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Short Note

Transmission wavefield velocity analysis

Jeff Shragge

INTRODUCTION

Transmission wavefields contain important information on subsurface velocity profiles. This is evident from the many tomography-based techniques that invert for perturbations on an assumed background velocity using the transmission wavefield response. One of the more successful approaches is waveform tomography (Pratt and Worthington, 1989; Woodward, 1992), an approach to non-linear velocity inversion problem that iteratively obtains an estimate of velocity perturbations by minimizing the difference between forward-modeled waveforms and acquired data through residual back-projection. A commonly used and accurate way to forward model synthetic data is using two-way finite-differences. Inversion of the correspondingly large matrices required for 2-D waveform inversion is typically done using a memory-intensive LU decomposition approach (Štekl and Pratt, 1998). Current computer memory limitations preclude the use of this approach on typical 3-D seismic volumes (Operto et al., 2006).

Wave-equation migration velocity analysis (WEMVA) is another velocity inversion technique (Sava and Biondi, 2004). This procedure back-projects wavefield perturbations derived from variations in migrated image volume (i.e. angle-gathers) to image velocity perturbations. Unlike typical waveform inversion approaches, this procedure is often implemented with one-way phase-only wavefield extrapolation for forward modeling, and is applied to the back-scattered reflection response. However, nothing precludes using a WEMVA-like formalism in inverting transmission wavefields for velocity perturbations. One potential benefit is that because the phase-only extrapolation operator is stated explicitly, one can represent scattering as a matrix operation that provides a direct link between a velocity perturbation and the gradient field.

In this paper, I derive a WEMVA-like framework for modeling transmission wavefields. I then use the waveform inversion objective function (Pratt and Worthington, 1989) to develop the equations appropriate for transmission wavefield waveform inversion using one-way extrapolation operators. Finally, I demonstrate that forward modeling in generalized coordinate systems (Sava and Fomel, 2005) does not pose any theoretical difficulties for the inversion process.

WEMVA FORWARD MODELING

Following Sava and Biondi (2004), I develop equations for imaging by wavefield extrapolation based on recursive continuation of the wavefields $\mathcal{U}$ from a given depth level to the next by means of an extrapolator operator $\mathbf{E}$

$$\mathcal{U}_{z+\Delta z} = \mathbf{E}_z[\mathcal{U}_z], \tag{1}$$

where $\mathbf{E}_z[] = \mathrm{e}^{\mathrm{i}k_z \Delta z}$, k_z is extrapolation wavenumber, and Δz is the depth step. Throughout this paper, I use a notation where $\mathbf{A}[x]$ denotes that operator $\mathbf{A}$ is applied to a field x. Subscripts z and $z+\Delta z$ correspond to quantities associated with depth levels z and $z+\Delta z$, respectively.

Using this operator notation, a data wavefield $\mathcal{D}$ can be recursively extrapolated through a medium described by model parameters (i.e. slowness). This operation can be written explicitly in matrix form,

$$\begin{bmatrix} \mathbf{1} & 0 & 0 & ... & 0 & 0 \\ -\mathbf{E}_0 & \mathbf{1} & 0 & ... & 0 & 0 \\ 0 & -\mathbf{E}_1 & \mathbf{1} & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & -\mathbf{E}_{n-1} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathcal{U}_0 \\ \mathcal{U}_1 \\ \mathcal{U}_2 \\ \vdots \\ \mathcal{U}_n \end{bmatrix} = \begin{bmatrix} \mathcal{D}_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{2}$$

where $\mathbf{1}$ is an identity operator, and fields without subscripts (e.g. $\mathcal{U}$ and $\mathcal{D}$) refer to complete wavefields. Equation 2 is written more compactly as

$$(\mathbf{1}-\mathbf{E})\,\mathcal{U} = \mathcal{D}, \tag{3}$$

where $(\mathbf{1}-\mathbf{E})$ is a Green's function $\mathbf{G}_0(\mathbf{x}',\mathbf{x})$ between levels $\mathbf{x}$ and $\mathbf{x}'$ generated by wavefield extrapolation. The Green's function satisfies the following adjoint definitions,

$$(\mathbf{1}-\mathbf{E}) = \mathbf{G}_0(\mathbf{x}',\mathbf{x}) = \mathbf{G}_0^{\dagger}(\mathbf{x},\mathbf{x}'), \tag{4}$$

$$(\mathbf{1}-\mathbf{E})^{-1} = \mathbf{G}_0^{\dagger}(\mathbf{x}',\mathbf{x}) = \mathbf{G}_0(\mathbf{x},\mathbf{x}'), \tag{5}$$

where superscripts $^{-1}$ and † indicate the inverse and adjoint operation (i.e. complex transpose), respectively.

Source wavefields well-modeled by a delta function exhibit the following relationships,

$$\mathcal{U}(\mathbf{x},\mathbf{s}) = (\mathbf{1}-\mathbf{E})^{-1}\,\mathcal{D} = \mathbf{G}_0(\mathbf{x},\mathbf{x}')\delta(\mathbf{x}'-\mathbf{s}) = \mathbf{G}_0(\mathbf{x},\mathbf{s}), \tag{6}$$

where $\mathbf{G}_0(\mathbf{x},\mathbf{s})$ describes the propagation from source point $\mathbf{s}$ throughout the domain denoted by $\mathbf{x}$. Note that the choice of $\mathbf{s}$ is arbitrary and an equivalent development applies for a receiver Green's function $\mathbf{G}_0(\mathbf{x},\mathbf{r})$,

$$\mathbf{G}_0(\mathbf{x},\mathbf{r}) = \mathbf{G}_0(\mathbf{x},\mathbf{x}')\delta(\mathbf{x}'-\mathbf{r}) = (\mathbf{1}-\mathbf{E})^{-1}\,\delta(\mathbf{x}'-\mathbf{r}), \tag{7}$$

where $\mathbf{r}$ is receiver location.

Introducing velocity perturbations

If a velocity perturbation is applied at some depth level, a perturbed wavefield $\Delta\mathcal{U}$ can be derived from the background wavefield by application of the chain rule to equation 1,

$$\Delta\mathcal{U}_{z+\Delta z} = \mathbf{E}_z[\Delta\mathcal{U}_z] + \Delta\mathcal{V}_{z+\Delta z}, \tag{8}$$

where $\Delta\mathcal{V}_{z+\Delta z}$ represents the scattered wavefield generated at $z+\Delta z$ by the interaction of the velocity model at depth z. Field $\Delta\mathcal{U}_{z+\Delta z}$ is the accumulated wavefield perturbation corresponding to the slowness perturbations at all levels above. It is computed by extrapolating the wavefield perturbations from the level above $\Delta\mathcal{U}_z$, plus the scattered wavefield at this level, $\Delta\mathcal{V}_{z+\Delta z}$.

Equation 8 is also a recursive equation that can be written in matrix form

$$\begin{bmatrix} \mathbf{1} & 0 & 0 & ... & 0 & 0 \\ -\mathbf{E}_0 & \mathbf{1} & 0 & ... & 0 & 0 \\ 0 & -\mathbf{E}_1 & \mathbf{1} & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & -\mathbf{E}_{n-1} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \Delta\mathcal{U}_0 \\ \Delta\mathcal{U}_1 \\ \Delta\mathcal{U}_2 \\ \vdots \\ \Delta\mathcal{U}_n \end{bmatrix} = \tag{9}$$

$$\begin{bmatrix} 0 & 0 & 0 & ... & 0 & 0 \\ \Delta\mathbf{E}_0 & 0 & 0 & ... & 0 & 0 \\ 0 & \Delta\mathbf{E}_1 & 0 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & \Delta\mathbf{E}_{n-1} & 0 \end{bmatrix} \begin{bmatrix} \mathcal{U}_0 \\ \mathcal{U}_1 \\ \mathcal{U}_2 \\ \vdots \\ \mathcal{U}_n \end{bmatrix}, \tag{10}$$

or in more compact notation as,

$$(\mathbf{1}-\mathbf{E})\,\Delta\mathcal{U} = \Delta\mathbf{E}\mathcal{U}. \tag{11}$$

Operator $\Delta\mathbf{E}$ denotes a perturbation of the extrapolation operator $\mathbf{E}$, while quantity $\Delta\mathbf{E}\mathcal{U}$ represents a scattered wavefield and is a function of the medium perturbation given by the scattering relationship derived in Appendix A. For single scattering we write,

$$\Delta\mathcal{V}_{z+\Delta z} \equiv \Delta\mathbf{E}_z[\mathcal{U}_z] = \mathbf{E}_z[\mathbf{S}_z(\tilde{\mathcal{U}}_z)[\Delta s]], \tag{12}$$

where $\mathbf{S}_z$ is the scattering operator, and $\Delta\mathbf{s}$ is slowness perturbation. This expression yields a recursive relationship that can be written in matrix form:

$$\begin{bmatrix} \mathbf{1} & 0 & 0 & ... & 0 & 0 \\ -\mathbf{E}_0 & \mathbf{1} & 0 & ... & 0 & 0 \\ 0 & -\mathbf{E}_1 & \mathbf{1} & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & -\mathbf{E}_{n-1} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \Delta\mathcal{U}_0 \\ \Delta\mathcal{U}_1 \\ \Delta\mathcal{U}_2 \\ \vdots \\ \Delta\mathcal{U}_n \end{bmatrix} =$$

$$
\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ \mathbf{E}_0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \mathbf{E}_1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mathbf{E}_{n-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ \mathbf{S}_0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \mathbf{S}_1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mathbf{S}_{n-1} & 0 \end{bmatrix} \begin{bmatrix} \Delta s_0 \\ \Delta s_1 \\ \Delta s_2 \\ \vdots \\ \Delta s_n \end{bmatrix}, \tag{13}
$$

or in more compact notation

$$
(\mathbf{1}-\mathbf{E})\,\Delta\mathcal{U} = \mathbf{E}\mathbf{S}\Delta s, \tag{14}
$$

where vector Δs denotes slowness perturbations at all depths.

Finally, introducing

$$
\mathbf{L} = (\mathbf{1}-\mathbf{E})^{-1}\,\mathbf{E}\mathbf{S}, \tag{15}
$$

we can write a simple relationship between slowness Δs and wavefield $\Delta\mathcal{U}$ perturbations:

$$
\Delta\mathcal{U} = \mathbf{L}\Delta s. \tag{16}
$$

This expression represents the wavefield scattering caused by the interaction of the background wavefield with the a medium perturbation. The total modeled field Ψ_m is defined as,

$$
\Psi_m(\mathbf{r},\mathbf{s}) = \tilde{\mathcal{U}}(\mathbf{r},\mathbf{s}) + \Delta\mathcal{U}(\mathbf{r},\mathbf{s}), \tag{17}
$$

where $\tilde{\mathcal{U}}$ is the background wavefield modeled by equation 3.

WAVEFORM INVERSION PROBLEM

The goal of waveform inversion is to invert for the optimal set of velocity perturbations that minimize the difference between forward-modeled waveforms and acquired data. The first step in setting up the inverse problem is defining data residuals, $\Delta\Psi$,

$$
\Delta\Psi(\mathbf{r},\mathbf{s};\omega) = \Psi_m(\mathbf{r},\mathbf{s};\omega) - \Psi_d(\mathbf{r},\mathbf{s};\omega), \tag{18}
$$

where $\Psi_d(\mathbf{r},\mathbf{s})$ is the recorded data. The L_2 residual norm is used to set up an objective function,

$$
E = \frac{1}{2}\sum_{\mathbf{s}}\sum_{\mathbf{r}} \Delta\Psi^{\dagger}(\mathbf{r},\mathbf{s})\Delta\Psi(\mathbf{r},\mathbf{s}), \tag{19}
$$

that is minimized with respect to slowness perturbations $\Delta s^{\dagger}$

$$
\frac{\partial E}{\partial \Delta s^{\dagger}} = \frac{1}{2}\sum_{\mathbf{s}}\sum_{\mathbf{r}} \frac{\partial}{\partial \Delta s^{\dagger}} \left(\Delta\Psi^{\dagger} + \Delta s^{\dagger}\mathbf{L}^{\dagger}\right)(\Delta\Psi + \mathbf{L}\Delta s) = 0. \tag{20}
$$

This results in the following least-squares estimate of the slowness perturbations

$$\Delta s(\mathbf{x}) = -\left(\sum_{\mathbf{s}}\sum_{\mathbf{r}}\mathbf{L}^{\dagger}\mathbf{L}\right)^{-1}\sum_{\mathbf{s}}\sum_{\mathbf{r}}\mathbf{L}^{\dagger}\Delta\Psi. \tag{21}$$

From here on, the sum over all sources and receivers is implicitly assumed. Also, we discuss only the gradient vector and the filtering of the gradient by the inverse Hessian matrix $\left(\mathbf{L}^{\dagger}\mathbf{L}\right)^{-1}$ is implicitly assumed.

The adjoint gradient operator $\mathbf{L}^{\dagger}$ is a composite matrix consisting of a number of chained operators (from equation 15):

$$\mathbf{L}^{\dagger} = \mathbf{S}^{\dagger}\mathbf{E}^{\dagger}\left((1-\mathbf{E})^{-1}\right)^{\dagger}, \tag{22}$$

where scattering operator or at each extrapolation interval, S_z is defined by (see Appendix A),

$$S_z = \omega^2 \tilde{\mathcal{U}}_z \frac{\mathrm{i}s_0}{\sqrt{\omega^2 s_0^2 - |\mathbf{k}|^2}} = \omega^2 \tilde{\mathcal{U}}_z F_z, \tag{23}$$

where $\mathbf{F}_z$ is considered a filter. This allows us to write composite operator $\mathbf{L}^{\dagger}$ with scattering $\mathbf{S}^{\dagger}$ and filter $\mathbf{F}^{\dagger}$ matrices as,

$$\mathbf{L}^{\dagger} = \omega^2 \tilde{\mathcal{U}}^{\dagger}\mathbf{F}^{\dagger}\mathbf{E}^{\dagger}\left((1-\mathbf{E})^{-1}\right)^{\dagger} \tag{24}$$

Inserting this expression into equation 21 yields,

$$\Delta s \approx -\omega^2 \tilde{\mathcal{U}}^{\dagger}\mathbf{F}^{\dagger}\mathbf{E}^{\dagger}\left((1-\mathbf{E})^{-1}\right)^{\dagger}\Delta\Psi. \tag{25}$$

Thus, using the relationship in equations 6 and 7, leads to the following result,

$$\Delta s \approx -\omega^2 \mathbf{G}_0^{\dagger}(\mathbf{x},\mathbf{s})\mathbf{F}^{\dagger}(\mathbf{x})\mathbf{E}^{\dagger}(\mathbf{x})\mathbf{G}_0^{\dagger}(\mathbf{x},\mathbf{r})\Delta\Psi(\mathbf{r},\mathbf{s}). \tag{26}$$

Relationship to Pratt's approach to waveform inversion

Equation 26 is a direct statement of the waveform inversion procedure of Pratt and Worthington (1989) and Sirgue and Pratt (2004). However, the use of one-way operators leads to the definition of a explicit scattering operator and a slightly different gradient operator:

$$g(\mathbf{x}) \approx -\omega^2 \textstyle\sum_{\mathbf{s}}\sum_{\mathbf{r}} \Re\left[\mathbf{G}_0^{*}(\mathbf{x},\mathbf{s})\mathbf{G}_0^{*}(\mathbf{x},\mathbf{r})\Delta\Psi(\mathbf{r},\mathbf{s})\right], \quad \text{(Pratt)} \tag{27}$$

$$\Delta s(\mathbf{x}) \approx -\omega^2 \textstyle\sum_{\mathbf{s}}\sum_{\mathbf{r}} \Re\left[\mathbf{G}_0^{\dagger}(\mathbf{x},\mathbf{s})\mathbf{F}^{\dagger}(\mathbf{x})\mathbf{E}^{\dagger}(\mathbf{x})\mathbf{G}_0^{\dagger}(\mathbf{x},\mathbf{r})\Delta\Psi(\mathbf{r},\mathbf{s})\right]. \quad \text{(WEMVA)} \tag{28}$$

Note that the two approaches are similar: the wavefield residuals are back-projected from the source point through the model and correlated with the source Green's function. This approach, though, has the scattering matrix chained between the source and receiver Green's functions. This derives from the application of a differential operator directly on the phase of the extrapolation operator.

CONCLUSION

This paper introduces an approach to waveform inversion that builds from the WEMVA theory developed by Sava and Biondi (2004). The main differences between the this approach and that of Pratt and Worthington (1989) is that this formalism provides a scattering operator that permits a direct estimate of the slowness perturbation. Future work will implement this scheme into a transmission wavefield WEMVA scheme.

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APPENDIX A

This appendix develops a WEMVA scattering operator (Sava and Biondi, 2004) for use in transmission wavefield waveform inversion. The extrapolation operator, $\mathbf{E}$, is given by,

$$\mathbf{E}_z[] = \mathrm{e}^{\mathrm{i}k_z \Delta z}, \tag{A-1}$$

where k_z is the depth wavenumber and Δz is the extrapolation depth step. The extrapolation wavenumber in depth is given by

$$k_z = \sqrt{\omega^2 s^2 - |\mathbf{k}|^2}, \tag{A-2}$$

where ω is temporal frequency and $|\mathbf{k}|$ is the horizontal wavenumber magnitude.

The vertical wavenumber can be separated into two components, one corresponding to the background medium, $\tilde{k_z}$, and one corresponding to a perturbation, Δk_z, such that,

$$k_z = \tilde{k_z} + \Delta k_z. \tag{A-3}$$

In a first-order approximation, we can relate these two extrapolation wavenumbers by a Taylor-series expansion,

$$\begin{aligned} k_z &\approx \tilde{k_z} + \left.\frac{\mathrm{d}k_z}{\mathrm{d}s}\right|_{s=\tilde{s}} (s - \tilde{s}) \\ &\approx \tilde{k_z} + \omega \frac{w\tilde{s}}{\sqrt{\omega^2 \tilde{s}^2 - |\mathbf{k}|^2}} (s - \tilde{s}), \end{aligned} \tag{A-4}$$

where $s(x,z)$ is the slowness and $\tilde{s}(z)$ corresponds to a background slowness.

Within any depth slab we can extrapolate the wavefield from the top, either in the perturbed or in the background medium. The wavefields at the bottom of the slab, $\tilde{\mathcal{U}}_{z+\Delta z} = \mathcal{U}_z \mathrm{e}^{\mathrm{i}\tilde{k_z}\Delta z}$ and $\mathcal{U}_{z+\Delta z} = \mathcal{U}_z \mathrm{e}^{\mathrm{i}k_z \Delta z}$, related by,

$$\mathcal{U}_{z+\Delta z} \approx \tilde{\mathcal{U}}_{z+\Delta z} \mathrm{e}^{\mathrm{i}\Delta k_z \Delta z}. \tag{A-5}$$

Equation A-5 is a direct statement of the Rytov approximation, because the wavefields at the bottom of the slab correspond to different phase shifts related by a linear equation. Thus, we obtain the wavefield perturbation $\Delta\mathcal{V}$ at the bottom of the slab by subtracting the background wavefield $\tilde{\mathcal{U}}$ from the perturbed wavefield $\mathcal{U}$:

$$\begin{aligned} \Delta\mathcal{V}_{z+\Delta z} &\approx \mathcal{U}_{z+\Delta z} - \tilde{\mathcal{U}}_{z+\Delta z} \\ &\approx \left(\mathrm{e}^{\mathrm{i}\Delta k_z \Delta z} - 1\right) \tilde{\mathcal{U}}_{z+\Delta z} \\ &\approx \mathrm{e}^{\mathrm{i}\tilde{k_z}\Delta z} \left(\mathrm{e}^{\mathrm{i}\left.\frac{\mathrm{d}k_z}{\mathrm{d}s}\right|_{s=\tilde{s}} \Delta s_z \Delta z} - 1\right) \tilde{\mathcal{U}}_z. \end{aligned} \tag{A-6}$$

For the Born approximation, we further assume that the wavefield differences are small so that we linearize the exponential function according to $\mathrm{e}^{\mathrm{i}\Delta\phi} \approx 1 + \mathrm{i}\Delta\phi$. With this approximation we write the following downward continued scattered wavefield,

$$\Delta\mathcal{V}_{z+\Delta z} \approx \mathrm{e}^{\mathrm{i}\tilde{k_z}\Delta z} \left(\mathrm{i} \left.\frac{\mathrm{d}k_z}{\mathrm{d}s}\right|_{s=\tilde{s}} \Delta s_z \Delta z\right) \tilde{\mathcal{U}}_z, \tag{A-7}$$

which, in operator form is

$$\mathbf{S}_z(\tilde{\mathcal{U}}_z[\Delta s_z]) \approx \mathrm{i} \left.\frac{\mathrm{d}k_z}{\mathrm{d}s}\right|_{s=\tilde{s}} \Delta s_z \Delta z \tilde{\mathcal{U}}_z. \tag{A-8}$$

The Born operator may be implemented in the Fourier domain relative to a constant reference slowness in any individual slab. In this case,

$$\left.\frac{\mathrm{d}k_z}{\mathrm{d}s}\right|_{s=\tilde{s}} \approx \omega \frac{\omega s_0}{\sqrt{\omega^2 s_0^2 - (1-\eta^2)|\mathbf{k}|^2}}, \tag{A-9}$$

where η is a damping parameter to avoid division by zero.

Figure A-1 shows the amplitude weighting demanded by the filter in equation A-9 for five different frequencies for slowness 0.5 s/km.

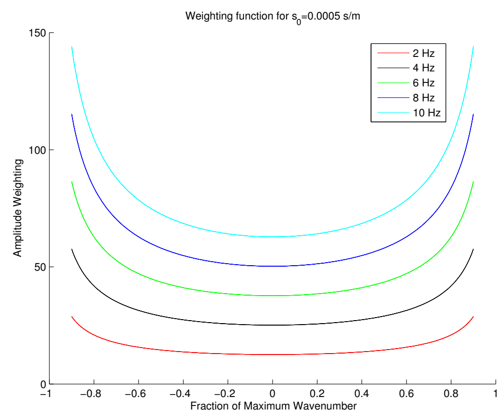

Figure A-1: Example of the Born amplitude weighting function demanded by the WEMVA theory for a slowness of 0.5 s/km and a damping factor of 0.001. jeff2-KXfilter [NR]

Waveform Inversion in Riemannian Space

Waveform extrapolation employing forward modeling in Riemannian coordinates (Sava and Fomel, 2005; Shragge, 2006) does not present a problem for general approach to waveform inversion developed herein because inversion does not take place in generalized coordinates. Rather, the calculated Green's functions are transformed back to global Cartesian grid through mapping pair

$$\mathbf{G}_0(\xi,\mathbf{s}) \approx \quad \mathbf{T}(\mathbf{x};\xi)\mathbf{G}_0(\mathbf{x},\mathbf{s}) \tag{A-10}$$

$$\mathbf{G}_0(\mathbf{x},\mathbf{s}) \approx \quad \mathbf{T}^{\dagger}(\mathbf{x};\xi)\mathbf{G}_0(\xi,\mathbf{s}) \tag{A-11}$$

where $\mathbf{T}$ is a transformation matrix that interpolates from the Riemannian space defined by ξ to global Cartesian space $\mathbf{x}$ that includes the transformation Jacobian. In practice, this is applied using weighted sinc interpolation. Thus, one may rewrite the adjoint of equation 26 in the following manner

$$\Delta s(\mathbf{x}) \approx -\omega^2 \sum_{\mathbf{s}} \sum_{\mathbf{r}} \mathbf{T}^{\dagger}(\mathbf{x};\xi)\mathbf{G}_0^{\dagger}(\xi;\mathbf{s})\mathbf{F}^{\dagger}(\xi)\mathbf{E}^{\dagger}(\xi)\mathbf{G}_0^{\dagger}(\xi;\mathbf{r})\Delta\Psi(\mathbf{r},\mathbf{s}). \tag{A-12}$$