seismic wave speeds in polar media at longer offsets than those originally intended for Thomsen's weak anisotropy formulation.

INTRODUCTION

Recently published work of the author (Berryman, 2007, 2008) shows how anisotropy due to cracks or large-scale fractures can be quantified using crack-influence parameters of Sayers and Kachanov (1991). Taking maximal advantage of this work requires use of exact or nearly exact formulas for anisotropic seismic wave speeds in field applications. Thomsen's weak anisotropy formulation (Thomsen, 1986), while accurate for short-offset data, is not adequate at longer offsets in strongly polar media. The point of the work summarized here is to arrive at a formulation almost as simple as Thomsen's, yet accurate enough to be useful at least to angles on the order of $\theta \simeq 45^o$.

EXACT RESULTS FOR VTI AND HTI MEDIA

If ρ is inertial mass density, and the coefficients c_{ij} are the usual elastic stiffnesses (for example, in isotropic media, $c_{11} = c_{22} = c_{33} = \lambda + 2\mu$, $c_{44} = c_{55} = c_{66} = \mu$, $c_{12} = c_{13} = c_{23} = \lambda$ with λ and μ being the well-known Lamé elastic parameters), then the exact results for VTI media with x_3-axis of symmetry (so $c_{11} = c_{22} \neq c_{33}$, $c_{13} = c_{23} \neq c_{12}$, $c_{44} = c_{55} \neq c_{66}$, and $c_{12} = c_{11} - 2c_{66}$) are given exactly by:

$$v_p^2(\theta) = \frac{1}{2\rho}\left\{c_{44} + \left[c_{11}\sin^2\theta + c_{33}\cos^2\theta\right] + R(\theta)\right\} \tag{1}$$

and

$$v_{sv}^2(\theta) = \frac{1}{2\rho}\left\{c_{44} + \left[c_{11}\sin^2\theta + c_{33}\cos^2\theta\right] - R(\theta)\right\}, \tag{2}$$

where

$$
\begin{aligned}
R^2(\theta) \;\; = \;\; & \Big[(c_{11} - c_{44}) \sin^2\theta \\
& - \;\; (c_{33} - c_{44}) \cos^2\theta \Big]^2 \\
& + \;\; 4\,(c_{13} + c_{44})^2 \sin^2\theta \cos^2\theta,
\end{aligned}
\tag{3}
$$

and where polar angle θ is measured from the vertical x_3-axis directed into the earth. Results for horizontally polarized shear waves in VTI media will not be treated in the following discussion. Results for HTI media (for example, from aligned vertical cracks) are obtained directly by substituting $\theta_H = \theta - \pi/2$ for any vertical plane that is also perpendicular to the vertical plane of symmetry.

Thomsen's anisotropy parameters ϵ, δ, γ can be related directly to the Sayers and Kachanov (1991) crack-influence parameters η_1 and η_2 for horizontal cracks according to $\epsilon_h = \frac{c_{11}-c_{33}}{2c_{33}} = \rho_c[(1+\nu_0)\eta_1+\eta_2]\frac{E_0}{1-\nu_0^2}\eta_2 G_0 \simeq \frac{2\rho_c\eta_2 G_0}{1-\nu_0} = \delta_h$, and $\gamma_h = \frac{c_{66}-c_{44}}{2c_{44}} = \rho_c\eta_2 G_0$, where Poisson's ratio ν_0, Young's modulus E_0, and shear modulus G_0 are the values for the assumed isotropic background medium. For the penny-shaped cracks (having penny radius a) considered (see Table 1), Thomsen's parameters ϵ and δ are always found to be equal to each other to lowest order in the crack density $\rho_c = na^3$, with $n = N/V$ being the crack number density.

Table 1			
Crack Parameters	NI Approx.	1st Model $\nu_0 = 0.00$	2nd Model $\nu_0 = 0.4375$
η_1 (GPa^{-1})	$-\frac{\nu_0\eta_2}{2(5-\nu_0)}$	0.0000	-0.0192
η_2 (GPa^{-1})	$\frac{8(1-\nu_0)(5-\nu_0)}{15(2-\nu_0)G_0}$	0.1941	0.3994

Table 1: Examples of the first crack-influence parameters (the two for lowest order in powers of ρ_c) estimated by Berryman and Grechka (2006), from simulations of Grechka.

EXTENDING THOMSEN'S FORMULAS TO LARGER OFFSETS FOR VTI AND HTI SYMMETRY

The most obvious problem with Thomsen's approximations to the wave speeds generally occurs in $v_{sv}(\theta)$. As noted in earlier work, the key issue is that Thomsen's approximation for $v_{sv}(\theta)$ is completely symmetric around $\theta = \pi/4 = 45^o$, while — unfortunately — this is usually not true of the actual wave speeds $v_{sv}(\theta)$. This inherent error may seem innocuous in itself since it is not immediately clear whether it affects the results for small angles of incidence ($< 15^o$) or not, but this inaccuracy clearly does lead to large over- or under-estimates of wave speeds in the neighborhood of both the extreme value (*i.e.*, a peak or a trough) located at $\theta = \theta_{ex}$ and also in the neighborhood of $\theta = 45^o \neq \theta_{ex}$. So these discrepancies can certainly become issues at offsets larger than the original design criterion having polar angles $\theta \leq 15^o$.

To improve this situation while still making use of a simple and practical approximation to the phase speed, we reconsider an approach originally proposed by Berryman (1979). In particular, notice that the square root of $R^2(\theta)$ in Eq. (3) can be exactly and conveniently rewritten as:

$$R(\theta) = [c_{11}\sin^2\theta + c_{33}\cos^2\theta - c_{44}]\sqrt{1-\zeta(\theta)}, \tag{4}$$

where

$$\zeta(\theta) \equiv \frac{[(c_{11}-c_{44})(c_{33}-c_{44}) - (c_{13}+c_{44})^2]\sin^2 2\theta}{[c_{11}\sin^2\theta + c_{33}\cos^2\theta - c_{44}]^2}. \tag{5}$$

To simplify this expression, first notice that $\zeta(\theta)$ has an absolute maximum (or minimum) value, which occurs when θ takes the value θ_m determined by

$$\tan^2\theta_m = \frac{c_{33}-c_{44}}{c_{11}-c_{44}} \equiv 1 - x_m, \tag{6}$$

where

$$x_m = \frac{c_{11}-c_{33}}{c_{11}-c_{44}} > 0. \tag{7}$$

The inequality in (7) is true for VTI media having horizontal fractures, since for this case the stiffness difference $c_{11}-c_{33}$ is known to be positive (as it is also known that ρ_c, η_2, and G_0 are all positive, while Poisson's ratio satisfies $\nu_0 \leq \frac{1}{2}$).

Then, the extreme value $\zeta_m \equiv \zeta(\theta_m)$ is itself given by

$$\begin{aligned} \zeta_m &= 1 - \tfrac{(c_{13}+c_{44})^2}{(c_{11}-c_{44})(c_{33}-c_{44})} \\ &= \tfrac{2(\epsilon-\delta)c_{33}}{c_{11}-c_{44}} \\ &= \tfrac{2(\epsilon-\delta)v_p^2(0)}{v_p^2(0)(1+2\epsilon)-v_s^2(0)}, \end{aligned} \tag{8}$$

where the second and third expressions in (8) relate ζ_m to the difference between the Thomsen parameters ϵ and δ, and also to $v_p(0)$ and $v_s(0)$. In general, $(\epsilon-\delta)$ — and therefore ζ_m — can take values positive, negative, or zero. Furthermore, $\zeta(\theta)$ may be rewritten as

$$\zeta(\theta) = \frac{2\zeta_m}{1+\chi(\theta)}, \tag{9}$$

where

$$\chi(\theta) = \frac{1}{2}\left[\frac{\tan^2\theta}{\tan^2\theta_m} + \frac{\tan^2\theta_m}{\tan^2\theta}\right]. \tag{10}$$

It is always true that $\zeta(\theta) \leq 1$. [Note that $\zeta_m \geq 0$ for all layered media since $\epsilon - \delta \geq 0$ for layered elastic media (Postma, 1955; Backus, 1962; Berryman, 1979). However, such a simple constraint is *not* known for other types of anisotropic systems.] The square root in equation (4), can be expanded to first order as

$$\sqrt{1-\zeta(\theta)} \simeq 1 - \frac{\zeta(\theta)}{2} = 1 - \frac{\zeta_m}{1+\chi(\theta)}. \tag{11}$$

Approximate results for $v_p(\theta)$ and $v_{sv}(\theta)$ are therefore:

$$v_p^2(\theta) \simeq \frac{1}{\rho}\Big(\left[c_{11}\sin^2\theta + c_{33}\cos^2\theta\right] - \frac{\zeta_m[(c_{11}-c_{44})\sin^2\theta + (c_{33}-c_{44})\cos^2\theta]}{2[1+\chi(\theta)]}\Big) \tag{12}$$

and

$$v_{sv}^2(\theta) \simeq \frac{1}{\rho}\Big(c_{44} + \frac{\zeta_m[(c_{11}-c_{44})\sin^2\theta + (c_{33}-c_{44})\cos^2\theta]}{2[1+\chi(\theta)]}\Big). \tag{13}$$

The only approximation made in arriving at equations (12) and (13) was the approximation of the square root shown in (11).

Although this simple approach is the one most commonly used, the analysis presented is not really limited to using only the first order Taylor approximation in (11). Other researchers (Fowler, 2003; Pederson et al., 2007) have explored rational approximations to such square roots, but we now choose to take a rather different approach.

Progress is made by noting that the quantity $\frac{1}{2}[1+\chi(\theta)]$ may be rewritten as:

$$\frac{1}{2}[1+\chi(\theta)] = \frac{1}{4}\left(\frac{\tan\theta}{\tan\theta_m} + \frac{\tan\theta_m}{\tan\theta}\right)^2 = \frac{(\tan^2\theta + \tan^2\theta_m)^2}{4\tan^2\theta\tan^2\theta_m}. \tag{14}$$

To simplify this expression, first multiply numerator and denominator of (14) by $\cos^4\theta\cos^4\theta_m$. The denominator of the result is then proportional to $\sin^2 2\theta\sin^2 2\theta_m$, while the numerator is now proportional to the square of the quantity $\cos^2\theta\cos^2\theta_m\times(\tan^2\theta+\tan^2\theta_m) = \sin^2\theta\cos^2\theta_m + \sin^2\theta_m\cos^2\theta = \frac{1}{2}\left(1-\cos 2\theta\cos 2\theta_m\right)$. Combining these two results gives

$$\zeta(\theta) = \frac{\zeta_m \sin^2 2\theta_m \sin^2 2\theta}{[1-\cos 2\theta_m\cos 2\theta]^2}, \tag{15}$$

which (although this may not be immediately obvious) is just a more compact version of (5). Equation (15) is exact; no approximations were made in the transition from (5) to (15).

Dividing the expression in the text before Eq. (15) by $2\cos^2\theta_m$, we also have

$$\sin^2\theta + \tan^2\theta_m\cos^2\theta = \frac{[1-\cos 2\theta_m\cos 2\theta]}{2\cos^2\theta_m}. \tag{16}$$

Using the definitions $X_\pm \equiv \left[1 \pm \sqrt{1-\zeta(\theta)}\right]$, the exact expression for quasi-SV-wave speed can now be rewritten as

$$2\rho v_{sv}^2 = 2c_{44} + (c_{11}-c_{44})\left(\sin^2\theta + \tan^2\theta_m\cos^2\theta\right)X_-. \tag{17}$$

Similarly, the exact equation for quasi-P-wave speed becomes

$$2\rho v_p^2 = 2c_{44} + (c_{11}-c_{44})\left(\sin^2\theta + \tan^2\theta_m\cos^2\theta\right)X_+. \tag{18}$$

Both of these expressions are exact rearrangements of the original equations.

These results can be consolidated further by using the result (16), together with the definition

$$c_{11} + c_{33} - 2c_{44} = \frac{c_{11} - c_{44}}{\cos^2\theta_m} \equiv 2\Delta c. \tag{19}$$

So finally, we have two very compact expressions for the exact wave speeds:

$$\rho v_{sv}^2 = c_{44} + \frac{\Delta c}{2}\left[1 - \cos 2\theta_m \cos 2\theta\right]\left[1 - \sqrt{1 - \zeta(\theta)}\right] \tag{20}$$

for the quasi-SV-wave speed, and also the corresponding equation which is

$$\rho v_p^2 = c_{44} + \frac{\Delta c}{2}\left[1 - \cos 2\theta_m \cos 2\theta\right]\left[1 + \sqrt{1 - \zeta(\theta)}\right] \tag{21}$$

for the exact quasi-P-wave speed.

Extended formulas for HTI symmetry

To complete this analysis, we show how the results for HTI symmetry arise when the fractures/cracks are aligned and vertical. These results follow easily from our earlier work:

$$\begin{aligned} \bar{v}_p^2(\theta_H)/v_p^2(0)(1+2\epsilon) &\simeq 1 - \tfrac{2\epsilon}{1+2\epsilon}\sin^2\theta_H \\ &- \tfrac{\epsilon-\delta}{2(1+2\epsilon)}\tfrac{2\sin^2\theta_m \sin^2 2\theta_H}{[1+\cos 2\theta_m \cos 2\theta_H]}, \end{aligned} \tag{22}$$

from which we find:

$$\begin{aligned} \bar{v}_p(\theta_H)/v_p(0)\sqrt{1+2\epsilon} &\approx 1 - \tfrac{\epsilon}{1+2\epsilon}\sin^2\theta_H \\ &- \tfrac{\epsilon-\delta}{4(1+2\epsilon)}\tfrac{2\sin^2\theta_m \sin^2 2\theta_H}{[1+\cos 2\theta_m \cos 2\theta_H]}. \end{aligned} \tag{23}$$

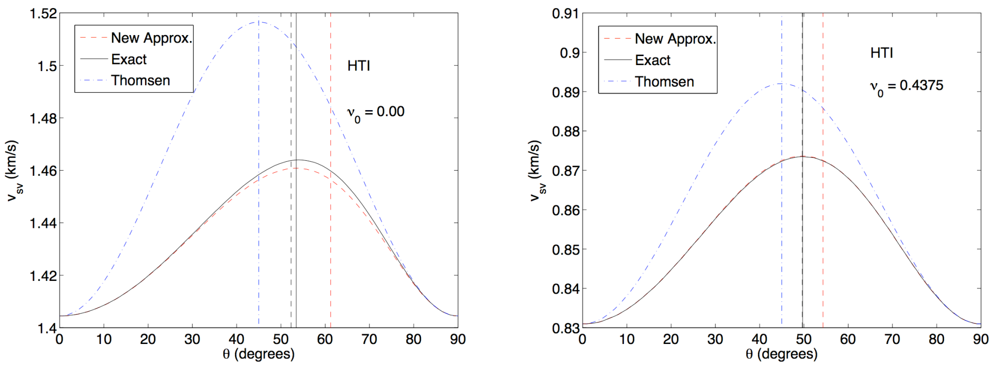
Similarly,

$$\begin{aligned} \bar{v}_{sv}^2(\theta_H)/v_s^2(0) &\simeq 1 + \left[v_p^2(0)/v_s^2(0)\right] \\ &\times \tfrac{\epsilon-\delta}{2}\tfrac{2\sin^2\theta_m \sin^2 2\theta_H}{[1+\cos 2\theta_m \cos 2\theta_H]}, \end{aligned} \tag{24}$$

from which follows:

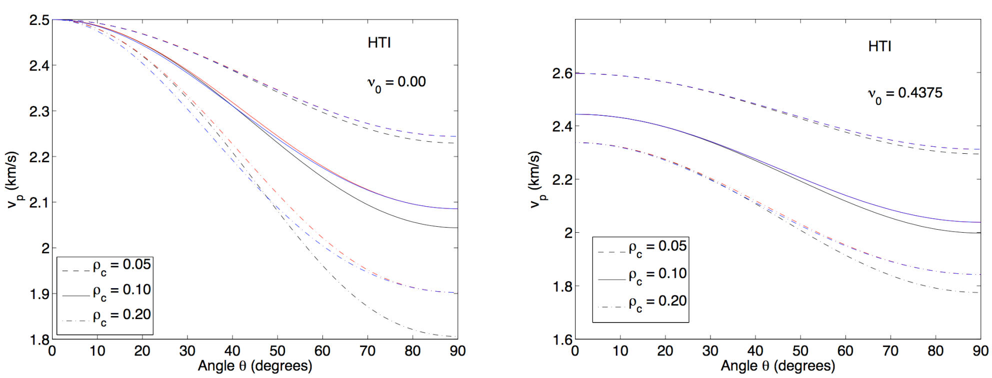
$$\begin{aligned} \bar{v}_{sv}(\theta_H)/v_s(0) &\approx 1 + \left[v_p^2(0)/v_s^2(0)\right] \\ &\times \tfrac{\epsilon-\delta}{4}\tfrac{2\sin^2\theta_m \sin^2 2\theta_H}{[1+\cos 2\theta_m \cos 2\theta_H]}. \end{aligned} \tag{25}$$

Again, these formulas reduce exactly to the equivalent Thomsen formulas for HTI symmetry if $\theta_m \rightarrow 45^o$. It should always be remembered, however, that these formulas apply only in planes perpendicular to the plane of the aligned fractures. For other angles of propagation, we must also account for the azimuthal dependence on angle ϕ, although this is in fact easy to do. Examples of these results are presented in the three Figures.



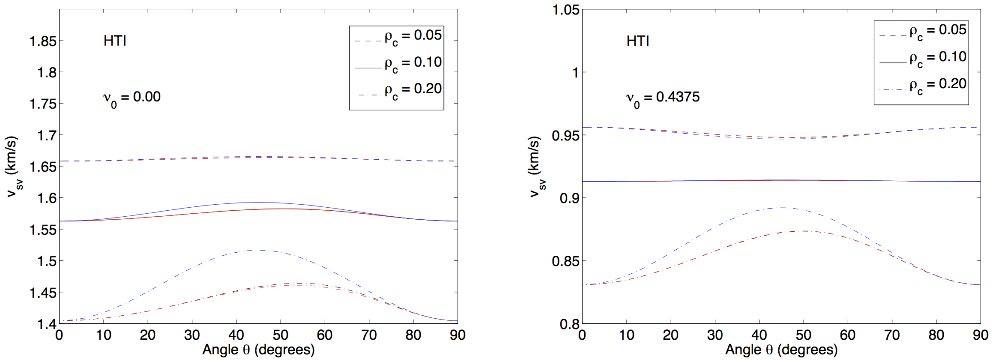
(a) (b)

Figure 1: For aligned vertical cracks having crack density $\rho_c = 0.2$ and HTI overall symmetry: two examples of anisotropic quasi-SV shear wave speeds for two values of Poisson's ratio ν_0 of the host medium: top $\nu_0 = 0.00$, bottom $\nu_0 = 0.4375$. Background wave speeds are top $v_s = 1.77$ km/s, bottom $v_s = 1.00$ km/s. Speeds in red are those for the new approximation. The corresponding exact result is then overlain in black. Finally, Thomsen's weak anisotropy curves are overlain in blue. Two of the vertical lines indicate locations of the true peaks of these curves: Thomsen's approximation is the dashed blue line and it is always at 45^o. The solid black line is the true peak of the exact expression. The value of θ_m from equation (6) is the vertical line shown in red. The other two dashed black vertical lines are two estimators (not treated here) of the peak of our approximate curve; these two estimators are often nearly indistinguishable. [**NR**]



(a) (b)

Figure 2: Quasi-P-wave speeds for the same two model reservoirs, for three choices of crack density $\rho_c = 0.05, 0.10, 0.20$. Background P-wave speed for the first model is $v_p = 2.50$ km/s and for the seco2nd model $v_p = 3.00$ km/s. As before, speeds in red are those for the new approximation. The corresponding exact result is then overlain in black. Finally, Thomsen's weak anisotropy curves are overlain in blue. [**NR**]



(a) (b)

Figure 3: Quasi-SV shear-wave speeds for the same two model reservoirs, for three choices of crack density $\rho_c = 0.05, 0.10, 0.20$. The two values of Poisson's ratio ν_0 of the host medium considered are: top $\nu_0 = 0.00$, bottom $\nu_0 = 0.4375$. Background shear-wave speed for the first model is $v_s = 1.77$ km/s and for the second model $v_s = 1.00$ km/s. As before, speeds in red are those for the new approximation. The corresponding exact result is then overlain in black. Finally, Thomsen's weak anisotropy curves are overlain in blue. Plots for $v_{sh}(\theta)$ are not dependent on the formulation of the new scheme and, therefore, are not displayed. [**NR**]

DISCUSSION AND CONCLUSIONS

The preceding analysis presented for the phase velocity equations does not depend on the source of the anisotropy, and therefore can be applied to layered media, etc., as well as to fractured media as we have done here.

The Kachanov (1980) and Sayers and Kachanov (1991) crack-influence parameters are ideally suited to analyzing the role of fracture mechanics in producing anisotropic elastic constants for aligned fractures in a reservoir exhibiting VTI or HTI symmetry. When this approach is combined with poroelastic analysis through the use of Skempton's coefficient [Skempton (1954), see also Berryman (2007)], it becomes comparatively easy to analyze a wide range of complicated situations that may raise in reservoir analysis, such as trying to deduce whether the fractures are dry/drained, or fluid-saturated/undrained. Skempton's coefficient B introduces a single parameter that varies from 0 to 1 as fluid properties change from being negligible to being very strong influences on the fracture compliance – and therefore on the Thomsen seismic parameters.

Another important observation from the modeling presented is that the Thomsen weak anisotropy formulation is valid for crack densities up to about $\rho_c \simeq 0.05$, but should be replaced by more accurate approximations, or (better yet) exact calculations whenever possible if the crack density is much above 0.05. When the crack density is $\rho_c \simeq 0.1$, or higher, then higher accuracy approximations are essential. Conversely, if the crack density ρ_c *estimated* from seismic data using the weak anisotropy formulation is in fact larger than $\rho_c \simeq 0.05$, one conclusion we might reach is that a more accurate method is required both to verify and properly quantify the result.

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