ONE-WAY WAVE EQUATIONS BY FACTORIZATION INTO PSEUDO-DIFFERENTIAL OPERATORS FOR THE VARIABLE COEFFICIENT CASE

David Brown

In the following discussion, it is assumed that the reader is familiar with at least the concept of wave equation factorization and knows something about the deviation of one-dimensional one-way wave equations in terms of asymptotic expansions in the time derivative (see SEP 13).

Begin by making the following definitions:

$$D_x = i\partial_x, \quad D_z = i\partial_z, \quad D_t = -i\partial_t$$

Fourier transform dual variables are:

$$t \rightarrow w$$

$$x \rightarrow \eta \quad \text{(instead of } k_x\text{)}$$

$$z \rightarrow \xi \quad \text{(instead of } k_z\text{)}$$

Then the scalar wave equation may be written

$$-Lu = (D_{xx} + D_{zz} - \frac{1}{v^2} D_{tt})\, u = 0 \tag{1}$$

where u is the state variable. We write then

$$-L(x,z,D_x,D_z,D_t) = D_{xx} + D_{zz} - \frac{1}{v^2(x,z)} \tag{2}$$

and for the fourier transform of L,

$$-L(x,z,\eta,\xi,\omega) = \eta^2 + \xi^2 - \frac{1}{v^2(x,z)}\, \omega^2 \tag{3}$$

To get the one-way wave equations with z as evolution direction, we want to make a factorization of (3) into the following form:

$$-L(x,z,\eta,\xi,\omega) = [\xi + \lambda(x,z,\eta,\omega)][\xi - \lambda(x,z,\eta,\omega)]. \qquad (4)$$

Note particularly that λ does not depend on ξ . The second factor is the operator for the one way wave equation. We see that this is so because it may stand alone while the first operator cannot since it operates on the second operator.

In order to determine the pseudo-differential operator, we will assume that it may be expanded in an asymptotic series with terms of decreasing order of homogeneity, i.e.,

$$\lambda(x,z,\eta,\omega) = \lambda^1(x,z,\eta,\omega) + \lambda^0(x,z,\eta,\omega) + \lambda^{-1}(x,z,\eta,\omega) + \ldots, \qquad (5)$$

where λ^j is a homogeneous function in η and ω of order j, i.e., $\lambda^j = O(|\eta|^j + |\omega|^j)$ for η, ω large. (See Engquist and Majda, 1977).

Before continuing, we will need to review (or learn) the rules for multiplication of pseudo-differential operators. [See also Nirenberg: Lectures on Linear Partial Differential Equations, (1973)].

First of all, we need a formal definition of the pseudo-differential operator $P(x,D_x)$. This is done by relating it

$$P(x,D_x)\ U(x) \equiv \int e^{i\xi x}\, P(x,\xi)\ u\ (\xi)\ d\xi$$

$$= \iint e^{i\xi x}\, e^{-i\xi y}\, P(x,\xi)\ u(y)\ d\xi dy. \qquad (6)$$

Here $u(\xi) = u(k_x)$ is the fourier transform of $u(x)$, and y is a dummy variable.

To "multiply" two pseudo-differential operators together, we can apply the rule given by (6). Let

$$R(x,D_x) = Q(x,D_x) \cdot P(x,D_x),$$

by which we mean that $R(x,D_x)\,u(x)$ should give the same result as $Q(x,D_x)\cdot P(x,D_x)u(x)$. From (6),

$$R(x,D_x) = Q(x,D_x)\iint e^{i\eta(z-y)}P(z,\eta)\;u(y)\;d\eta dy$$

$$= \iiiint e^{i\xi(x-y)}Q(x,\xi)e^{i\eta(z-y)}P(z,\eta)u(y)\,d\xi dz d\eta dy.$$

Matching terms using (6) again,

$$R(x,\eta) = \iint e^{i(\eta-\xi)(z-x)}Q(x,\xi)P(z,\eta)dzd\xi \tag{7}$$

We can take Q outside the integral if we expand it in a power series about $Q(x,\eta)$:

$$Q(x,\xi) = Q(x,\eta) + \sum_{k=1}^{\infty} \frac{1}{k!}\,\partial_\eta^k\,Q(x,\eta)(\xi-\eta)^k$$

We get:

$$R(x,\eta) = \sum_k \frac{1}{k!}\,\partial_\eta^k\,Q(x,\eta)\iint e^{i(\eta-\xi)(z-x)}P(z,\eta)(\xi-\eta)^k dzd\xi$$

Integrating over z,

$$R(x,\eta) = \sum_k \frac{1}{k!}\,\partial_\eta^k\,Q(x,\eta)\int e^{i(\xi-\eta)x}\,\breve{P}(\xi-\eta,\eta)(\xi-\eta)^k d\xi$$

$$= \sum_k \frac{1}{k!}\,\partial_\eta^k\,Q(x,\eta)\int e^{i(\xi-\eta)x}\,\breve{P}(\xi-\eta,\eta)(\xi-\eta)^k d(\xi-\eta),$$

and then over $(\eta-\xi)$,

$$R(x,\eta) = \sum_k \frac{1}{k!}\,\partial_\eta^k\,Q(x,\eta)\cdot D_x^k\,P(x,\eta), \tag{8}$$

where we have used the theorem for the derivative of a Fourier transform to get the final result.

Let us return now to the problem of determining $\lambda(x,z,\eta,\omega)$. The approach we take is to substitute λ' for λ in equation (4) and demand that the equality be satisfied to first order in η and ω. We then add the second term in the asymptotic series and require that the resulting product give the true differential operator to zeroth order. The process is recursive, i.e., each term can be determined from the previously determined ones. The first approximation will come from

$$[\xi + \lambda^1(x,y,\eta,\omega)][\xi - \lambda^1(x,y,\eta,\omega)] = -L(x,y,\xi,\eta,\omega) + O(|\eta|+|\omega|). \tag{9}$$

Using the formula in equation (8), we get

$$\xi^2 + \lambda^1\xi - \xi\lambda^1 - D_x\lambda^1 - (\lambda^1)^2 = \xi^2 + \eta^2 - \frac{\omega^2}{v^2} + O(|\eta|+|\omega|)$$

The fourth term on the left is $O(|\eta|+|\omega|)$ and can be neglected, hence

$$\lambda^1 = \sqrt{\frac{\omega^2}{v^2} - \eta^2} \tag{10}$$

or Fourier tranforming,

$$\lambda^1(x,z,D_x,D_t) = \sqrt{\frac{1}{v^2} D_t^{\,2} - D_x^{\,2}} \tag{11}$$

Thus, the first approximation gives the one-way wave equation

$$(\partial x - \sqrt{\frac{1}{v^2}\partial_t^{\,2} - \partial_x^{\,2}}^{\,1}) \quad u = 0. \tag{12}$$

We get the second approximation by including λ^0 in the expansion for λ:

$$[\xi + \lambda^1 + \lambda^0][\xi - \lambda^1 - \lambda^0] = -L(x,y,\xi,\eta,\omega) + O(|\eta|^0+|\omega|^0),$$

from which

$$\xi^2 - (\xi\lambda^1 + D_x\lambda^1) - (\xi\lambda^0 + D_x\lambda^0) + (\lambda^1\xi)$$

$$- (\lambda^1\lambda^1 + \partial\eta\lambda^1 D_y\lambda^1 + \frac{1}{2}\partial^2\eta\,\lambda^1 D_y^2\,\lambda^1 + \ldots) + (\lambda^0\xi)$$

$$- (\lambda^1\lambda^0 + \partial\eta\,\lambda^1 D_y\lambda^0 + \ldots) - (\lambda^0\lambda^1 + \partial\eta\,\lambda^0 D_y\lambda^1 + \ldots)$$

$$- (\lambda^0\lambda^0 + \partial\eta\,\lambda^0 D_y\lambda^0 + \ldots) = \xi^2 + \eta^2 - (\omega^2/v^2) + O(|\eta|^0 + |\omega|^0).$$

Using (10), we get

$$\lambda^0 = -(D_x\lambda^1 + \partial\eta\,\lambda^1 D_y\lambda^1)/2\lambda^1 \,, \tag{13}$$

where terms of $O(|\eta|^0 + |\omega|^0)$ and lower have been neglected. We can follow the same process to get λ^{-1}:

$$\lambda^{-1} = -(D_x\lambda^0 + \partial\eta\,\lambda^1 D_y\lambda^0 + \partial\eta\,\lambda^0 D_y\lambda^1 + \frac{1}{2}\partial^2\eta\lambda^1 D_y^2\,\lambda^1 + \lambda^0\lambda^0)/2\lambda^1 \,. \tag{14}$$

The one-way wave equation for the second approximation is found by using (10) and (13). Its Fourier transform (i.e., the dispersion relation) is given by

$$\xi - \left(\frac{\omega^2}{v^2} - \eta^2\right)^{1/2} + \frac{\left(\frac{v_z}{2v^3}\right) i\omega^2\left(\frac{\omega^2}{v^1} - \eta^2\right)^{1/2} - \left(\frac{v_x}{2v^3}\right) i\omega^2\eta}{[(\omega^2/v^2) - \eta^2]^{3/2}} = 0 \,, \tag{15}$$

and the pseudo-differential equation itself is given by

$$\left[\partial_z - \frac{1}{v}\partial_t\left(1 - v^2\partial^{tt}_{xx}\right)^{1/2} + \frac{\frac{v_z}{2v}\left(1 - v^2\,\partial^{tt}_{xx}\right)^{1/2} - \frac{v_x}{2}\,\partial^t_x}{(1 - v^2\,\partial^{tt}_{xx})^{3/2}}\right] u = 0. \tag{16}$$

The one-way wave equation suggested by Engquist (SEP-13), can be obtained by expanding the square-roots and dropping the appropriate higher-order terms. We see that the equation proposed by Engquist is not unique, i.e., we can make expansions of arbitrarily high order by, say, making a Padé expansion of the square-root terms. The one-way wave equation for the third approximation can be determined also using equations (14), (15), and (10) to find $\lambda^{-1}(x,z,\eta,\omega)$. The equation will then be given by

$$[D_z - \lambda^1(x,z,D_x,D_t) - \lambda^0(x,z,D_x,D_t) - \lambda^{-1}(x,z,D_x,D_t)]u = 0. \qquad (17)$$

The algebra is rather tedious, and is left as an exercise for the reader.

References

Engquist, B. and A. Majda (1977), "Absorbing Boundary conditions for the Numerical Simulation of Waves," *Math. of Computation* 31.

Nirenberg, Louis (1973), *Lectures on Linear Partial Differential Equations*, Regional Conf. Ser. Math., Am. Math. Soc., No. 17, Providence, Rhode Island.