Ignoring density in waveform inversion

Roland Gunther and Biondo Biondi[1]

ABSTRACT

We study the effectiveness of velocity-only time-domain waveform inversion for inverting synthetic data modeled with both velocity and density contrasts. We present a detailed review of the Born approximation for the constant-density acoustic wave equation and its application to the inversion of velocity models for seismic reflection data. We create synthetic models with both constant and variable density and compare the effectiveness of velocity-only waveform inversion in each case. Results from this simple test suggest that density contrasts can hamper the reconstruction of velocity perturbations.

INTRODUCTION

Velocity models for processing seismic reflection data are usually derived from traveltime tomography or other methods that depend on detection of moveout in picked reflection events. Picking is time consuming and prone to error and makes use of only a subset of the information available in a dataset. Waveform inversion provides an alternative approach for deriving velocity models. As an automatic algorithm, waveform inversion is less dependent on human input. The goal of the inversion is to match both data phase and amplitude, so it is theoretically possible to recover subtle local variations that are too small to lead to measurable moveouts in gathers.

Though the method is conceptually appealing, several barriers have prevented waveform inversion from becoming viable for real data: it is only able to recover anomalies that are either very small in magnitude or that have similar spatial wavelengths as the seismic data; it is computationally expensive, especially if based on time-domain modeling; and when the physics of wave-propagation is simplified to reduce cost and complexity, the inversion may not converge to a useful solution. In this report we investigate the last issue.

Early formulations (Lailly, 1984; Tarantola, 1984) describe the method as simultaneous inversions for the source function, the density field, and the bulk modulus field. Woodward (1990) and Luo and Schuster (1991) choose to invert only for a velocity field. Due to the limited geometries of seismic reflection surveys, there is an ambiguity between velocity and density: a velocity anomaly, a density anomaly, or a combination of the two can all create reflections, and near-vertical-incidence waves do not contain much information to distinguish between these cases.

[1]**e-mail:** rgunther@stanford.edu

Here we first present a simplified formulation of waveform inversion, based on Tarantola (1984), and show the results of inverting a small synthetic dataset modeled with the same physics—the constant-density acoustic wave equation—as used in the inversion. Then we show results from a velocity-only inversion of a dataset modeled with both velocity and density contrasts.

REVIEW OF WAVEFORM INVERSION

Our implementation is based on the constant-density acoustic wave equation

$$\left(\nabla^2 - \frac{1}{v(\vec{r})^2}\frac{\partial^2}{\partial t^2}\right)\Psi(\vec{r},t) = 0, \tag{1}$$

where ψ is a pressure-field solution, $\vec{r}$ are the model coordinates (x and z for the two-dimensional case), and t is time. Though the implementation uses time-domain finite differences, it is more convenient to express the equation in frequency ω and slowness σ:

$$\left(\nabla^2 + \omega^2\sigma(\vec{r})^2\right)\Psi(\vec{r},t) = 0. \tag{2}$$

The Born approximation

Given a solution $\Psi(\vec{r},t)$ to equation (2), is it possible to recover $\sigma(\vec{r})$? The Born approximation, named after physicist Max Born, was first developed for scattering theory in quantum mechanics. Applied to seismology, the first-order approximation provides a linear, and thus invertible, relationship between a small change in the slowness model and a resulting small change in the wavefield. We split the model into a background slowness $\sigma_0(\vec{r})$ and a small slowness perturbation $\Delta\sigma(\vec{r})$, where

$$\sigma(\vec{r}) = \sigma_0(\vec{r}) + \Delta\sigma(\vec{r}). \tag{3}$$

The wavefield depends on slowness squared, so here we bring in the first approximation, which is not yet the Born approximation:

$$\sigma(\vec{r})^2 \approx \sigma_0(\vec{r})^2 + 2\sigma_0(\vec{r})\Delta\sigma(\vec{r}). \tag{4}$$

To achieve an approximate relation linear with $\Delta\sigma$, we first substitute (4) into the wave equation:

$$\left(\nabla^2 + \omega^2\sigma(\vec{r})^2\right)\Psi(\vec{r},t) \approx \left(\nabla^2 + \omega^2\sigma_0(\vec{r})^2 + 2\omega^2\sigma_0(\vec{r})\Delta\sigma(\vec{r})\right)\Psi(\vec{r},t). \tag{5}$$

This approximation is then divided into halves, with only one side depending on $\Delta\sigma$:

$$(\nabla^2 + \omega^2\sigma_0(\vec{r})^2)\Psi(\vec{r},t) \approx -2\omega^2\sigma_0(\vec{r})\Delta\sigma(\vec{r})\Psi(\vec{r},t). \tag{6}$$

Much as the slowness field was split into two parts, the wavefield now is divided into a background wavefield Ψ_0 and a scattered wavefield $\Delta\Psi$ such that

$$\Psi(\vec{r},\omega) = \Psi_0(\vec{r},\omega) + \Delta\Psi(\vec{r},\omega), \tag{7}$$

where, by definition, Ψ_0 is the solution for the background wavefield, or

$$\left(\nabla^2 + \omega^2 \sigma_0(\vec{r})^2\right) \Psi_0(\vec{r}, \omega) = 0. \tag{8}$$

Substituting the divided wavefield (7) into the approximate wave equation (6), and using the fact that the background wavefield is an exact solution for the background velocity, we can write

$$\left(\nabla^2 + \omega^2 \sigma_0(\vec{r})^2\right) \Delta\Psi(\vec{r}, \omega) \approx -2\omega^2 \sigma_0(\vec{r}) \Delta\sigma(\vec{r}) \Psi(\vec{r}, t). \tag{9}$$

At this point, we have an implicit relation between a small change $\Delta\sigma$ in the model and the resulting scattered wavefield $\Delta\Psi$. Ideally, we would like to have an explicit expression for $\Delta\sigma$ as a function of the background and scattered wavefields. Such an expression cannot be written directly; instead, we can find an expression for $\Delta\Psi$ as a function of $\Delta\sigma$. This expression is an integral over potential scatterers convolved with the Green's function $G_0(\vec{r}, \omega; \vec{r}')$, the response at point $\vec{r}'$ and frequency ω for a point source at point $\vec{r}$. The subscript indicates that the Green's function is defined for the background wavefield. We build up the integral expression by starting with the formal definition of the Green's function, which is the solution of the wave equation with a delta-function source:

$$\left(\nabla^2 + \omega^2 \sigma_0(\vec{r})^2\right) G_0(\vec{r}, \omega; \vec{r}') = \delta(\vec{r} - \vec{r}'). \tag{10}$$

Both sides of this definition are multiplied by $-2\omega^2 \sigma_0 \Delta\sigma \Psi$ and integrated with respect to $\vec{r}'$:

$$-\int d\vec{r}' 2\omega^2 \sigma_0(\vec{r}') \Delta\sigma(\vec{r}') \Psi(\vec{r}', \omega) \left[(\nabla^2 + \omega^2 \sigma_0(\vec{r})^2) G_0(\vec{r}, \omega; \vec{r}') = \delta(\vec{r} - \vec{r}') \right]. \tag{11}$$

The Laplacian operator is taken with respect to $\vec{r}$, not $\vec{r}'$, so the left side of the expression can be simplified by moving the integral inside the operator; on the right side, the delta function sifts the original function out of the integral, leaving

$$\left(\nabla^2 + \omega^2 \sigma_0(\vec{r})^2\right) - \int d\vec{r}' 2\omega^2 \sigma_0(\vec{r}') \Delta\sigma(\vec{r}') G_0(\vec{r}, \omega; \vec{r}') \Psi(\vec{r}', \omega) = -2\omega^2 \sigma_0(\vec{r}) \Delta\sigma(\vec{r}) \Psi(\vec{r}, \omega). \tag{12}$$

Comparing with (9), we can see that the integral represents a solution for $\Delta\Psi$, allowing us to write

$$\Delta\Psi(\vec{r}, \omega) \approx -\int d\vec{r}' 2\omega^2 \sigma_0(\vec{r}') \Delta\sigma(\vec{r}') G_0(\vec{r}, \omega; \vec{r}') \Psi(\vec{r}', \omega). \tag{13}$$

Unfortunately, the scattered wavefield is still a function of the entire—and unknown—wavefield Ψ. The first-order Born approximation asserts that when the scattered wavefield is small compared to the background wavefield, the interaction between scattering points can be ignored. This is equivalent to replacing Ψ with Ψ_0 on the right-hand side, leaving

$$\Delta\Psi(\vec{r}, \omega) \approx -\int d\vec{r}' 2\omega^2 \sigma_0(\vec{r}') \Delta\sigma(\vec{r}') G_0(\vec{r}, \omega; \vec{r}') \Psi_0(\vec{r}', \omega). \tag{14}$$

This approximation now provides a linear relationship between a small change in the model and the resulting small wavefield perturbation.

Application to seismic inversion

Application of the Born approximation as expressed in (14) requires knowledge of the residual wavefield everywhere in the image space. Unfortunately, the full wavefield, and thus the residual wavefield, is only known at the receivers. For simplicity we assume that receivers are located at all x-locations on the surface, or that the receiver wavefield is unaliased and can be perfectly recovered. We define $w_n(\vec{r}, \omega, s)$ as the background wavefield for shot s at the n^{th} iteration. This wavefield is computed by forward modeling the shot field through the n^{th} slowness model. The data residual $\Delta d_n(x, \omega, s)$ is computed by selecting the background wavefield at $z = 0$ and subtracting from the recorded data $d(x, \omega, s)$, or

$$\Delta d_n(x, \omega, s) = d(x, \omega, s) - \left| w_n(\vec{r}, \omega, s) \right|_{z=0}. \tag{15}$$

The objective of the inversion is to minimize the l^2 norm of Δd.

Our implementation uses the linear forward operator to compute a step length at each iteration. Substituting w_n for the background wavefield and selecting only the scattered field at the receivers, the frequency domain expression for the linear forward operator becomes

$$\Delta d_n(x, \omega, s) = \left| - \int d\vec{r}' 2\omega^2 \sigma_0(\vec{r}') \Delta\sigma(\vec{r}') G_n(\vec{r}, \omega; \vec{r}') w_n(\vec{r}', \omega, s) \right|_{z=0}. \tag{16}$$

Since we use time-domain finite-difference modeling, it is useful to express the operator in the time domain. The $-\omega^2$ factor is applied as a second time derivative to the w_n wavefield and the multiplication of w_n, and G_n becomes a convolution along the time dimension, yielding

$$\Delta d_n(x, t, s) = \left| \int d\vec{r}' 2\sigma_0(\vec{r}') \Delta\sigma(\vec{r}') G_n(\vec{r}, t; \vec{r}') * \ddot{w}_n(\vec{r}', t, s) \right|_{z=0}. \tag{17}$$

The forward operator is implemented in two steps. First, the background wavefield w_n is computed by propagating the source field forward in time. Next, the background wavefield is scaled by $-2\sigma_n \Delta\sigma$ and used as a new source field that is also propagated forward in time.

The gradient direction $\Delta\sigma$ for each step of the inversion is computed using the adjoint of the forward operator. The independent variables used in the forward operator are $\vec{r}$, $\vec{r}'$, ω, and s. The forward operator integrates over $\vec{r}'$ and selects data at $z = 0$, so the adjoint is expressed by integrating over the remaining variables and injecting data, expressed here as multiplying with a delta function, at $z = 0$:

$$\Delta\sigma_n(\vec{r}') = - \iiint ds d\omega d\vec{r} 2\omega^2 \sigma_n(\vec{r}') w_n^*(\vec{r}', \omega, s) G_n^*(\vec{r}, \omega; \vec{r}') \delta(z) \Delta d_n(x, \omega, s) \tag{18}$$

This integral represents reverse-time migration of the data residual. We show a simplified expression by defining a new wavefield res_n that represents the propagation of the data residual. The time axis of the Green's function is reversed due to the complex conjugate in the frequency domain:

$$res_n(\vec{r}', t, s) = \int d\vec{r} G_n(\vec{r}, -t; \vec{r}') * \delta(z) \Delta d_n(x, t, s). \tag{19}$$

In practice, the integral is computed by forward propagating the time-reversed data residual. Due to Green's function reciprocity, integration over $\vec{r}$ is equivalent to the integration over $\vec{r}'$ in (17). The wavefield res_n is then substituted into the time-domain expression for the adjoint operator where integration over frequencies is exchanged for integration over time, and the time axis of the background wavefield is reversed:

$$\Delta\sigma_n(\vec{r}') = 2\sigma_n(\vec{r}') \iint dsdt\ \ddot{w}_n(\vec{r}', -t, s) \cdot res_n(\vec{r}', t, s). \tag{20}$$

With both the forward and adjoint linear seismic modeling expressions defined, we have all of the building blocks needed to invert for σ. We use a non-linear variation of conjugate gradients following Claerbout (2004). The method differs from linear conjugate gradients in that for each iteration the operators, which depend on w_n, change and the data residual Δd_n is recomputed.

APPLICATION TO MODELS WITH AND WITHOUT DENSITY

To test effects of density on waveform inversion, we construct two earth models, one with reflectors simulated by velocity spikes and one with reflectors simulated by density spikes. We add a Gaussian anomaly to both velocity models to test whether our constant-density implementation of waveform inversion can recover long-wavelength velocity perturbations.

Figure 1 shows the slowness field (a) for the constant-density model. Ten horizontal stripes with +1% change in slowness act as reflectors that generate events in the data. Though the goal of waveform inversion is to invert long-wavelength velocity perturbations, the inversion also needs to recover high-frequency perturbations in order to match the data. We also add a +1% Gaussian anomaly to the model. We provide the inversion with a constant-slowness initial model (b) that matches the background velocity of the actual model. After 185 iterations, the inversion (c) recovers both the reflectors and the Gaussian anomaly. Vertical slices through the middle of the slowness model and inversion result (d) show that the inversion comes close to correctly estimating the magnitude of the slowness spikes, especially in the region unaffected by the anomaly. The inversion under-estimates the magnitude of the slowness anomaly, and it smears the anomaly vertically. These two effects tend to counteract each other since a slowness perturbation with a large spatial extent but small magnitude can introduce similar delays as a spatially small perturbation with a large magnitude.

Figure 2 shows the slowness field (a) and the density field (b) for the variable-density model. In this example we introduce the same +1% Gaussian anomaly to the velocity model, but we simulate reflectors with density contrasts instead of slowness contrasts. Since slowness and density changes create reflections of opposite polarity, we add -1% spikes to the density model. After 300 iterations, the inversion introduces horizontal stripes into the velocity model to account for the density reflectors and partially recovers the Gaussian anomaly (c). Overall, the inversion result is noisier, and the norm of the data residual is larger than for the previous example. Vertical slices through the model and inversion result (d) show that the inversion introduces negative changes at the top and bottom of the anomaly, which is entirely positive. This example illustrates that data effects due to density can inhibit the ability of waveform inversion to recover velocity anomalies.

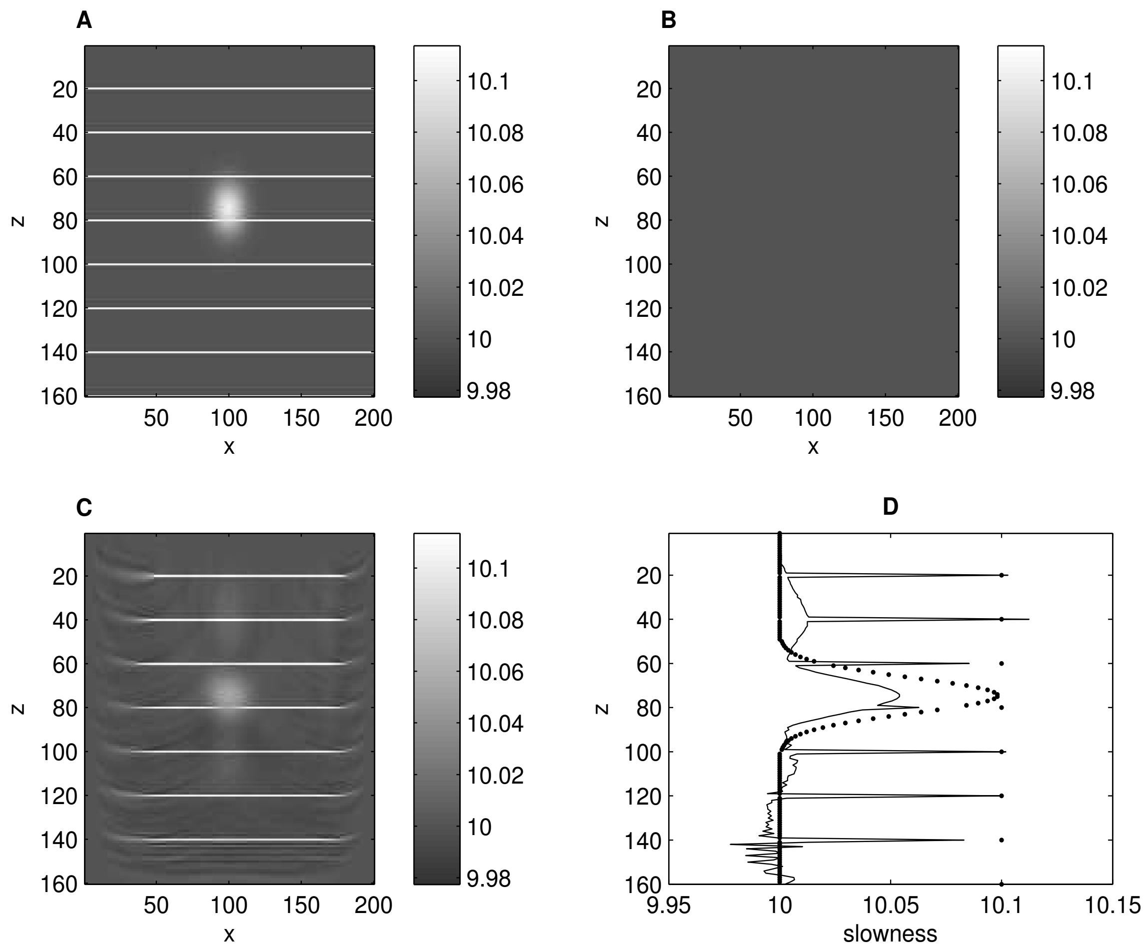

Figure 1: Numerical experiment for a constant-density earth: (a) slowness model use to compute the data; (b) starting slowness model; (c) inversion result after 185 iterations; and (d) slices through the model (dotted line) and inversion result (solid line).

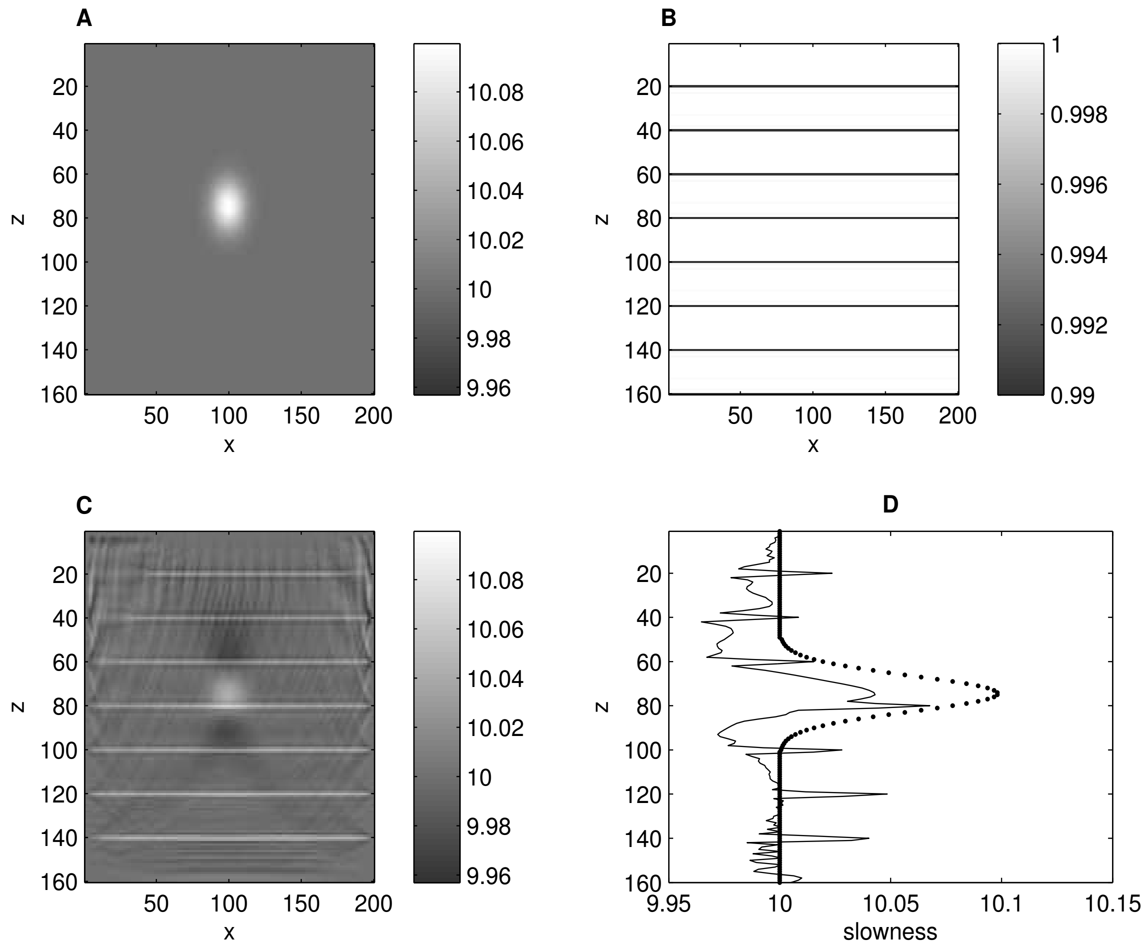

Figure 2: Numerical experiment for a variable-density earth. Data is modeled for a constant-background slowness field (a) containing a Gaussian anomaly but no reflectors. The reflectors are instead embedded in the density field (b). After 300 iterations, the inversion (c) attempts to fit the reflection with velocity spikes but still manages to recover some of the anomaly. (d) Slices through the model (dotted line) and inversion result (solid line) show that the anomaly is recovered less effectively than in the previous example.

CONCLUSION

As with travel-time based velocity analysis methods, the primary purpose for waveform inversion is to find a velocity model for imaging. However, waveform inversion needs to match data phase and amplitude, not just travel-times, and the examples in this report show that inverting for just velocity can be dangerous when density variations are significant. A possible solution is the joint inversion of density and velocity as proposed by Tarantola (1984). Further research is needed to determine whether seismic data typically contain the information necessary to constrain both fields.

REFERENCES

Claerbout, J., 2004, Image estimation by example: Geophysical soundings image construction: Stanford Exploration Project.

Lailly, P., 1984, The seismic inversion problem as a sequence of before stack migrations: Conference on Inverse Scattering: Philadelphia, SIAM, **49**, 1259–1266.

Luo, Y. and G. T. Schuster, 1991, Wave-equation travel-time inversion: Geophysics, **56**, 645–653.

Tarantola, A., 1984, Inversion of seismic refleciton data in the acoustic approximation: Geophysics, **49**, 1259–1266.

Woodward, M., 1990, Wave equation tomography: **62**.