Efficient 3-D wavefield extrapolation with Fourier finite-differences and helical boundary conditions

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Summary

Fourier finite-difference (FFD) migration combines the complementary advantages of the phase-shift and finite-difference migration methods. However, as with other implicit finite-difference algorithms, direct application to 3-D problems is prohibitively expensive. Rather than making the simple $x-y$ splitting approximation that leads to extensive azimuthal operator anisotropy, I demonstrate an alternative approximation, that retains azimuthal isotropy without the need for additional correction terms.

Helical boundary conditions allow the critical 2-D inverse-filtering step to be recast as 1-D inverse-filtering. A spectral factorization algorithm can then factor this 1-D filter into a (minimum-phase) causal component and a (maximum-phase) anti-causal component. This factorization provides an LU decomposition of the matrix, which can then be inverted directly by back-substitution. The cost of this approximate inversion remains $O(N)$ where N is the size of the matrix.

Method

Three-dimensional FFD (Ristow and Ruhl, 1994) extrapolation is based on the equation,

$$\frac{\partial P}{\partial z} = i\left[\sqrt{\frac{\omega^2}{c^2}+\nabla^2_{x,y}} + \left(\frac{\omega}{v}-\frac{\omega}{c}\right) + \frac{\omega}{v}\left(1-\frac{c}{v}\right)\frac{\frac{v^2}{\omega^2}\nabla^2_{x,y}}{a+b\frac{v^2}{\omega^2}\nabla^2_{x,y}}\right]P, \tag{1}$$

where $v=v(x,y,z)$ is the medium velocity, c is a reference velocity ($c \leq v$), and a and b are coefficients subject to optimization. The first term describes a simple Gazdag phase-shift; the second term describes the split-step correction (Stoffa et al., 1990); and the third term describes an additional correction that can be applied as an implicit finite-difference operator (Claerbout, 1985).

In areas with strong lateral velocity variations ($c/v \approx 0$), FFD reduces to a finite-difference migration, while in areas of weak lateral velocity variations ($c/v \approx 1$), FFD retains the steep-dip accuracy advantages of phase-shift migration. As a full-wave migration method, FFD also correctly handles finite-frequency effects.

For constant lateral velocity, the finite-difference term in equation (1) can be rewritten as the following matrix equation,

$$(\mathbf{I}+\alpha_1\mathbf{D})\,\mathbf{q}_{z+1} = (\mathbf{I}+\alpha_2\mathbf{D})\,\mathbf{q}_z \tag{2}$$

$$\mathbf{A}_1\mathbf{q}_{z+1} = \mathbf{A}_2\mathbf{q}_z \tag{3}$$

where $\mathbf{D}$ is a finite-difference representation of the x,y-plane Laplacian, $\nabla^2_{x,y}$. Scaling coefficients, α_1 and α_2, are complex and depend both on the ratio, ω/v, and the ratio c/v.

The right-hand-side of equation (3) is known. The challenge is to find the vector $\mathbf{q}_{z+1}$ by inverting the matrix, $\mathbf{A}_1$. For 2-D problems, only a tridiagonal matrix must be inverted; whereas, for 3-D problems the matrix becomes blocked tridiagonal. For most applications, direct inversion of such a matrix is prohibitively expensive, and so approximations are required for the algorithm to remain cost competitive with other migration methods.

A partial solution is to split the operator to act sequentially along the x and y axes. Unfortunately this leads to extensive azimuthal operator anisotropy, and necessitates expensive additional phase correction operators.

The blocked-tridiagonal matrix of the 3-D extrapolation, $\mathbf{A}_1$, represents a two-dimensional convolution operator. Following Rickett et al.'s (1998) approach to factoring the 45° equation, I apply helical boundary conditions (Claerbout, 1998b), to simplify the structure of the matrix, reducing the 2-D convolution to an equivalent problem in one dimension.

For example, through the process illustrated in Figure 1, helical boundary conditions allow the two-dimensional 5-point Laplacian filter, d, to be expressed as an equivalent one-dimensional filter of length $2N_x+1$ as follows

$$d = \begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} \xrightarrow{\text{helical boundary conditions}} (1,\, 0,\, \ldots\, 0,\, 1,\, -4,\, 1,\, 0,\, \ldots\, 0,\, 1)$$

Unfortunately, the complex scale-factor, α_1, means $\mathbf{A}_1$ is symmetric, but not Hermitian, so the filter, a_1, is not an autocorrelation function, and standard spectral factorization algorithms will fail. Fortunately, however, the Kolmogoroff method can be extended to factor any cross-spectrum into a pair of minimum phase wavelets and a delay (Claerbout, 1998a).

With this algorithm, the 1-D convolution filter of length $2N_x+1$ can be factored into a pair of (minimum-phase) causal and (maximum-phase) anti-causal filters, each of length N_x+1. Fortunately, filter coefficients drop away rapidly from either end, and in practice, small-valued coefficients can be safely discarded.

By reconsituting the matrices representing convolution with these filters, I obtain an approximate LU decomposition of the original matrix. The lower and upper-triangular factors can then be inverted efficiently by recursive back-substitution.

While we have only described the factorization for $v(z)$ velocity models, the method can also be extended to handle lateral variations in velocity. For every value of ω/v and c/v, we precompute the factors of the 1-D helical filters, a_1 and a_2. Filter coefficients are stored in a look-up table. We then extrapolate the wavefield by non-stationary convolution, followed by non-stationary polynomial division. The convolution is with the spatially variable filter pair corresponding to a_2. The polynomial division is with the filter pair corresponding to a_1. The non-stationary polynomial division is exactly analogous to time-varying deconvolution, since the helical boundary conditions have converted the two-dimensional system to one-dimension.

Examples

Figure 2 compares time-slices through impulses response of FFD migration (with $c/v = 0.8$) for the splitting approximation, (a), and the helical factorization, (b). Figure 3 shows extracts from a three-dimensional FFD depth migration of a zero-offset subset from the SEG/EAGE salt dome dataset.

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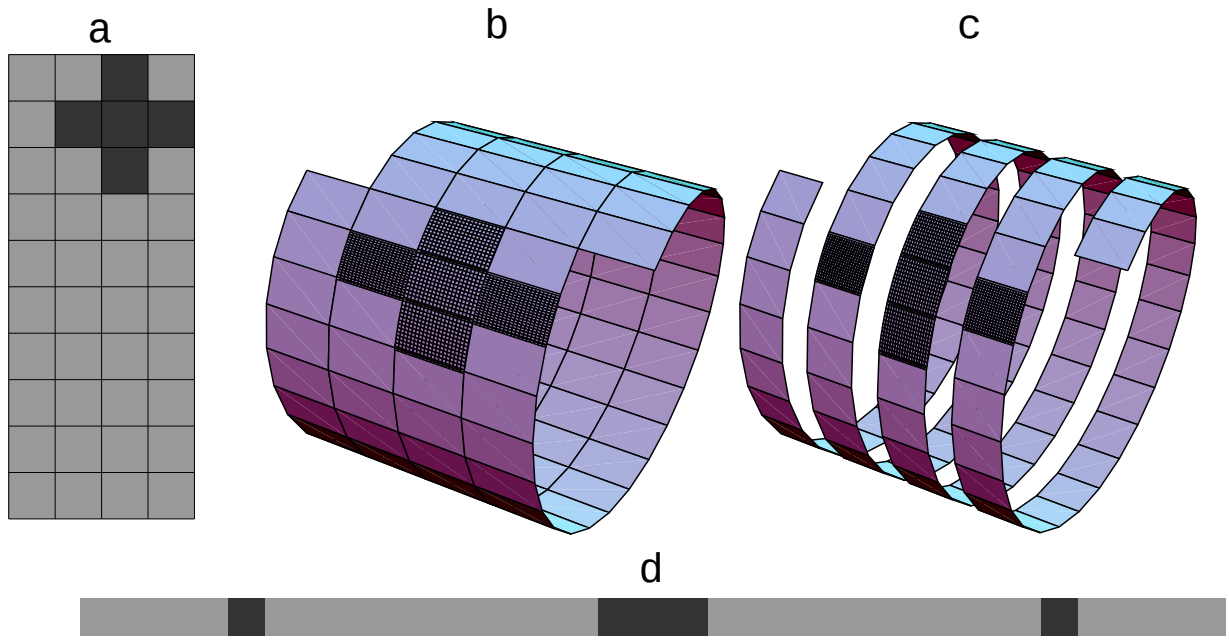

Figure 1: Illustration of helical boundary conditions mapping a two-dimensional function (a) onto a helix (b), and then unwrapping the helix (c) into an equivalent one-dimensional function (d). (Figure by Sergey Fomel).

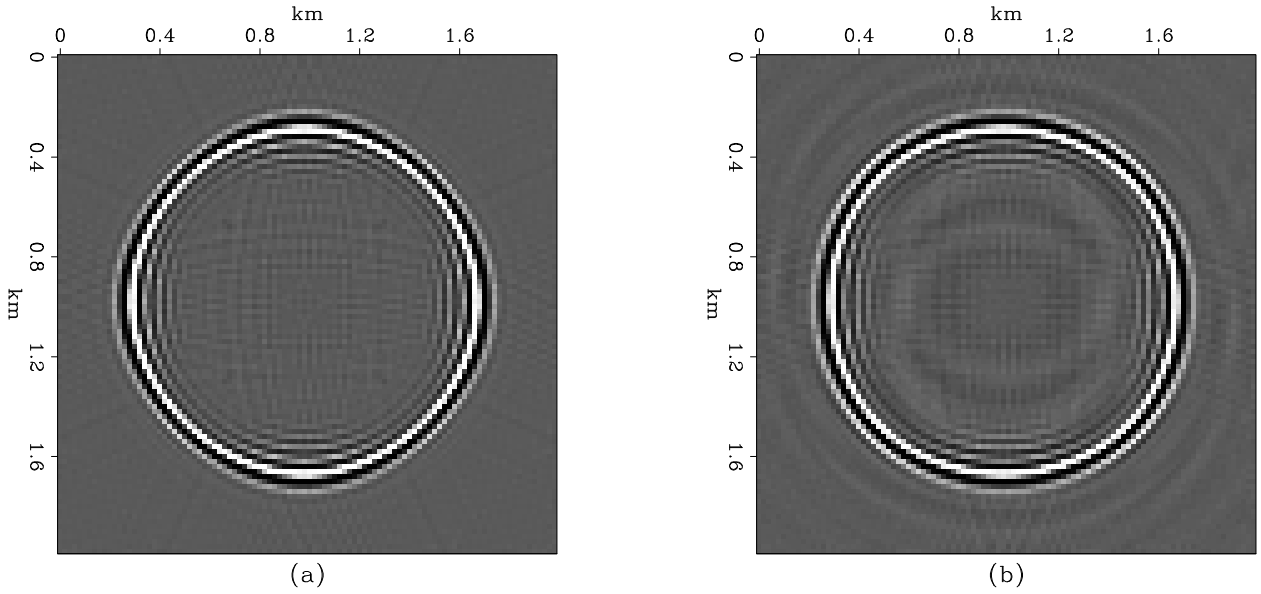

Figure 2: Depth-slices of centered impulse response corresponding to a dip of 45° for $c/v = 0.8$. Panel (a) shows the result of employing an $x - y$ splitting approximation, and panel (b) shows the result of the helical factorization. Note the azimuthally isotropic nature of panel (b).

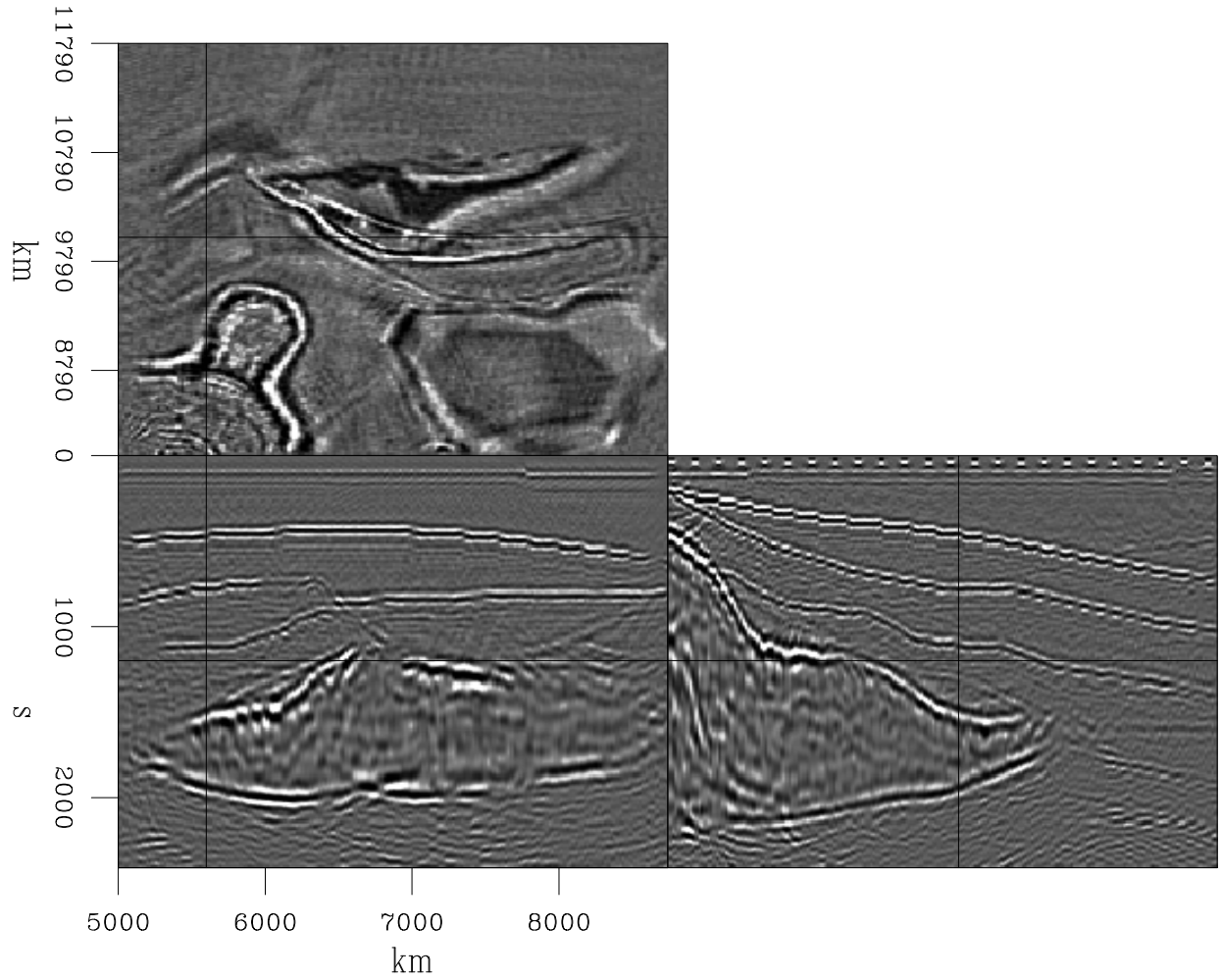

Figure 3: Migration of a three-dimensional zero-offset subset from the SEG/EAGE salt dome dataset by Fourier finite-differences with helical boundary conditions.