An acoustic wave equation for anisotropic media

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SUMMARY

A wave equation, derived under the acoustic medium assumption for P-waves in transversely isotropic media with a vertical symmetry axis (VTI media), though physically impossible, yields good kinematic approximation to the familiar elastic wave equation for VTI media. The VTI acoustic wave equation is fourth-order and has two sets of complex conjugate solutions. One set of solutions is just perturbations of the familiar acoustic wavefield solutions for isotropic media for incoming and outgoing waves. The second set describes an unwanted wave type that propagates at speeds slower than the P-wave for the positive anisotropy parameter, η, and grows exponentially, becoming unstable, for negative values of η. Luckily, most η values corresponding to anisotropies in the subsurface have positive values which is in the stability range of the acoustic equation. Placing the source in an isotropic layer, a common occurrence in marine surveys where the water layer is isotropic, eliminates most of the energy of this additional wave type. Numerical examples prove the usefulness of this acoustic equation in simulating wave propagation in complex models.

INTRODUCTION

The wave equation is the central ingredient in defining and constraining wave propagation in a given medium. No other constraint, such as the eikonal or raytracing equations, is as conclusive and elaborate (includes all traveltime and amplitude aspects) as the full wave equation. Having such an equation in a simple form for transversely isotropic media with a vertical symmetry axis will help us get a better grip on wave propagation in such media.

In anisotropic media, the acoustic wave equation does not describe a physical phenomenon. This is because acoustic media cannot be anisotropic. If the shear wave velocity equals zero, the medium is rendered isotropic. However, if we ignore the physical aspects of the problem, an acoustic equation for VTI media can be extracted by simply setting the shear wave velocity to zero. Though physically impossible, kinematically the equations resulting from setting the shear wave velocity to zero yield good approximations of the elastic equations.

Alkhalifah and Tsvankin (1995) showed that time-related processing for P-waves, including dip-moveout correction and time migration, in transversely isotropic (TI) media with a vertical symmetry axis (VTI media) depends just on two parameters: the zero-dip NMO velocity [$V_{\rm nmo}(0)$], and an anisotropy parameter η that is a special combination of Thomsen's (1986) parameters.

In an earlier paper (Alkhalifah, 1997), I have derived a simple dispersion equation that relates the vertical slowness to the horizontal one in transversely isotropic media. The simplicity of this acoustic equation is a direct result of setting the shear wave velocity to zero. Although the equation results in an approximation, the accuracy is far within the typical accuracies expected in practical geophysical applications. Simply stated, the equation is exact within the confines of seismic error tolerance. This equation served as the starting point for the development of an acoustic wave equation that describes P-wave propagation in VTI media. In this expanded abstract, I derive the acoustic wave equation for VTI media using the dispersion relation. Numerical simulations of wave propagation using finite difference techniques demonstrate the accuracy and efficiency of the VTI acoustic wave equation, especially in comparison with the elastic wave equation.

THE VTI ACOUSTIC WAVE EQUATION

Recently, I have derived a simple equation that relates the vertical slowness, p_z, to the horizontal one, p_r, in VTI media, based on set-

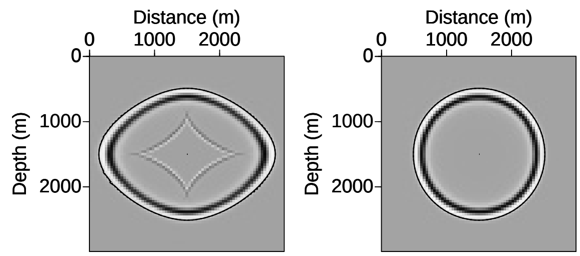

Figure 1: The wavefield at 1 s resulting from a source at center of the section for a VTI medium with η=0.4 (left), and for an isotropic medium (right). In both cases, the velocity (vertical velocity in VTI media) is 1000 m/s. The solid black curves are the solutions of the eikonal equation for both media with the shear wave velocity set to zero, and the dashed curves (not apparent) are the solutions when the shear wave velocity equals half the P-wave velocity. The two eikonal solution curves coincide in the isotropic case and practically coincide in the VTI case.

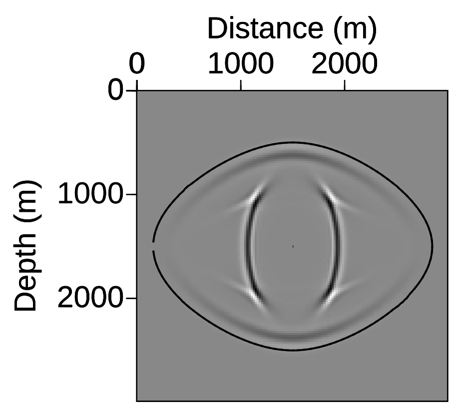

Figure 2: The z-component of the elastic wavefield at 1 s caused by a source at time 0 located at the center. The VTI model is the same as in Figure 1, with v=1000 m/s and η=0.4. The solid curve corresponds to the solution of the acoustic eikonal equation for the same medium.

ting the vertical shear wave velocity to zero (Alkhalifah, 1997). In such media, the slowness surface in the horizontal plane is circular (isotropic), and therefore, p_r can be replaced by $\sqrt{p_x^2+p_y^2}$, where the slowness vector, **p**, has components in the Cartesian coordinates given by p_x, p_y, and p_z. As a result, the migration dispersion relation in 3-D media is

$$p_z^2 = \frac{v^2}{v_v^2}\left(\frac{1}{v^2} - \frac{p_x^2+p_y^2}{1-2v^2\eta(p_x^2+p_y^2)}\right). \quad (1)$$

Using $\mathbf{k} = \omega\mathbf{p}$, where **k** is the wavenumber vector with components in the Cartesian coordinates (k_x, k_y, k_z), and ω is the angular frequency, equation (1) becomes

$$k_z^2 = \frac{v^2}{v_v^2}\left(\frac{\omega^2}{v^2} - \frac{\omega^2(k_x^2+k_y^2)}{\omega^2-2v^2\eta(k_x^2+k_y^2)}\right). \quad (2)$$

Multiplying both sides of equation (2) with the wavefield in the Fourier domain, $F(k_x,k_y,k_z,\omega)$, as well as using inverse Fourier transform on k_z, k_x, k_y, and ω ($k_z \rightarrow -i\frac{d}{dz}$, $k_x \rightarrow -i\frac{\partial}{\partial x}$, $k_y \rightarrow -i\frac{\partial}{\partial y}$, and $\omega \rightarrow i\frac{\partial}{\partial t}$) yields the acoustic wave equation for VTI media is given by

$$\frac{\partial^4 F}{\partial t^4} - (1+2\eta)v^2\left(\frac{\partial^4 F}{\partial x^2 \partial t^2} + \frac{\partial^4 F}{\partial y^2 \partial t^2}\right) =$$
$$v_v^2\frac{\partial^4 F}{\partial z^2 \partial t^2} - 2\eta v^2 v_v^2\left(\frac{\partial^4 F}{\partial x^2 \partial z^2} + \frac{\partial^4 F}{\partial y^2 \partial z^2}\right). \quad (3)$$

This equation is a fourth-order partial differential equation in t. Setting $\eta = 0$, $v = v_v$, and substituting $P = \frac{\partial^2 F}{\partial t^2}$ yields the acoustic equation for isotropic media

$$\frac{\partial^2 P}{\partial t^2} = v^2\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2}\right). \quad (4)$$

Rewriting equation (3) in terms of $P(x,y,z,t)$ instead of $F(x,y,z,t)$, yields

$$\frac{\partial^2 P}{\partial t^2} = (1+2\eta)v^2\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right) + v_v^2\frac{\partial^2 P}{\partial z^2} -$$
$$2\eta v^2 v_v^2\left(\frac{\partial^4 F}{\partial x^2 \partial z^2} + \frac{\partial^4 F}{\partial y^2 \partial z^2}\right), \quad (5)$$

where

$$F(x,y,z,t) = \int_0^t dt' \int_0^{t'} P(x,y,z,\tau)d\tau.$$

In the numerical implementation, for convenience, I rely on the equation (5).

For comparison, the 2-D elastic wave equation, which is best described in VTI media using the density-normalized elastic coefficients, $A_{ijkl}(=C_{ijkl}/\rho)$, is given by (Aki and Richards, 1980)

$$\frac{\partial^2 u_x}{\partial t^2} = A_{1111}\frac{\partial^2 u_x}{\partial x^2} + (A_{1133}+A_{1313})\frac{\partial^2 u_z}{\partial x \partial z} + A_{1313}\frac{\partial^2 u_x}{\partial z^2},$$

and

$$\frac{\partial^2 u_z}{\partial t^2} = A_{3333}\frac{\partial^2 u_z}{\partial z^2} + (A_{1133}+A_{1313})\frac{\partial^2 u_x}{\partial x \partial z} + A_{1313}\frac{\partial^2 u_z}{\partial x^2},$$

where u_x and u_z are the components of the wavefield vector, **u**, in two dimensions. Solving the elastic wave equation in heterogeneous media requires applying finite-difference computation to two equations (three equations in 3-D media) corresponding to the components of the wavefield. Calculating the wavefield for each component is almost as computationally involved as calculating the acoustic wavefield. This method also incurs the additional expense of input, output, and storage of the wavefield and the corresponding medium parameters in the elastic medium case.

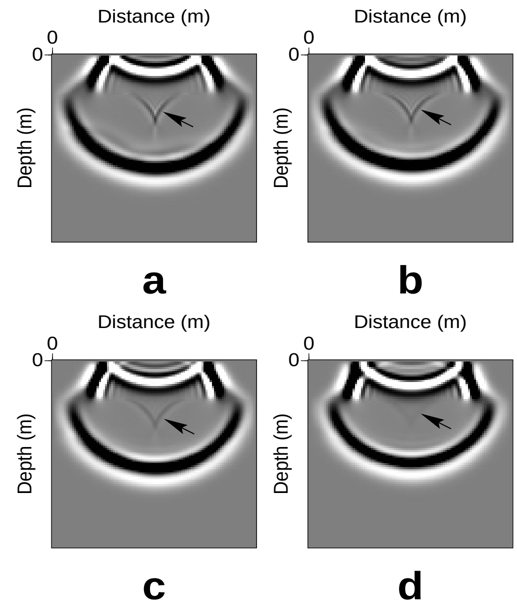

Figure 3: The wavefield at 0.13 s caused by a source at a central lateral location. The model consists of two layers with an interface at depth 100 m. The first layer is isotropic with $v = 1500$ m/s, and the second layer is VTI with v=2000 m/s and $\eta = 0.1$. The source depth varies with (a) the source at depth 100 m (on the isotropic-VTI interface), (b) the source at depth 95 m, (c) the source at depth 90 m, and (d) the source at depth 80 m. The arrows point to the additional wave as it decays gradually with increasing distance between the source and the VTI layer.

In addition, the solution of the elastic wave equation contains both P- and S-waves, whereas the acoustic equation yields only P-waves. The presence of S-waves in the solution of elastic wave equation makes that equation less desirable when used for modeling P-wave propagation in zero-offset conditions, such as when the exploding reflector assumption is used.

FINITE DIFFERENCE SOLUTIONS OF THE WAVE EQUATION

For simplicity, I use the acoustic wave equation (5), which is second order in t, as opposed to equation (3) [fourth order in derivates of t]. The finite-difference equations for VTI media are subjected to the same constraints and rules used in the isotropic case (such as the CFL condition) to avoid numerical dispersion and instability.

Figure 1 shows the wavefield at time 1 second caused by an impulse force excited at time 0. On the left side, the medium is homogeneous and VTI with v_v=1 km/s, v=1 km/s, and η=0.4. On the right side, the medium is isotropic with v_v=1 km/s, v=1 km/s, and η=0. Both wave-

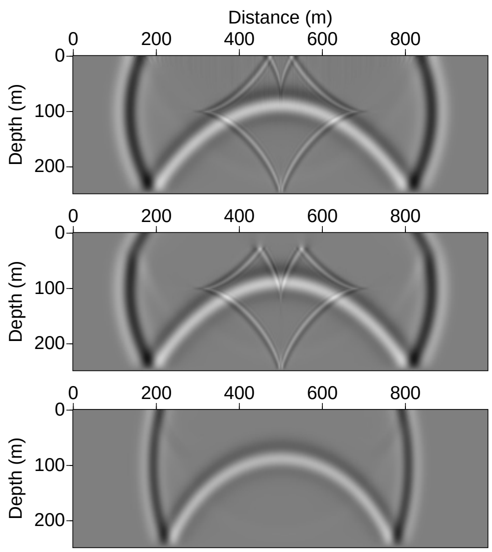

Figure 4: Snapshots of the wavefield at 0.2 s caused by a source at depth 100 m in a central lateral location of 500 m. Top: A snapshot corresponding to a homogeneous VTI medium with η=0.2. Middle: A snapshot for the same medium but with a thin isotropic layer of 25 m thickness at the top. Bottom: A snapshot for an isotropic medium with η=0 throughout. The velocity for all models is the same at 2000 m/s.

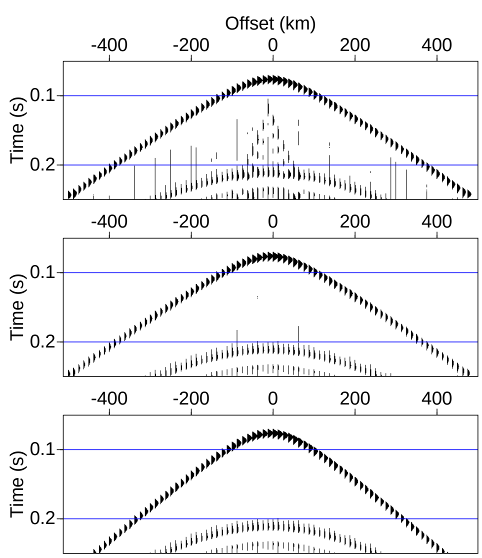

Figure 5: Common-shot gathers for the models described in Figure 4 from top to bottom, respectively. The geophones are placed 10 m under the surface. Top: the gathers generated for the homogeneous VTI medium with η=0.2. Middle: the gathers generated for the same homogeneous VTI medium, but with a thin isotropic layer of 25 m thickness at the top. Bottom: the gathers generated for the isotropic medium.

fields are calculated using the second-order finite difference applied to the new acoustic wave equation. In the case of VTI, an additional wave type appears in the section and travels at a speed that is lower than the P-wave velocity. This artifact is the additional solution, mentioned earlier, that behaves like a wave for positive η and exponentially decays or grows for negative η. Such an artifact does not appear in the solution for the isotropic medium. Therefore, we may want to place the source in an isotropic layer and take advantage of the evanescent nature of this wave in isotropic media. The black curves in Figure **??** correspond to solutions of the eikonal equation. The solid curves correspond to the solutions that use the acoustic assumption, in which the shear wave velocity equals zero, while the dashed curves correspond to a shear wave velocity equal to half the P-wave velocity. As expected, in the isotropic case both curves exactly coincide and are therefore indistinguishable. In the VTI case, differences between the two curves exist, but are hardly noticeable. The independence of the eikonal equation on the shear wave velocity in VTI medium is in agreement with the results obtained in an earlier study (Alkhalifah, 1997).

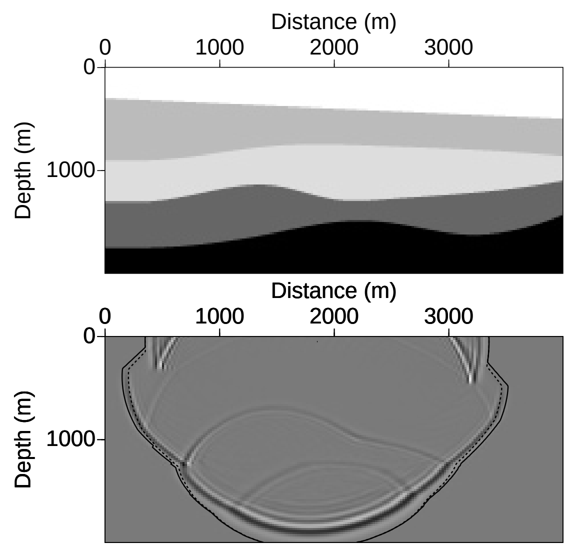

Figure 6: Top: A velocity model consisting of five layers with velocity equal 1500, 1900, 1700, 2400, and 3000 m/s from top to bottom. The corresponding η for the VTI model are 0, 0.1, 0.2, 0.15, and 0.05 from top to bottom. Bottom: The wavefield at 1 s caused by a source at distance 1850 m and depth 50 m for the VTI model. The solid black curve is the solution of the eikonal equation for the VTI model, and the dashed curve is the solution for the corresponding isotropic model.

Figure 2 shows the z-component of the elastic wavefield (computed using the elastic wave equation) for the same model used in Figure 1. The solid curve is the solution of the acoustic eikonal equation. Kinematically, for P-waves, the acoustic and elastic wavefields are similar. Dynamically, they differ considerably; the elastic wavefield includes S-waves (the slower waves), here with triplication, a phenomenon common to S-waves in strongly anisotropic media.

ELIMINATING THE ARTIFACT

To demonstrate some of the features of this additional wave type (the artifact in Figure 1), Figure 3 shows a snapshot taken at 0.13 s of the finite-difference solution of the acoustic VTI wave equation for a two-layered model with an interface at depth 100 m. The first layer is isotropic with velocity equal 1500 m/s, and the second layer is VTI with velocity equal to 2000 m/s and η=0.1. In Figure 3a, the source is placed on the boundary between the isotropic and VTI layers at depth 100 m. As a result, no artifact appears in the isotropic layer, whereas the artifact is clearly apparent in the VTI layer. In Figure 3b, the source is placed in the isotropic layer at depth 95 m from the surface. The resulting additional wave is weaker than in the previous case, where the source was directly interacting with the VTI layer. A greater distance between the source and the VTI layer, as in Figures 3c and 3d, causes the artifact to decay. In fact, when the source is placed at depth 80 m (Figure 3d), only 20 m from the VTI layer, the additional wave practically disappears.

Putting the receivers in the isotropic layer all but assures that no such artifacts appear on synthetic sections. The basic concept is that these waves do not travel in the isotropic layer. Figure 4 shows snapshots of the wavefield for a VTI homogeneous model (top section), a VTI model with a thin isotropic layer of 25 m thickness at top (middle section), and a purely isotropic homogeneous model (bottom section). All three models have a constant velocity of 2000 m/s. As shown earlier, the artifact, illustrated by the diamond shaped wave, appears only when the medium is VTI, as is the case with the top two sections. However, this wave does not travel in the thin isotropic layer in the middle section; it actually reflects at the isotropic-VTI boundary. Therefore, by placing the geophones in this thin isotropic layer, we will not record the artifact.

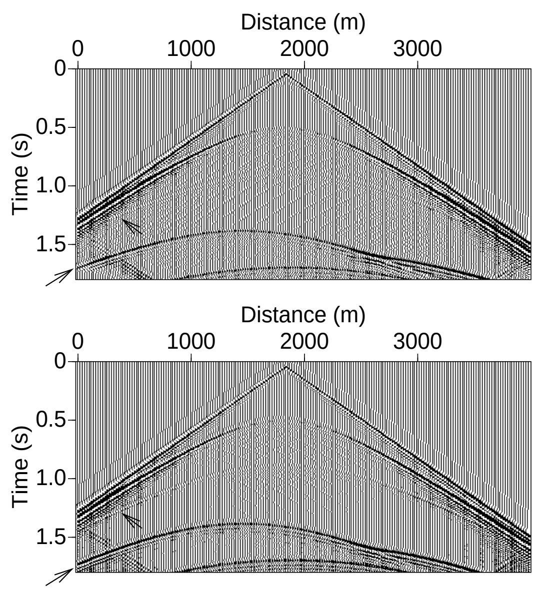

Figure 7: Common-shot gathers for the model in Figure 6 using geophones that span the 4-km distance, buried at depth 50 m from the surface. On top is a shot gather corresponding to the VTI model; on the bottom, a shot gather corresponding to the isotropic model. Both synthetic gathers were calculated using the acoustic wave equation (**??**). The arrows point to some of the differences in energy arrival times between the two media.

Figure 5 shows common-shot gathers corresponding to the models described in Figure 4. The horizontal line of receivers, in all cases, are placed at depth 10 m from the surface. Therefore, for the VTI model case with the thin isotropic layer (middle section), the artifact, which is apparent for the purely VTI model, disappears. The wave traveltimes, on the other hand, are barely affected by this thin isotropic layer. The purely isotropic model (bottom section) results in slower wave arrival times than those in the VTI models.

A PRACTICAL SUBSURFACE MODEL

Figure 6 shows a velocity model and the corresponding wavefield for a source excited near the surface at lateral distance 1850 m from the origin. The wavefield is calculated using the finite-difference method applied to equation (5) with absorbing boundary conditions. The curves correspond to solutions of the eikonal equation for the VTI model (solid curve) and an equivalent isotropic model (dashed curve). Both models have the same vertical and NMO velocities, and as a result the two wavefront curves coincide at the zero angle from the vertical. The biggest difference between the two wavefronts occurs near horizontal wave propagation, where the influence of the different η values affects the wavefront the most. The corresponding elastic curve (plotted in gray, but indistinguishable) coincide with the acoustic one for the VTI model. The model was constructed so that the source and receivers are placed in the water layer, which conveniently, is isotropic.

Figure 7 shows common-shot gathers corresponding to the model in Figure 6 computed using geophones placed near the surface. The geophones cover the whole 4-km lateral distance. The top gather in Figure 7 corresponds to the VTI model, and the bottom gather corresponds to the isotropic model. The differences (indicated by arrows) are concentrated at later times, because the largest anisotropies are at depth. Figure 7 also demonstrates the importance of anisotropy in processing; such differences in traveltimes, as well as amplitudes, will considerably hamper isotropic processing when anisotropy similar to that modeled here is ignored.

CONCLUSIONS

Though physically impossible, the acoustic wave equation for P-waves in transversely isotropic media with a vertical symmetry axis (VTI media) yields good kinematic approximations to the familiar elastic wave equation for VTI media. The fourth-order nature of this acoustic equation results in two sets of complex conjugate solutions. One set of solutions are just perturbations of the familiar acoustic wavefield solutions in isotropic media for incoming and outgoing waves. The second set describes a wave type that propagates at speeds slower than the P-wave for the positive anisotropy parameter, η, and grows exponentially, becoming unstable, for negative values of η. Most η values corresponding to anisotropies in the subsurface are likely to have positive values. Placing the source or receivers in an isotropic layer, a common occurrence in marine surveys where the water layer is isotropic, will eliminate most of the energy of this additional wave type. Numerical examples, provided in this paper, prove the usefulness of this acoustic equation in simulating wave propagation in VTI media.

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