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Origin of Gassmann's equations

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ABSTRACT

A short tutorial on the derivation of Gassmann's equations is provided.

INTRODUCTION

Gassmann's relations are receiving more attention as seismic data are increasingly used for reservoir monitoring. Correct interpretation of underground fluid migration from seismic data requires a quantitative understanding of the relationships among the velocity data and fluid properties in the form of fluid substitution formulas, and these formulas are very commonly based on Gassmann's equations. Nevertheless, confusion persists about the basic assumptions and the derivation of Gassmann's (1951) well-known equation in poroelasticity relating dry or drained bulk elastic constants to those for fluid saturated and undrained conditions. It is frequently stated, for example, but quite incorrect to say that Gassmann *assumes* the shear modulus is constant, *i.e.*, mechanically independent of the presence of the saturating fluid. This note clarifies the situation by presenting an unusually brief derivation of Gassmann's relations that emphasizes the true origin of the constant shear modulus *result*, while also clarifying the role played by the shear modulus in the derivation of the better understood result for the bulk modulus.

DERIVATION FOR ISOTROPIC POROUS MEDIA

I now present a very concise, but nevertheless complete, derivation of Gassmann's famous results. For the sake of simplicity, the analysis that follows is limited to isotropic systems, but it can be generalized with little difficulty to anisotropic systems (Gassmann, 1951; Brown and Korringa, 1975; Berryman, 1998). Gassmann's (1951) equations relate the bulk and shear moduli of a saturated isotropic porous, monomineralic medium to the bulk and shear moduli of the same medium in the drained case and shows furthermore that the shear modulus *must be* mechanically independent of the presence of the fluid. An important implicit assumption is that there is no chemical interaction between porous rock and fluid that affects the moduli; if such effects are present, we assume the medium is drained (rather than dry) but otherwise

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neglect chemical effects for this argument. Gassmann's paper is concerned with the quasistatic (low frequency) analysis of the elastic moduli and that is what we emphasize here also. Generalization to higher frequency effects and complications arising in wave propagation due to frequency dispersion are well beyond the scope of what we present.

In contrast to simple elasticity with stress tensor σ_{ij} and strain tensor e_{ij}, the presence of a saturating pore fluid in porous media introduces the possibility of an additional control field and an additional type of strain variable. The pressure p_f in the fluid is the new field parameter that can be controlled. Allowing sufficient time (equivalent to a low frequency assumption) for global pressure equilibration will permit us to consider p_f to be a constant throughout the percolating (connected) pore fluid, while restricting the analysis to quasistatic processes. The change ζ in the amount of fluid mass contained in the pores is the new type of strain variable, measuring how much of the original fluid in the pores is squeezed out during the compression of the pore volume while including the effects of compression or expansion of the pore fluid itself due to changes in p_f. It is most convenient to write the resulting equations in terms of compliances s_{ij} rather than stiffnesses c_{ij}, so for an isotropic porous medium (chosen only for the sake of its simplicity) the basic equation to be considered takes the form:

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ -\zeta \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & s_{12} & -\beta \\ s_{12} & s_{11} & s_{12} & -\beta \\ s_{12} & s_{12} & s_{11} & -\beta \\ -\beta & -\beta & -\beta & \gamma \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ -p_f \end{pmatrix}. \tag{1}$$

The constants β and γ appearing in the matrix on the right hand side will be defined later. It is important to write the equations this way rather than using the inverse relation in terms of the stiffnesses, because the compliances s_{ij} appearing in (1) are simply and directly related to the drained constants λ_{dr} and μ_{dr} (the Lamé parameters for the isotropic porous medium in the drained case) in the same way they are related in normal elasticity (the matrix s_{ij} is just the inverse of the matrix c_{ij}), whereas the individual stiffnesses c^*_{ij} (the $*$ superscript indicates the constants for the saturated case) obtained by inverting the equation in (1) must contain coupling terms through the parameters β and γ that depend on the porous medium and fluid compliances. Using the standard relations for the isotropic moduli, I find that

$$s_{11} = \frac{1}{E_{dr}} = \frac{\lambda_{dr} + \mu_{dr}}{\mu_{dr}(3\lambda_{dr} + 2\mu_{dr})} = \frac{1}{9K_{dr}} + \frac{1}{3\mu_{dr}} \tag{2}$$

and

$$s_{12} = -\frac{\nu_{dr}}{E_{dr}} = \frac{1}{9K_{dr}} - \frac{1}{6\mu_{dr}}, \tag{3}$$

where the drained Young's modulus E_{dr} is defined in terms of the drained bulk modulus K_{dr} and shear modulus μ_{dr} by the second equality of (2) and the drained Poisson's ratio is determined by

$$\nu_{dr} = \frac{\lambda_{dr}}{2(\lambda_{dr} + \mu_{dr})}. \tag{4}$$

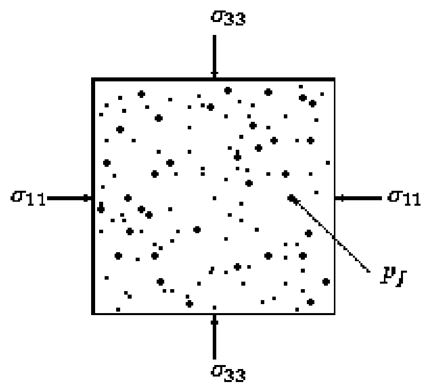

Figure 1: Vertical and horizontal applied stresses are given by σ_{33} and σ_{11}, respectively. The pore pressure is p_f. jim2-poroelast3 [NR]

The fundamental results of interest (Gassmann's equations) are found by considering the saturated (and undrained) case such that

$$\zeta \equiv 0, \tag{5}$$

which — by making use of (1) — implies that the pore pressure must respond to external applied stresses according to

$$p_f = -\frac{\beta}{\gamma}(\sigma_{11}+\sigma_{22}+\sigma_{33}). \tag{6}$$

Equation (6) is often called the "pore-pressure buildup" equation (Skempton, 1954). Then, using this result to eliminate both ζ and p_f from (1), I obtain

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \end{pmatrix} = \begin{pmatrix} s_{11}^* & s_{12}^* & s_{12}^* \\ s_{12}^* & s_{11}^* & s_{12}^* \\ s_{12}^* & s_{12}^* & s_{11}^* \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{pmatrix} = \left[\begin{pmatrix} s_{11} & s_{12} & s_{12} \\ s_{12} & s_{11} & s_{12} \\ s_{12} & s_{12} & s_{11} \end{pmatrix} - \frac{\beta^2}{\gamma} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right] \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{pmatrix}, \tag{7}$$

where s_{ij}^* is the desired compliance including the effects of the trapped fluid, while s_{ij} is the compliance in the absence of the fluid. Since for elastic isotropy there are only two independent coefficients (s_{11} and s_{12}), I find that (7) reduces to one expression for the diagonal compliance

$$s_{11}^* = s_{11} - \frac{\beta^2}{\gamma}, \tag{8}$$

and another for the off-diagonal compliance

$$s_{12}^* = s_{12} - \frac{\beta^2}{\gamma}. \tag{9}$$

If K^* and μ^* are respectively the undrained bulk and shear moduli, then (2) and (3) together with (8) and (9) imply that

$$\frac{1}{9K^*} + \frac{1}{3\mu^*} = \frac{1}{9K_{dr}} + \frac{1}{3\mu_{dr}} - \frac{\beta^2}{\gamma}, \tag{10}$$

and

$$\frac{1}{9K^*} - \frac{1}{6\mu^*} = \frac{1}{9K_{dr}} - \frac{1}{6\mu_{dr}} - \frac{\beta^2}{\gamma}. \tag{11}$$

Subtracting (11) from (10) shows immediately that $1/2\mu^* = 1/2\mu_{dr}$ or equivalently that

$$\mu^* = \mu_{dr}. \tag{12}$$

Thus, the first *result* of Gassmann is that, for purely mechanical effects, the shear modulus for the case with trapped fluid (undrained) is the same as that for the case with no fluid (drained). Then, substituting (12) back into either (10) or (11) gives one form of the result commonly known as Gassmann's equation for the bulk modulus:

$$\frac{1}{K^*} = \frac{1}{K_{dr}} - \frac{9\beta^2}{\gamma}. \tag{13}$$

I want to emphasize that the analysis presented shows clearly that (12) is a definite *result* of this analysis, *not an assumption*. In fact, we must have (12) in order for (13) to hold, and furthermore, if (13) holds, then so must (12). Thus, monitoring any changes in shear modulus with changes of fluid content (say through shear velocity measurements) provides a test of both Gassmann's assumptions (homogeneous frame, no chemical effects, & low frequencies) and results.

To obtain one of the more common forms of Gassmann's result for the bulk modulus, first note that

$$3\beta = \frac{1}{K_{dr}} - \frac{1}{K_g} \equiv \frac{\alpha}{K_{dr}}, \tag{14}$$

where K_g is the grain modulus of the solid constituent present and α is the Biot-Willis parameter (Biot and Willis, 1957). Furthermore, the parameter γ is related through (6) to Skempton's pore-pressure buildup coefficient B, so that

$$\frac{3\beta}{\gamma} = B. \tag{15}$$

Substituting these results into (13) gives

$$K^* = \frac{K_{dr}}{1-\alpha B}, \tag{16}$$

which is another form (Carroll, 1980) of Gassmann's standard result for the bulk modulus.

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