One Way Elastic Wave Equations

by Bjorn Engquist

Let us start from the following formulation of the elastic wave equation

$$\begin{aligned} \rho\, u_{tt} &= (\lambda+2\mu)\, u_{xx} + (\lambda+\mu)\, w_{xz} + \mu\, u_{zz} \\ \rho\, w_{tt} &= \mu\, w_{xx} + (\lambda+\mu)\, u_{xz} + (\lambda+2\mu)\, w_{zz} \end{aligned} \qquad (1)$$

where u and w are the horizontal and vertical displacements respectively, and where the density ρ and the Lamé parameters λ and μ are constant.

We will modify equation (1) such that z can be used as evolution direction. In this way upgoing and downgoing waves are described separately, similar to Claerbout's treatment of acoustic waves (see Claerbout). Compare also the study of elastic waves in frequency domain (Landers, Claerbout). Equation (1) must essentially have time as evolution direction and cannot be used for extrapolation of waves in space.

We rewrite (1) in vector notation

$$U_{tt} = D_1 U_{xx} + H\, U_{xz} + D_2\, U_{zz} \qquad (2)$$

$$U = \begin{pmatrix} u \\ w \end{pmatrix}, \quad D_1 = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} d_2 & 0 \\ 0 & d_1 \end{pmatrix}$$

$$d_1 = (\lambda+2\mu)/\rho, \quad d_2 = \mu/\rho, \quad h = (\lambda+\mu)/\rho$$

$$(\lambda, \rho > 0, \quad \mu \geq 0)$$

Our one way elastic wave equation will have the form

$$U_{tz} + S_1 U_{xz} + S_2 U_{tt} + S_3 U_{tx} + S_4 U_{xx} = 0 \tag{3}$$

where S_i ($i = 1, 2, 3, 4$) are 2x2 matrices. In our examples some of the matrices vanish or contain zeros.

We will derive different sets of S_i corresponding to different orders of approximation or to the use of other dependent variables than u and w. Well posedness will be proved for a particular equation and for a physically reasonable range of λ and μ. That is, we will give bounds on the growth of the solution showing that the equation can be used for calculations. Finally, the dispersion relation for some of the equations are studied and also used for numerical well posedness analysis.

The dispersion relation of (2) is

$$\omega^2 L(\omega, k_x, k_z) = -I\omega^2 + D_1 k_x^{\,2} + H k_x k_z + D_2 k_z^{\,2} = 0 \tag{4}$$

We want dispersion relations for downgoing and upgoing waves

$$\omega L_+(\omega, k_x, k_z) = 0 \quad \text{and} \quad \omega L_-(\omega, k_x, k_z) = 0 \ ,$$

approximating (4), and corresponding to differential equations which can be solved as initial value problems both in z and t. As we have mentioned, the equation (2) cannot be solved in a bounded way as initial value problem in z. With L_+ and L_- approximating

L we mean that $L_+ \hat{U} = 0$ or $L_- \hat{U} = 0$ should imply that $L\hat{U}$ is small, say of order $O(|\frac{k_x}{\omega}|^3)$ or $O(|\frac{k_x}{\omega}|^4)$. In this way a solution to our approximate wave equations will be close to a solution of (2) for waves which travel essentially in the z direction. Let us concentrate on downgoing waves and insert a trial expansion

$$L_+ = \frac{k_z}{\omega} I + A + B \frac{k_x}{\omega} + C (\frac{k_x}{\omega})^2 + D (\frac{k_x}{\omega})^3 = 0$$

into L=0 (we have $i\omega$ as the dual of t).

$$-I + D_1 (\frac{k_x}{\omega})^2 - H \frac{k_x}{\omega} (A + B \frac{k_x}{\omega} + C (\frac{k_x}{\omega})^2 + D (\frac{k_x}{\omega})^3) +$$

$$+ D_2 (A + B (\frac{k_x}{\omega}) + C (\frac{k_x}{\omega})^2 + D (\frac{k_x}{\omega})^3)^2 = O (|\frac{k_x}{\omega}|^4)$$

When the factors in front of 1 , $\frac{k_x}{\omega}$, $(\frac{k_x}{\omega})^2$ and $(\frac{k_x}{\omega})^3$ are annihilated we have

$$- I + D_2 A^2 = 0 \qquad (5)$$

$$- H A + D_2 (AB + BA) = 0 \qquad (6)$$

$$D_1 - H B + D_2 (B^2 + AC + CA) = 0 \qquad (7)$$

$$- H C + D_2 (AD + DA + BC + CB) = 0 \qquad (8)$$

If $L_+ \hat{U} = 0$ for some ω , k_x and k_z and some vector $\hat{U}$, then

$$\frac{k_z}{\omega} \hat{U} = - (A + B \frac{k_x}{\omega} + C (\frac{k_x}{\omega})^2 + D (\frac{k_x}{\omega})^3) \hat{U}$$

Hence, if (5)-(8) are valid, then $L\hat{U}$ is of order $O (|\frac{k_x}{\omega}|^4)$

Let us now solve (5)-(8).

(5): $$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = (D_2^{-1})^{1/2} \quad (\text{for } \mu > 0)$$

$$A_{11} = \frac{1}{\sqrt{d_2}}, \quad A_{22} = \frac{1}{\sqrt{d_1}}, \quad A_{12} = A_{21} = 0$$

The solution $A = -(D_2^{-1})^{1/2}$ corresponds to upgoing waves (L_-) .

(6): $$AB + BA = D_2^{-1} H A = \begin{pmatrix} 0 & \frac{h A_{22}}{d_2} \\ \frac{h A_{11}}{d_1} & 0 \end{pmatrix}$$

$$B_{12} = \frac{h A_{22}}{d_2 (A_{11} + A_{22})} = \frac{h}{\sqrt{d_2} (\sqrt{d_1} + \sqrt{d_2})}$$

$$B_{21} = \frac{h A_{11}}{d_1 (A_{11} + A_{22})} = \frac{h}{\sqrt{d_1} (\sqrt{d_1} + \sqrt{d_2})}$$

$$B_{11} = B_{22} = 0$$

(7): $$AC + CA = - D_2^{-1} D_1 + D_2^{-1} H B - B^2$$

$$C_{11} = \left(- \frac{d_1}{d_2} + \frac{h B_{21}}{d_2} - B_{12} B_{21} \right) / 2 A_{11} =$$

$$= - \frac{d_1}{2\sqrt{d_2}} + \frac{h^2}{2\sqrt{d_2} (\sqrt{d_1} + \sqrt{d_2})^2}$$

$$C_{22} = \left(- \frac{d_2}{d_1} + \frac{h B_{12}}{d_1} - B_{12} B_{21} \right) / 2 A_{22} =$$

$$= - \frac{d_2}{2\sqrt{d_1}} + \frac{h^2}{2\sqrt{d_1} (\sqrt{d_1} + \sqrt{d_2})^2}$$

$$C_{12} = C_{21} = 0$$

(8): $AD + DA = D_2^{-1} HC - BC - CB$

$$D_{12} = \left(\frac{h C_{22}}{d_2} - B_{12} (C_{11} + C_{22}) \right) / (A_{11} + A_{22})$$

$$D_{21} = \left(\frac{h C_{11}}{d_1} - B_{21} (C_{11} + C_{22}) \right) / (A_{11} + A_{22})$$

$$D_{11} = D_{22} = 0$$

If L_+ is used directly to determine a differential equation, that equation will contain third derivatives. We can instead consider a truncated dispersion relation

$$L_+' = \frac{k_z}{\omega} I + A + B \frac{k_x}{\omega} + C \left(\frac{k_x}{\omega}\right)^2 \qquad (9)$$

which corresponds to the differential equation

$$U_{tz} + A U_{tt} + B U_{tx} + C U_{xx} = 0 \qquad (10)$$

That is $S_1 = 0$, $S_2 = A$, $S_3 = B$ and $S_4 = C$ in formula (3). This is the equation for which we will prove well posedness.

The error between L and L' is of order $0 (|\frac{k_x}{\omega}|^3)$. By using a matrix $S_1 \neq 0$ we can formally derive an approximation of one order higher.

$$L_+'' = (I + S_1 \frac{k_x}{\omega}) \frac{k_z}{\omega} + S_2 + S_3 \frac{k_x}{\omega} + S_4 \left(\frac{k_x}{\omega}\right)^2 \qquad (11)$$

The relation $L''_+ = 0$ is equivalent to

$$\frac{k_z}{\omega} + (I + S_1 \frac{k_x}{\omega})^{-1} (S_2 + S_3 \frac{k_x}{\omega} + S_4 (\frac{k_x}{\omega})^2) = 0$$

$$\frac{k_z}{\omega} + S_2 + (S_3 - S_1 S_2) \frac{k_x}{\omega} + (S_4 - S_1 S_3 + S_1{}^2 S_2)^2 (\frac{k_x}{\omega})^2$$

$$+ (- S_1 S_4 + S_1{}^2 S_3 - S_1{}^3 S_2) (\frac{k_x}{\omega})^3 = O (|\frac{k_x}{\omega}|^4)$$

If L_+ and L''_+ are compared we have

$$A = S_2$$

$$B = S_3 - S_1 S_2$$

$$C = S_4 - S_1 S_3 + S_1{}^2 S_2 \qquad (= S_4 - S_1 B)$$

$$D = - S_1 S_4 + S_1{}^2 S_3 - S_1{}^3 S_2 \qquad (= - S_1 C)$$

and

$$\begin{aligned} S_1 &= - D C^{-1} \\ S_2 &= A \\ S_3 &= B - D C^{-1} A \\ S_4 &= C - D c^{-1} B \end{aligned} \qquad (12)$$

We can also work with retarded time for these equations as we can do for the scalar wave equation. The elastic equations, however, describe two wave speeds and we can only hope to eliminate one of them.

$$t' = t - c\,z$$

$$k_z \to k'_z - c\,\omega'$$

$$S'_2 = S_2 - c\,I$$

$$S'_3 = S_3 - c\,S_1$$

In addition to the change of independent variables there is the possibility of using dependent variables other than the displacements u and w. As soon as the original equations with time as evolution direction have the structure of (2), we can apply our analysis.

If, for example u_z and w_x are used as dependent variables the dispersion relation (4) is changed by the following transformation

$$\begin{pmatrix} \hat{u}_z \\ w_x \end{pmatrix} = \begin{pmatrix} i\,k_z\,\hat{u} \\ i\,k_x\,\hat{w} \end{pmatrix}$$

$$\begin{pmatrix} i\,k_z & 0 \\ 0 & i\,k_x \end{pmatrix} \omega^2 L\,(\omega, k_x, k_z) \begin{pmatrix} -\frac{i}{k_z} & 0 \\ 0 & -\frac{i}{k_x} \end{pmatrix} = 0$$

This alters the coefficient matrices such that the mixed derivatives vanish.

$$D_1 = \begin{pmatrix} d_1 & 0 \\ h & d_2 \end{pmatrix}, \quad H = 0$$

$$D_2 = \begin{pmatrix} d_2 & h \\ 0 & d_1 \end{pmatrix}$$

When $H = 0$ we have B and D_1 and hence, also S_1 and S_3 equal zero.

$$S_2 = A = (D_2^{-1})^{1/2} = \begin{pmatrix} \frac{1}{d_2} & -\frac{h}{d_1 d_2} \\ & \frac{1}{d_1} \end{pmatrix}^{1/2} =$$

$$= \begin{pmatrix} \frac{1}{\sqrt{d_2}} & -\frac{h}{\sqrt{d_1}\sqrt{d_2}(\sqrt{d_1}+\sqrt{d_2})} \\ & \frac{1}{\sqrt{d_1}} \end{pmatrix}$$

(The square root can be determined for all matrices with positive eigenvalues.)

The coefficients in the matrix $S_4 = C$ are then determined from the system of linear equations

$$AC + CA = -D_2^{-1} D_1$$

In the same way we can derive one way wave equations for other pairs of variables like the stresses τ_{zz} and τ_{zx} or the combination w_x and the stress τ_{zx} . The latter pair is used by Claerbout and Clayton in this report. They derive the coefficient matrices starting from a formulation of the elastic wave equation other than (1).

The question of well posedness remains. We will prove that (10) can be solved in a bounded way as an initial value problem both in t and z if $\mu > \lambda/2$.

In order to study the problem where z is the evolution direction we transform (10) in t and x .

$$\hat{U}_z = i\left(-\omega A - k_x B - \frac{k_x^2}{\omega} C\right)\hat{U}$$

$$\hat{U}(\omega, k_x, 0) = \hat{f}(\omega, k_x) \qquad \text{(initial data)}$$

The matrix B has positive off diagonal elements and can be symmetrized by a diagonal transformation

$$S = \begin{pmatrix} \sqrt{B_{21}} & 0 \\ 0 & \sqrt{B_{12}} \end{pmatrix}, \quad S B S^{-1} = \begin{pmatrix} 0 & \sqrt{B_{12}B_{21}} \\ \sqrt{B_{12}B_{21}} & 0 \end{pmatrix}$$

Since A and C are diagonal the matrix

$$Q = -S\left(\omega A + k_x B + \frac{k_x^2}{\omega} C\right) S^{-1}$$

is symmetric. Hence, we have

$$(S\hat{U})_z = i Q (S\hat{U})$$

$$S\hat{U}(\omega, k_x, 0) = \exp(i Q z) S \hat{f}(\omega, k_x)$$

The L_2-norm of an exponential of an antisymmetric matrix is bounded by 1 and we have from Parseval's relation

$$\| U(t,x,z) \| \leq \text{Constant} \; \| f(x,t) \|$$

For the stability in time we transform (10) in x and z .

$$\hat{U}_{tt} + i\,(k_z A^{-1} + k_x A^{-1} B)\,\hat{U}_t - k_x^{\,2} A^{-1} C\,\hat{U} = 0$$

$$\hat{U}(0, k_x, k_z) = \hat{f}(k_x, k_z) \qquad (13)$$

$$\hat{U}_t(0, k_x, k_z) = \hat{f}(k_x, k_z)$$

We can symmetrize $(k_z A^{-1} + k_x A^{-1} B)$ in the same way as above, since $A^{-1}B$ has positive off diagonal elements. If we can show that the diagonal matrix $A^{-1}C$ has negative elements the well posedness is guaranteed by the following lemma.

Lemma: A system of ordinary differential equations $U_{tt} = R U_t - S U$ where $R = -R^*$, $S \geq \mu \geq 0$, $||S|| / \mu \leq C_1$ has a solution with the following bounds

$$||U(t)|| \leq C\,(||U(0)|| + ||U_t(0)||) \qquad 0 \leq t \leq T$$

(The constant C depends only on C_1 and T ; R^* denotes the transpose of R .)

Before we prove the lemma, let us check on $A^{-1}C$. The diagonal matrix A^{-1} has positive elements.

$$C_{11} = -\frac{d_1}{2\sqrt{d_2}} + \frac{h^2}{2\sqrt{d_2}(\sqrt{d_1}+\sqrt{d_2})^2} < -\frac{d_1}{2\sqrt{d_2}} + \frac{d_1^{\,2}}{2\sqrt{d_2}(\sqrt{d_1})^2} < 0$$

for $\mu > 0$

$$C_{22} = -\frac{d_2}{2\sqrt{d_1}} + \frac{h^2}{2\sqrt{d_1}(\sqrt{d_1}+\sqrt{d_2})^2}$$

$C_{22} < 0$ is equivalent to the following

$$d_2 - \frac{h^2}{(\sqrt{d_1} + \sqrt{d_2})^2} > 0$$

$$d_2 (d_1 + d_2 + 2\sqrt{d_1 d_2}) - h^2 > 0$$

$$2\mu\sqrt{(\lambda + 2\mu)\mu} + 2\mu^2 - \lambda^2 - \mu\lambda > 0$$

$$\mu > \lambda/2$$

Hence $\mu > \lambda/2$ is necessary for $A^{-1}C$ to be negative definite which we need in our proof.

Proof of lemma: When $R = 0$ we are back to a system $U_t = RU$ where R is anti-symmetric and U is bounded. Hence, we assume $S \neq 0$ and rewrite the equation as a system

$$\begin{aligned} v_t &= Rv - Su \\ u_t &= v \end{aligned} \qquad (14)$$

The matrix S is positive definite, since $\eta \geq \|S\| / C_1 > 0$ and, hence, it has a square root and we can make the transformation

$$\begin{pmatrix} I & 0 \\ 0 & S^{1/2} \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix}_t = \begin{pmatrix} I & 0 \\ 0 & S^{1/2} \end{pmatrix} \begin{pmatrix} R & -S \\ I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & S^{-1/2} \end{pmatrix} *$$

$$* \begin{pmatrix} I & 0 \\ 0 & S^{1/2} \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix}$$

$$\begin{pmatrix} v \\ S^{1/2} u \end{pmatrix}_t = \begin{pmatrix} R & -S^{1/2} \\ S^{1/2} & 0 \end{pmatrix} \begin{pmatrix} v \\ S^{1/2} u \end{pmatrix}$$

The matrix $\begin{pmatrix} R & -S^{1/2} \\ S^{1/2} & 0 \end{pmatrix}$ is antisymmetric. Its exponential is then bounded by 1 and we have

$$||v(t)||^2 + ||S^{1/2}\, u(t)||^2 \leq ||v(0)||^2 + ||S^{1/2}\, u(0)||^2$$

For $||S|| \geq 1$ we get

$$||S^{1/2}\, u(t)||^2 \leq ||v(0)||^2 + ||S||\ ||u(0)||^2$$

$$||u(t)||^2 \leq \frac{1}{\eta}\, ||v(0)||^2 + \frac{||S||}{\eta}\, ||u(0)||^2$$

$$||u(t)|| \leq C\, (\, ||v(0)|| + ||u(0)||\,)$$

If $||S|| \leq 1$ we can write the system (14) in the following way

$$\begin{pmatrix} v \\ u \end{pmatrix}_t = \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix} + \begin{pmatrix} 0 & S \\ I & 0 \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix}$$

$e^{Rt}\, \bar{v} = v$ gives

$$\begin{pmatrix} e^{Rt} & \bar{v}_t \\ & u_t \end{pmatrix} = \begin{pmatrix} 0 & S \\ I & 0 \end{pmatrix} \begin{pmatrix} e^{Rt} & \bar{v} \\ & u \end{pmatrix}$$

$$\begin{pmatrix} \bar{v} \\ u \end{pmatrix}_t = \begin{pmatrix} 0 & e^{-Rt} & S \\ e^{Rt} & & 0 \end{pmatrix} \begin{pmatrix} \bar{v} \\ u \end{pmatrix}$$

$||e^{-Rt}\, S|| \leq 1$, $||e^{Rt}|| \leq 1$ implies

$$(\, ||v(t)||^2 + ||u(t)||^2\,)^{1/2} \leq \exp(t) \quad (\, ||v(0)||^2 + ||u(0)||^2\,)^{1/2}$$

$$||u(t)|| \leq C\, (\, ||u(0)|| + ||v(0)||\,)$$

(Here C depends on t_{max})

This ends the proof of the lemma.

Let us go back to the dispersion relation $L'_{+} = 0$, (9), and plot the two real roots k_z/ω of (det denotes determinant)

$$\det (L'_{+} (\omega, k_x, k_z)) = 0 \tag{15}$$

as functions of k_x/ω (Figure 1 -- the constants ρ , λ and μ all have value 1). The curves correspond to the pressure and shear waves.

In Figure 2 the corresponding curves for retarded time ($t' = t - z/\sqrt{d_1}$, $k_z = k'_z - \omega' / \sqrt{d_1}$) are displayed. The dashed circles correspond to the dispersion relation $L = 0$ [see (4)]. As expected the fit between the two sets of curves is best for small $|k_x/\omega|$.

These figures also give us necessary conditions for well posedness of (10). We can see that for all real ω , k_x ($\omega \neq 0$) there are two real roots k_x such that (15) is valid. Since $\det(L'_{+})$ is a second order polynomial in k_z (or k_z/ω) , there are no other roots.

Let us assume that there existed a complex root $k_z = a + ib$ ($b \neq 0$) . This means that there would exist $\hat{U} \neq 0$ such that $L'_{+} \hat{U} = 0$ and hence, also a solution

$$\exp (i (\omega t + k_z z + k_x x)) \hat{U}$$

to (10) with norm $|\exp(-bz)| \, \|\hat{U}\|$. If $b > 0$ for ω , k_x it is negative for $-\omega$, $-k_x$. The absolute value of b grows if ω and k_x are multiplied by a large constant, since the dispersion relation is homogeneous in ω , k_x and k_z . This tells us that if there is a complex root there is also an exponentially growing solution that can grow with arbitrary rate.

Our Figures 1 and 2 indicate that the one way wave equations are well posed when regarded as initial value problems in z. The full elastic equations (2), however, have exponentially growing modes for large k_x/ω. It is only for small k_x/ω that we have the four real roots k_z/ω which are needed since $\det(L)$ is a fourth order polynomial in k_z/ω.

We can perform a similar study for the t direction using Figure 3. The real roots ω/k_z of (15) are plotted as functions of k_x/k_z. The equation (15) is fourth order as a polynomial in ω and we have four solutions ω/k_z for all k_x/k_z suggesting well posedness. ($\omega = k_x = 0$ is a double root.)

With retarded time $t' = t - cz$ where $c = 1/\sqrt{d_1}$, equation (15) is a third order polynomial. Figure 4 also shows a stable picture with three real roots. When $c > 1/\sqrt{d_1}$ there were complex roots in all experiments. Examples are given in Figures 5 and 6. The experiments also indicated that $\mu < \lambda/2$ caused equation (10) to be ill posed. For $u \geq \lambda/2$ we got the right number of real roots as expected.

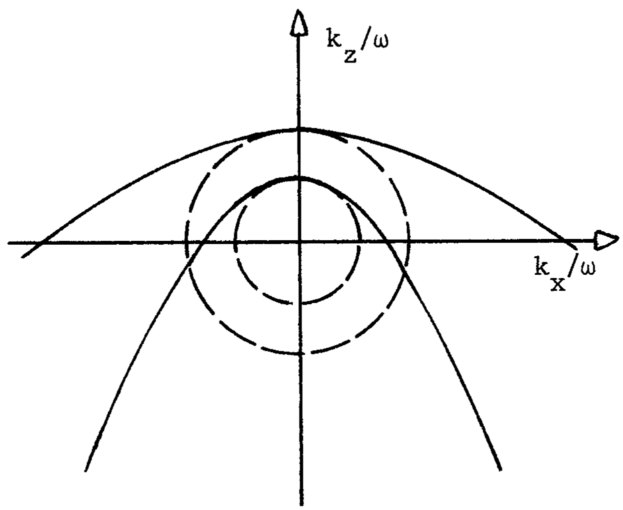

Figure 1. $c = 0$

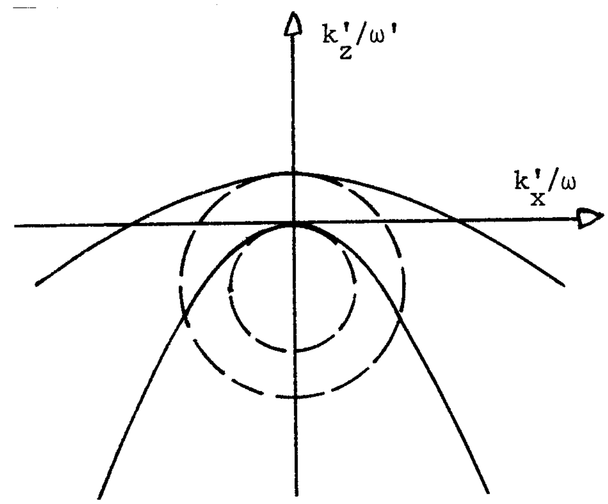

Figure 2. $c = 1/\sqrt{d_1}$

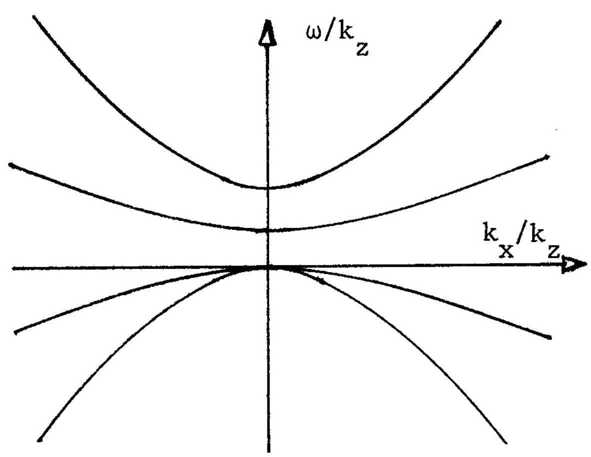

Figure 3. $c = 0$

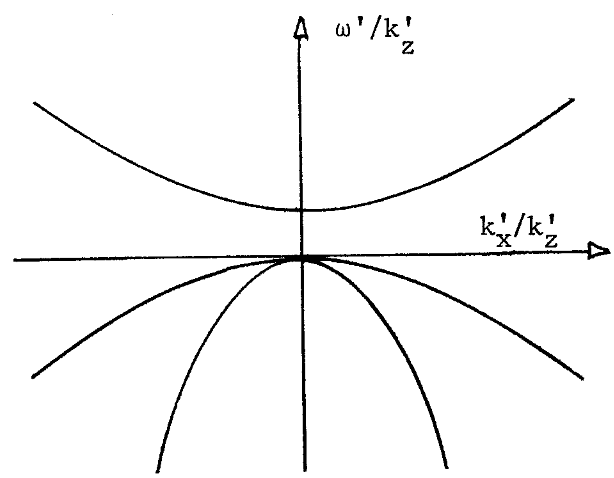

Figure 4. $c = 1/\sqrt{d_1}$

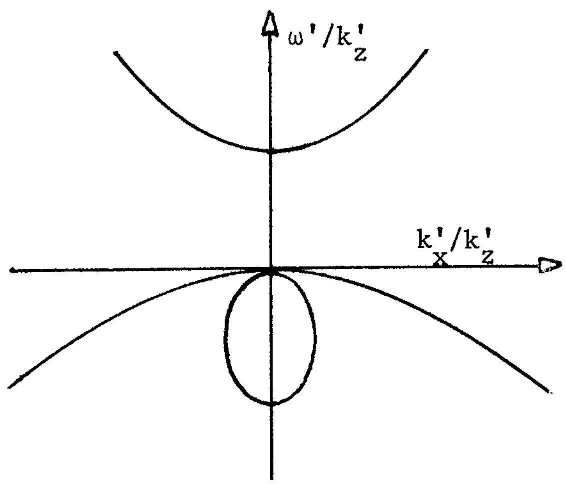

Figure 5. $2c = 1/\sqrt{d_1} + 1/\sqrt{d_2}$

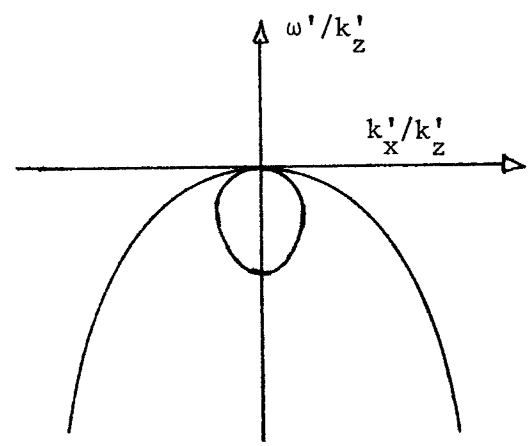

Figure 6. $c = 1/\sqrt{d_2}$

Retarded time $t' = t - c\,z$

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[1] Claerbout, Jon F.: Fundamentals of Geophysical Data Processing: With Application to Petroleum Prospecting, McGraw Hill, 1976.

[2] Landers T. and Jon F. Claerbout: Numerical Calculation of Elastic Waves in Laterally Inhomogeneous Media, J. of Geophys. Res., Vol. 77, No. 8, pp. 1476-1482.