RADIATIVE EQUILIBRIUM IN ACOUSTIC LAYERED MEDIA

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John Burg recently told me of a new theorem of his that describes the distribution of radiant energy in an acoustic layered medium. For the so-called earthquake geometry depicted in Fig. 1 (copied from Fig. 8-8, *Fundamentals of*

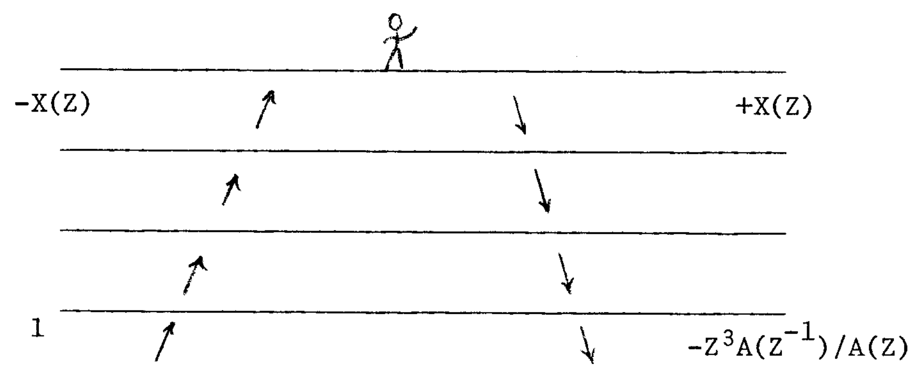

FIGURE 1.—An impulse incident from beneath a stack of layers. A(Z) is the Levinson prediction error filter.

Geophysical Data Processing), it turns out that the total energy in any one layer is the same as that in any other layer. Of course the power spectrum varies from layer to layer, but its integral, the total power, is the same for each layer. The theorem is proved by analysis of the deterministic case of an impulse incident from below. With white noise incident from below, the thermodynamic concept of uniform distribution of energy results.

This theorem should not be confused with conservation of energy. Energy conservation says that the energy flux $F(\omega)$ is constant in any source-free region; that is, $F(\omega)$ does not depend on k where

$$F(\omega) \quad = \quad Y_k(U_k^* U_k - D_k^* D_k) \ , \tag{1}$$

where D_k is the downgoing wave, U_k is the upgoing wave, and Y_k is the admittance of the k^{th} layer. Equation (1) is valid for each and every frequency in any source-free region. The power spectra of sources in other regions are irrelevant.

Radiative equilibrium says that

$$\text{const} = Y_k \int (U_k^* U_k + D_k^* D_k)\, d\omega \,. \tag{2}$$

It will be established only for the restricted geometry of Fig. 1 where the incident radiation is white. It is almost always untrue if the incident radiation is non-white.

In my physics education, the concepts of thermodynamics were taken as either postulates or as observations. For me this is the first time I have seen a purely algebraic deduction based on simple deterministic mechanics lead to an apparently thermodynamic conclusion.

Unfortunately, this deduction is by no means algebraically trivial, and brevity requires me to begin from various equations in *Fundamentals of Geophysical Data Processing* ("FGDP") rather than from first principles. The necessary equations are

$$\frac{Y_k}{Y_1} = \prod_{i=1}^{k-1} \frac{t_i}{t_i'} \,, \tag{3}$$

where t_i is the transmission coefficient across the i^{th} interface from below and t_i' is the same from above. Equation (3) may be derived from FGDP equations 8-1-1, 8-1-2, and 8-1-4.

Further, from 8-2-4, we have the layer matrix

$$\begin{bmatrix} U \\ D \end{bmatrix}_{k+1} = \frac{Z^{-1/2}}{t_k} \begin{bmatrix} 1 & c_k Z \\ c_k & Z \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_k \,. \tag{4}$$

And the Kunetz equation 8-4-4,

$$[1 + R_k(Z) + R_k(Z^{-1})]A_k(Z)\; A_k(Z^{-1}) = \prod_{i=1}^{k-1} t_i' \, t_i \,, \tag{5}$$

where R_k is the reflection seismogram from a model with k layers, and A_k is the Levinson prediction error filter.

From (8-3-5) and (8-3-15), we see that the coefficient of Z to the zero of $A(Z)$ is +1 and we recall that $A(Z)$ is minimum phase, so we can make the causal expansion

$$[A(Z)]^{-1} \quad = \quad 1 + b_1 Z + b_2 Z^2 + \ldots \; . \tag{6}$$

And from exercise (5), p. 157, we have

$$- X_k(Z) \quad = \quad \frac{Z^{(k-1)/2} \prod_{i=1}^{k-1} t_i}{A_k(Z)} \; . \tag{7}$$

Figure 2 is a convenient rearrangement of Fig. 1, where the incident waveform is chosen to get an impulsive waveform at the surface.

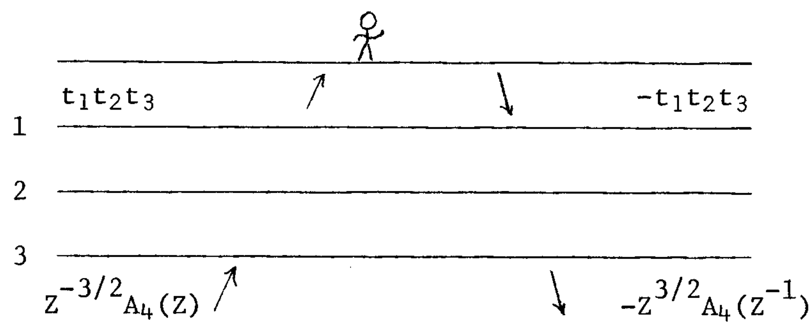

FIGURE 2.—An arrangement of incident radiation that produces a single impulse at the surface.

We can readily determine the waveforms in the second layer of Fig. 2 by means of the layer matrix of Eq. (4). We get

$$Z^{-1/2} t_2 t_3 \begin{bmatrix} 1 & -c_1 Z \\ c_1 & -Z \end{bmatrix} = \frac{Z^{-1/2}}{t_1} \begin{bmatrix} 1 & c_1 Z \\ c_1 & Z \end{bmatrix} \begin{bmatrix} t_1 t_2 t_3 \\ -t_1 t_2 t_3 \end{bmatrix} . \tag{8}$$

We observe that the up- and downgoing waves in the top layer are proportional to the first forward and reverse Levinson prediction error filters. This suggests we establish a general result of this nature. We presume

$$\begin{bmatrix} U \\ D \end{bmatrix}_k = \frac{\prod_{i=1}^{N-1} t_i}{\prod_{i=1}^{k-1} t_i} \begin{bmatrix} Z^{-(k-1)/2} & A_k(Z) \\ -Z^{(k-1)/2} & A_k(Z^{-1}) \end{bmatrix} . \tag{9}$$

Inserting into Eq. (4),

$$\begin{bmatrix} Z^{-k/2} & A_{k+1}(Z) \\ Z^{k/2} & A_{k+1}(Z^{-1}) \end{bmatrix} = Z^{-1/2} \begin{bmatrix} 1 & c_k Z \\ c_k & Z \end{bmatrix} \begin{bmatrix} Z^{-(k-1)/2} & A_k(Z) \\ -Z^{(k-1)/2} & A_k(Z^{-1}) \end{bmatrix}, \tag{10}$$

or

$$A_{k+1}(Z) = A_k(Z) - c_k Z^{-k} A_k(1/Z) , \tag{11}$$

which being the Levinson recursion confirms the presumption (9).

The geometry we actually want to work with is Fig. 1 and not Fig. 2. To find out the waveforms internal to Fig. 1, all we need to do is divide (9) through by the incident source $Z^{-(N-1)/2} A_N(Z)$, obtaining

$$\begin{bmatrix} U \\ D \end{bmatrix}_k = \frac{\prod^{N-1} t}{\prod^{k-1} t} \; \frac{Z^{(N-1)/2}}{A_N(Z)} \begin{bmatrix} Z^{-(k-1)/2} & A_k(Z) \\ -Z^{(k-1)/2} & A_k(Z^{-1}) \end{bmatrix} . \tag{12}$$

We have now assembled everything we need to substitute into (2) to check the theorem. Let the power resident in the k^{th} layer be denoted by P_k. First observe that by the power flux theorem for this geometry $U^*U = D^*D$. Thus, from (2),

$$P_k = 2\, Y_K \int U(1/Z)\; U(Z)\; d\omega . \tag{13}$$

First let us get the impedance in terms of transmission coefficients. Equation (3) may be manipulated to achieve

$$Y_k = Y_1 \prod_{i=1}^{k-1} \frac{t}{t'} = Y_N \left(\prod^{N-1} \frac{t'}{t} \right) \left(\prod^{k-1} \frac{t}{t'} \right). \tag{14}$$

Insert (14) and (12) into (13):

$$P_k = 2\, Y_N \frac{\prod^{N-1} t' \prod^{k-1} t}{\prod^{N-1} t \prod^{k-1} t'} \int \frac{\prod^{N-1} t^2}{\prod^{k-1} t^2} \; \frac{A_k(1/Z)\; A_k(Z)}{A_N(1/Z)\; A_N(Z)}\, d\omega . \tag{15}$$

Cancel some transmission coefficients,

$$P_k = 2Y_N \frac{\prod^{N-1} t't}{\prod^{k-1} t't} \int \frac{A_k(1/Z)\ A_k(Z)}{A_N(1/Z)\ A_N(Z)}\, d\omega\ . \tag{16}$$

Eliminate A_N using (5) with $k = N$:

$$P_k = \frac{2Y_N}{\prod^{k-1} t't} \int [1 + R_N(Z) + R_N(1/Z)]\ A_k(Z^{-1})\ A_k(Z)\ d\omega\ . \tag{17}$$

But the reflection seismogram R_N agrees with R_k at least to the k^{th} coefficient. Thus (5) enables us to simplify (17) further to

$$P_k = 2\,Y_N\ , \tag{18}$$

and the theorem is proven.