Chapter 3. The Wave Equation Approach to Multiples Modelling and Suppression

This chapter will be devoted to the implementation of the scalar wave equation as a descriptor of seismic wave fields. We will start by considering coordinate transformations that yield approximate equations for propagating up and downgoing waves in free space. Following this, we will couple up and downgoing waves to obtain equations valid for solving the forward or inverse problems. Finally, the theory will be illustrated with some synthetic examples.

The equations which we will deduce and solve in this chapter are

$$U'_{z't'} = -\frac{v}{2}\sec^3\theta U'_{x'x'} - c(x'z')D''_{t'}(x'-2\tan\theta\, z', z', t'-2\cos\theta\,\frac{z'}{v}) \quad (3\text{-}1a)$$

and

$$D''_{z''t''} = \frac{v}{2}\sec^3\theta\, D''_{x''x''} \quad (3\text{-}1b)$$

where $c(x',z')$ is the reflection coefficient, v the compressional velocity, θ the propagation angle and the subscripts denote partial derivatives. These equations represent the essence of the wave modelling and data processing schemes of this thesis and are thought to yield the most accurate deterministic multiple reflection suppression method of reflection seismic data processing. Nevertheless, it was still necessary to make many approximations. The derivation of (3-1a,b) will elucidate the accuracy of the approach. After completing it we will show how these equations are solved.

The inclusion of the wave equation in our scheme comes as a natural extension of our previous ray theory approximation. In effect, if we delete the term containing the second derivative in x, which is responsible for the diffraction of the acoustic energy, we are left within the framework of a ray approximation

$$U'_{z't'} = -c\, D''_{t'} \tag{3-2a}$$

$$D''_{z''t''} = 0 \tag{3-2b}$$

The first equation for U then can be integrated by t'. Representing the remaining derivative in z' through the difference $U_1 - U_2$ (omitting for simplicity the x', t' variables and assuming $Dz' = 1$), equation (3-2a) becomes $U_2 = U_1 + c_1\, D_1$, which is equation (2-5a) of Chapter 2. On the other hand, equation (3-2b) implies D'' = constant, which corresponds to equation (2-5b). It is the possibility of including new properties such as diffractions and geometrical spreading which represents a major advantage over the simplified model of Chapter 2. However, important elements such as shear waves and absorption are still neglected.

We differentiate between two kinds of equations: the uncoupled and the coupled equations. The former refers to the equation that controls the propagation of each separate wave field (U or D) through a homogeneous region with no reflection or transmission effects. The latter describes the propagation through inhomogeneous media, where reflection coefficients couple up and downgoing waves.

Actually we have already obtained these two types of equations in Chapter 2, where (2-5b) was an uncoupled equation for D-waves and (2-5a) a coupled equation for U-waves. From their definition, as well as from the experience of Chapter 2, it follows that the structure of the coupled equations will be highly dependent on the model that we choose for the propagating medium.

3.1. Uncoupled Equations and Coordinate Transformations

There are two main objectives which we wish to accomplish through the coordinate transformation. First we want a transformation that yields separate equations for downgoing and upcoming waves. Second, we want a transformation which takes care of all spatial and temporal translations of the wave field, leaving the wave equation to do only diffraction.

It is not difficult to show that both objectives can be accomplished through the transformation

$$x' = x \pm z \tan\theta \qquad (3\text{-}3a)$$

$$z' = z \qquad (3\text{-}3b)$$

$$t' = \pm \frac{x \sin\theta}{v} \pm \frac{z \cos\theta}{v} + t \qquad (3\text{-}3c)$$

where the sign "-" corresponds to downgoing waves, the "+" to upcoming waves and θ is the propagation angle (from the vertical) for a plane wave. If we refer only to downgoing waves and, as before, denote partial derivatives through subscripts so that

$$\frac{\partial^2}{\partial_x \partial_z} U = \partial_{xz} U = U_{xz} \quad (3\text{-}4)$$

then the Jacobian of this transformation can be written as

$$x'_{x,z,t} = 1, \ -\tan\theta, \ 0 \quad (3\text{-}5a)$$

$$z'_{x,z,t} = 0, \ 1, \ 0 \quad (3\text{-}5b)$$

$$t'_{x,z,t} = -\sin\theta/v \ , \ -\cos\theta/v \ , \ 1 \quad (3\text{-}5c)$$

By using (3-5), we can now transform the 2-D scalar wave equation

$$P_{xx} + P_{zz} - (1/v^2)\, P_{tt} = 0 \ , \quad (3\text{-}6)$$

where P is the pressure, into the new coordinate system. Defining Q as the transformed wave field and noting that the wave field is invariant under coordinate transformations ($P(x,z,t) = Q(x',z',t')$), equation (3-6) becomes

$$v \sec^2\theta \ Q_{x'x'} + v\, Q_{z'z'} - 2\, v \tan\theta\, Q_{x'z'} - 2 \cos\theta \ Q_{z't'} = 0 \quad (3\text{-}7)$$

The intermediate steps leading to (3-7) can be found in Appendix 1. In order to achieve the separation into U and D waves, we would like the obtained equation to be first order in z' . The standard procedure, known as paraxial or parabolic approximation, is to drop the $Q_{z'z'}$ term. The dispersion relationship of the remaining equation shows that its validity is then limited to an aperture angle of approximately $\pm$ 15 degrees off the main direction of propagation θ . Besides, the same dispersion relation indicates that the term proportional to $Q_{x'z'}$ is only significant for angles of propagation larger than 40 degrees, which is, in any case, sort of an upper limit for the other more general approximations involved in the theory. Neglecting then both terms, equation (3-7) can be finally written as

$$Q_{z't'} = (v/2) \sec^3\theta \, Q_{x'x'} \tag{3-8a}$$

If we desire a better approximation to the wave equation than (3-8a), we could estimate $Q_{z'z'}$ from (3-8a) (after integrating by t' and differentiating by z') and substitute back into (3-7). However, to keep the discussion simple, we will leave equation (3-8a) as it is. It is interesting to notice that by choosing $z' = z \sec^3 \theta$ we get:

$$Q_{z't'} = (v/2) \, Q_{x'x'} \tag{3-8b}$$

where the leading coefficient of $Q_{x'x'}$ is no longer angle dependent.

3.2. Coupled Equations

The equations that we obtained in 3.1 referred to waves propagating through a homogeneous region where up and downgoing waves are uncoupled. However, in order to solve the forward or inverse problem we have to consider the fact that the U and D wave fields will couple through the reflection coefficients of the medium.

For the case of stratified media, Claerbout [5] showed that the coupled equations for U and D waves can be written as

$$U_z = -(i\omega/v)\cos\phi\, U - (I_z/2I)\,(U+D) \tag{3-9a}$$

$$D_z = (i\omega/v)\cos\phi\, D - (I_z/2I)\,(U+D) \tag{3-9b}$$

In equations (3-9) U and D have been Fourier transformed in x and t, ϕ is interpreted as the deviation angle from the normal direction of propagation θ, and I is the impedance defined as

$$I = \rho\, v/\cos\phi \tag{3-10}$$

with ρ being the density. The use of equations (3-9) in our case is not totally justifiable since they are obtained by requiring that the medium characteristics be z-dependent only, whereas our theory allows for small lateral variations of the reflection coefficients. Nevertheless we will assume that they represent a reasonable approximation in the case of slowly varying media. This assumption will be reinforced at the end by the fact that, in the limiting case of a ray approximation, the coupled equations to be deduced from (3-9) give equations identical to those obtained in the previous chapter. The idea is then, to estimate ϕ as well as the Fourier transforms of U and D in relation to the frame of the waves of interest (U or D) . These estimations are substituted in (3-9) and the obtained equation is Fourier transformed back into the original frame. We will start by making the same assumptions of Chapter 2 in relation to the propagating medium, that is, we will ignore transmission losses (eliminates U from the second term in (3-9a)) and intrabed multiples (eliminates the second term completely in (3-9b)). Thus, as before, the only coupling that remains is in the upcoming wave equation

$$U_z = -(i\omega/v)\cos\phi\, U - (I_z / 2I)\, D \tag{3-11}$$

We illustrate the above procedure with the transformation (3-5) for upcoming waves

$$x' = x + z\tan\theta \tag{3-12a}$$

$$z' = z \tag{3-12b}$$

$$t' = \frac{x\sin\theta}{v} + \frac{z\cos\theta}{v} + t \tag{3-12c}$$

By requiring as before, that the wave fields be invariant $(P(x,z,t) = U'(x',z',t'))$ and by using the Jacobian corresponding to (3-12), the first derivatives of U can be expressed in the new frame as

$$U_x = U'_{x'} - U'_{t'}\,(\sin\theta)/v \tag{3-13a}$$

$$U_z = U'_{x'}\tan\theta + U'_{z'} + U'_{t'}\,(\cos\theta)/v \tag{3-13b}$$

$$U_t = U'_{t'} \tag{3-13c}$$

In order to estimate the Fourier transform of U_z and the wave-number-frequency relationships in both frames (observer and upcoming), we introduce in (3-13) a monochromatic solution of the type

$$\exp(ikx - i\omega t) \tag{3-14}$$

We then obtain

$$ik\,U = (\,ik' + i\omega'\,(\sin\theta)/v\,)\,U' \tag{3-15a}$$

$$U_z = U'_{z'} + (\,ik'\tan\theta - i\omega'\,(\cos\theta)\,/\,v\,)\,U' \tag{3-15b}$$

$$-i\omega\,U = -i\omega'\,U' \tag{3-15c}$$

The cosine of ϕ in terms of the wavenumber k appears as:

$$\cos\phi = [\,1 - (\frac{kv}{\omega})^2\,]^{1/2} = [\,1 - (\frac{k'v+\omega'\sin\theta}{\omega'})^2\,]^{1/2} \tag{3-16}$$

Expanding (3-16) to the second order about $k'v/\omega'$, we obtain

$$\cos\phi \cong \cos\theta - \frac{k'v}{\omega'}\tan\theta - (\frac{k'v}{\omega'})^2\,\frac{1}{2\cos^3\theta} \tag{3-17}$$

Its inverse to first order is

$$\cos^{-1}\phi \cong \cos^{-1}\theta\,[\,1 + \frac{k'v}{\omega'}\cdot\frac{\sin\theta}{\cos^2\theta}\,] \tag{3-18}$$

Inserting (3-13b) and (3-17) into (3-11) we obtain:

$$U'_{z'} = -\frac{v}{2}\sec^3\theta\,\frac{k'^2}{i\,\omega}\,U' - \frac{1}{2}\,\frac{I_z}{I}\,D \tag{3-19}$$

If now we differentiate (3-10) and substitute (3-18) into the expression for I and I_z, we get:

$$\frac{I_z}{I} = \frac{(\rho v)_z}{(\rho v)} + \frac{k'}{\omega'}\sec\theta\,\tan\theta\,v_z \tag{3-20}$$

The first term of this relation is associated with the reflection coefficient for vertical propagation, while the second term accounts for its angular dependence. Finally, to obtain the upcoming wave equation in time domain, we insert (3-20) into (3-19) and inverse Fourier transform

$$U'_{z't'} = -\frac{v}{2}\sec^3\theta\, U'_{x'x'} - \frac{1}{2}\frac{(\rho v)_{z'}}{(\rho v)} D''_{t''} - \frac{1}{2} v_{z'} \sec\theta \tan\theta\, D''_{x''} \quad (3\text{-}21)$$

This equation is the coupled version of (3-8). The absence of the $U_{x'z'}$ term indicates that the approximations that were made when computing cos ϕ and its inverse, left us within the scope of the approximation discussed in section 3.1. For practical purposes we would like equation (3-21) to be expressed in a single coordinate frame. That implies expressing D'' in terms of the upcoming coordinates x', z', t' . In order to do that we need the transformation between up and downgoing waves, which is:

$$x'' = x' - 2\, z' \tan\theta \quad (3\text{-}22a)$$

$$z'' = z' \quad (3\text{-}22b)$$

$$t'' = t' - 2\, z'(\cos\theta)\, /\, v \quad (3\text{-}22c)$$

Then equation (3-21) can be finally written as:

$$U'_{z't'} = -\frac{v}{2}\sec^3\theta\, U'_{x'x'} - \frac{1}{2}\frac{(\rho v)_{z'}}{(\rho v)} D''_{t'}(x'-2z'\tan\theta, z', t'-2z'\cos\theta/v) -$$

$$-\frac{1}{2} v_{z'} \sec\theta\tan\theta\, D''_{x'}\, (x'-2z'\tan\theta,\ z',\ t'-2z'\cos\theta/v\,)\,. \quad (3\text{-}23)$$

To solve the forward or inverse problem we can complement the transformation (3-12) and the coupled equation (3-23) with the corresponding uncoupled equation for downgoing waves:

$$D''_{z''t''} = (v/2)\sec^3\theta \, D''_{x''x''} \quad . \tag{3-24}$$

Equations (3-23) and (3-24) are the equivalents of equations (2-7) in Chapter 2.

3.3. Computer Algorithms

The pair of equations (3-23) and (3-24) obtained in the last section suffice to solve the forward and inverse problem within the scope of the approximations involved. We will simplify the discussion further by assuming that the reflection coefficient is independent of angle. Neglecting then the last term in equation (3-23) and expressing the vertical reflection coefficient $(\rho v)_z/2(\rho v)$ as $c(x,z)$, equations (3-23) and (3-24) become

$$U'_{z't'} = -\frac{v}{2}\sec^3\theta \, U'_{x'x'} - c(x'z')D''_{t'}(x'-2\tan\theta\, z', z', t'-2\cos\theta z'/v) \tag{3-25}$$

and

$$D''_{z''t''} = \frac{v}{2}\sec^3\theta \, D''_{x''x''} \tag{3-26}$$

In order to use these equations as practical operators capable of extrapolating wave fields, either we have to find integral solutions for U and D or we have to approximate them through finite elements or finite differences. We shall take the last alternative. The first step in this direction is to discretize the coordinates and wave variables as we did in Chapter 2

$$x = j\,Dx \; ; \quad z = k\,Dz \; ; \quad t = n\,Dt \tag{3-27}$$

and

$$U(x,z,t) = U(j\,Dt,\; k\,Dz,\; n\,Dt) \;\text{--->}\; U^{n}_{k,j} \tag{3-28}$$

Since in equation (3-25) we have to express the downgoing wave in terms of upcoming coordinates we will need, in addition, the discrete version of the transformation (3-22) between U and D waves

$$j'' = j' - 2(Dz/Dx)\tan\theta\, k' \tag{3-29a}$$

$$k'' = k' \tag{3-29b}$$

$$n'' = n' - 2\,\frac{Dz}{v\,Dt}\cos\theta\, k' \tag{3-29c}$$

If we now define the sampling intervals Dx , Dz and Dt such that

$$2\tan\theta\,(Dz/Dx) = f \tag{3-30a}$$

and

$$2\cos\theta\,\frac{Dz}{v\,Dt} = e \tag{3-30b}$$

where f and e are integers, the transformation (3-29) reduces to

$$j'' = j' - f\,k' \tag{3-31a}$$

$$k'' = k' \tag{3-31b}$$

$$n'' = n' - e\,k' \tag{3-31c}$$

The next step is to introduce the unit delay operator $Z = \exp(-i\omega Dt)$ and the unit shift operator in the z direction $W = \exp(-ik_z Dz)$, such that

$$Z\, D^n_{k,j} = D^{n-1}_{k,j} \quad \text{and} \quad W\, D^n_{k,j} = D^n_{k-1,j} \tag{3-32}$$

With these definitions we can obtain by following the Crank-Nicholson scheme discrete centered approximations of the derivatives in z and t

$$D_z = \frac{\delta_z}{Dz} D = \frac{2}{Dz} \frac{1-W}{1+W} D^n_{k,j} \tag{3-33a}$$

$$D_t = \frac{\delta_t}{Dt} D = \frac{2}{Dt} \frac{1-Z}{1+Z} D^n_{k,j} \tag{3-33b}$$

For the second derivative in x we can use

$$D_{xx} = \frac{\delta_{xx}}{(Dx)^2} D = \frac{D^n_{k,j-1} - 2D^n_{k,j} + D^n_{k,j+1}}{(Dx)^2} \tag{3-34}$$

A better approximation results if, instead, we discretize the second derivative as

$$D_{xx} = \frac{1}{(Dx)^2} \frac{\delta_{xx}}{1 - b\, \delta_{xx}} D \tag{3-35}$$

where b is a constant which when made equal to $1/12$ gives fourth order accuracy in x. This may be important when working with real seismograms, where the data tends to be undersampled in x. Otherwise, b can be used to simplify the difference equations.

Substituting these difference approximations into equations (3-25) we will have

$$\frac{(1-Z)(1-W)}{(1+Z)(1+W)} U^{n'}_{k',j'} = - a \frac{\delta_{xx}}{1-b\,\delta_{xx}} U^{n'}_{k',j'} - \frac{1}{2} c_{k'j'} \frac{1-Z}{1+Z} D^{n''}_{k'',j''} \quad (3\text{-}36)$$

where

$$a = \frac{v\ Dz\ Dt}{8(Dx)^2} \sec^3\theta \quad (3\text{-}37)$$

With the help of (3-31) we can express D in the upcoming coordinates. If in addition we define a source term according to

$$S^{n'}_{k',j'} = (1/2)\ c_{k',j'}\ D^{n''}_{k'',j''} = (1/2)\ c_{k',j'}\ D^{n'-ek'}_{k',j'-fk'} \quad (3\text{-}38)$$

and drop the primes, equation (3-36) becomes

$$(1-Z)(1-W)(1-b\delta_{xx})U^n_{k,j} = -a(1+Z)(1+W)U^n_{k,j} - (1-Z)(1+W)(1-b\delta_{xx})S^n_{k,j} \quad (3\text{-}39)$$

Upon substituting (3-33), this equation can be rewritten as

$$[\,1+(a-b)\delta_{xx}\,]\,U^n_{k,j} = [\,1-(a+b)\delta_{xx}\,]\,(U^n_{k+1,j} + U^{n-1}_{k,j}) - [\,1+(a-b)\delta_{xx}]\,U^{n-1}_{k+1,j} -$$

$$- (1-b\delta_{xx})\,[\,S^n_{k+1,j} + S^n_{k,j} - S^{n-1}_{k+1,j} - S^{n-1}_{k,j}\,] \quad (3\text{-}40)$$

Making $a = b$ (by properly choosing Dz or by dropping fourth order accuracy in x), equation (3-40) finally simplifies into an explicit equation

$$U^n_{k,j} = [\,1-2a\,\delta_{xx}\,]\,(U^n_{k+1,j} + U^{n-1}_{k,j}) - U^{n-1}_{k+1,j} - (1-a\,\delta_{xx})(S^n_{k+1,j} +$$

$$+ S^n_{k,j} - S^{n-1}_{k+1,j} - S^{n-1}_{k,j}) \quad (3\text{-}41)$$

Another way to represent the delay operators in equation (3-36), is by rewriting it as follows

$$(1-W)(1-b\delta_{xx})U^n_{k,j} = -a\,\frac{(1+Z)(1+W)}{1-Z}\,\delta_{xx}U^n_{k,j} - (1+W)(1-b\delta_{xx})S^n_{k,j} \quad (3\text{-}42)$$

Now we can expand $(1-Z)^{-1}$ in a power series

$$(1-Z)^{-1} = 1 + Z + Z^2 + Z^3 + \cdots \quad (3\text{-}43)$$

Substituting back into (3-42) and letting as before $a = b$, we finish with an integrated form of equation (3-36)

$$U^n_{k,j} = (1-2a\delta_{xx})U^n_{k+1,j} - 2a\delta_{xx}\sum_{i=1}^{n}(U^{n-i}_{k,j} + U^{n-i}_{k+1,j}) -$$

$$- (1-a\,\delta_{xx})\,(S^n_{k,j} + S^n_{k+1,j}) \quad (3\text{-}44)$$

Similarly, the corresponding difference approximations for the uncoupled equation (3-26) are

$$D^n_{k,j} = (1 - 2a\,\delta_{xx})\,(D^{n-1}_{k,j} + D^n_{k-1,j}) - D^{n-1}_{k-1,j} \quad (3\text{-}45)$$

and

$$D^n_{k,j} = (1 - 2a\,\delta_{xx})\,D^n_{k-1,j} - 2a\,\delta_{xx}\sum_{i=1}^{n}(D^{n-i}_{k,j} + D^{n-i}_{k-1,j}) \quad (3\text{-}46)$$

A detailed discussion about the stability of these and other related equations can be found in [9]. I will just mention the two most important constraints. First, the constant a has to be less or equal 4 ($a \leq 4$). Secondly, the only valid unknowns with time running forward are $U^{n+1}_{k,j}$ and $D^{n+1}_{k+1,j}$, while with time running backwards - $U^{n}_{k+1,j}$ and $D^{n}_{k,j}$. This last constraint is related to causality and has been discussed previously by Claerbout [5] and Riley [17].

Provided we guarantee stability, any of the equations presented here (3-41, 44, 45 and 46) can be used as a continuation equation to extrapolate up and downgoing waves from the surface back into the earth and vice-versa. I did not try to solve the inverse problem in the diffraction case, but the 2-D forward and 1-D inverse cases, indicate that the technique is closely related to that of Chapter 2. To implement the computer algorithm we have to supplement these equations with the corresponding initial and boundary conditions which, in principle, are identical to those of Chapter 2

$$U^{n}_{0,j} = R^{n}_{j} \qquad (3\text{-}47a)$$

$$D^{n}_{0,j} = B^{n}_{j} - R^{n}_{j} \qquad (3\text{-}47b)$$

$$U^{n}_{k,j} = 0 \quad \text{for} \quad n \leq k \qquad (3\text{-}47c)$$

where, as before, R is the recorded surface seismogram and B is the source waveform. In addition, due to the presence of the diffraction term, we will need side boundary conditions. Here we have several choices: 1) we can assume that U and D vanish at the side boundaries, 2) we can assume a zero slope of the wave fields or, 3) if we want to avoid reflections off the sides, we can try the more sophisticated absorbing boundary conditions discussed in [7]. For the computation of the synthetic examples shown in the next section, a zero slope condition was used. The remaining algorithm was close to that discussed by Riley with the exception that in the slanted case we have to consider an extra shift in x when comparing the U and D grids during the computation. Higher order algorithms are discussed in [10].

3.4. Synthetic Examples

Figure 3.1a shows the same model used in Chapter 2, that is, a dipping, undulating sea bottom, overlying a faulted reflector. The sequence illustrates only the forward problem. The vertical as well as the slanted seismograms are included. Although the computer algorithm used in this case was different from that of Chapter 2, the 1-D vertical and slanted seismograms (Figures 3.1b and 3.1c) are replicas of those obtained with the Noah algorithm. The only difference is in relation to the angle of propagation, which in this case allows for a better splitting of the peg-legs (PL_{11} through PL_{23}). The peg-legs at the right appear to separate due to lateral variations of the sea floor, while the peg-legs on the left clearly show distinct arrival times. The multiple reflections also show differences in arrival times when the vertical and the slant seismograms are compared. This is especially noticeable in the region where the second order multiple (M_2) intersects the faulted reflector (P_2). The oscillations of the multiples in the slant case are smaller in amplitude compared to those of the vertical seismogram.

In the 2-D vertical seismogram (Figure 3.1d) we can observe the diffraction hyperbolas, which are usually symmetrical, in the regions where the acoustic energy focuses. Reflections are present as well, at the side boundaries due to the zero slope boundary condition that was used. Also a relatively large amount of numerical dispersion can be noted, indicating the need for better difference approximations. Comparison with the 2-D slant seismogram shows the loss of symmetry in the diffraction hyperbolas, which tend to be skewed and higher in amplitude toward the side from which they are being illuminated. The separation of the peg-legs is not as clear as in the 1-D case due to the masking by the diffractions.

Figure 3.2 is a repeat of the previous sequence but for a model that mimics a bright spot. Again, the complex separation of the peg-legs in the slanted case is clearly observed. The intersection of the second order multiple (M_2) with the top of the spot, also indicates differences in arrival times in both cases (vertical and slanted). In the 2-D seismograms (3.2d and 3.2e), the diffractions have the same asymmetrical pattern of the previous example.