Velocity estimation with alternating gradiometry and wavefield reconstruction inversion

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ABSTRACT

We implement a wave equation based time-domain inversion that recovers an earth model from sparse seismic data and avoids pitfalls of other popular wave equation based solutions. This method iteratively solves for an optimal wavefield through Wavefield Reconstruction Inversion (WRI) that is used to update the earth model using gradiometry. We show examples of both WRI and gradiometry that illustrate their feasibility as individual inverse problems. Finally, we combine the methods into a joint solution and demonstrate that the optimal wavefield retrieved by WRI can be used as the input of gradiometry for updating the earth model in the direction of the true earth solution by gradiometry.

INTRODUCTION

Estimating elastic earth parameters is a topic of interest in the geophysical community for a variety of reasons, including imaging of deep geological layers, ground shaking earthquake hazard predictions, groundwater monitoring, and mineral exploration. The field of exploration seismology employs many inversion techniques that aim to estimate these earth parameters. These methods typically suffer from high computational costs, nonlinearities, and sensitivity to initial models.

A novel inversion scheme that alternately minimizes a data misfit term and a wave equation misfit term was introduced by Van Leeuwen and Herrmann (2013), De Ridder and Maddison (2017), and De Ridder et al. (2017). The scheme first reconstructs an estimate of the wavefield that fits (in a least-squares sense) both a wave equation (given an estimate of the medium parameters) as well as recorded data. This wavefield is then used to update the earth model parameters with a wave equation inversion employing gradiometry to measure wavefield gradients (Curtis and Robertsson, 2002; Langston, 2007a,b; De Ridder and Biondi, 2015; De Ridder and Curtis, 2017). When the estimate of earth parameters is held constant, the reconstruction of the wavefield becomes a linear problem with respect to the wavefield. Likewise, when the wavefield is held constant the gradiometry inversion becomes linear with respect to the earth model parameters. In this manner, the inversion scheme iteratively solves two linear inverse problems ultimately finding an optimal earth model and wavefield that match recorded data and obey some wave equation.

This technique is attractive because neither WRI nor gradiometry require forward or adjoint wave propagation. This property frees the entire method from the stability issues assosciated with wave propagation, allowing much coarser discretization in time compared to other wave equation based velocity estimation techniques such as Full Waveform Inversion (FWI) (Tarantola, 1984). The linearity of WRI and gradiometry also enable the solution to be found over multiple shots at once. Beyond computational advantages, the hope is that the competing data misfit and the wave equation misfit terms allow the inversion to avoid local minumum that plague other techniques.

The past implementations of this alternating inversion scheme focused on frequency domain formulations where the discretized Helmholtz equation is factorized to obtain a direct inverse solution for wavefields. This factorization will become unfeasible due to computational cost when applied to three dimensional problems. Here we solve the alternating inverse problems in the time domain using a linear conjugate gradient algorithm. We show inversion results with a synthetic experiment that assumes a constant density, acoustic, and isotropic earth model.

GRADIOMETRY

Here we wish to solve a wave equation inversion for earth model parameters using a given wavefield. We employ wavefield gradiometry to measure the spatial and temporal gradients of the wavefield in order to solve for some earth parameters (e.g. p-wave velocity, s-wave velocity, density, anisotropy). Here, we work with a two-dimensional acoustic isotropic wave equation to solve for the background slowness model given an observed pressure wavefield. The example illustrates that we can recover Gaussian anomalies given a known pressure wavefield.

Theory

Begin with the two-dimensional acoustic isotropic wave equation in the time domain:

$$\mathrm{s}(\mathbf{x})^2 \frac{\partial^2 \mathrm{p}(\mathbf{x},t)}{\partial t^2} - \bigtriangledown^2 \mathrm{p}(\mathbf{x},t) = f(\mathbf{x},t), \tag{1}$$

where $\mathbf{x}$ denotes the vector representing the two dimensional space-domain, t denotes time, $\mathrm{s}(\mathbf{x})$ denotes the slowness model, $\mathrm{p}(\mathbf{x},t)$ denotes the pressure wavefield, and $f(\mathbf{x},t)$ denotes the source function. To simplify the problem furthur, we assume the source is outside of the model domain, yielding:

$$\mathrm{s}(\mathbf{x})^2 \frac{\partial^2 \mathrm{p}(\mathbf{x},t)}{\partial t^2} = \bigtriangledown^2 \mathrm{p}(\mathbf{x},t). \tag{2}$$

This can be written as a simple linear system:

$$\mathbf{Am} = \mathbf{d}, \tag{3}$$

where:

$$\mathbf{A} = \frac{\partial^2 \mathrm{p}(\mathbf{x},t)}{\partial t^2}, \tag{4}$$

$$\mathbf{m} = \mathrm{s}(\mathbf{x})^2, \tag{5}$$

$$\mathbf{d} = \bigtriangledown^2 \mathrm{p}(\mathbf{x},t). \tag{6}$$

Note that the operator **A** relates the spatial and temporal derivatives of the wavefield linearly with the slowness squared. Given a background wavefield, an optimal slowness squared can be found by solving the minimization problem:

$$\arg\min_{\mathbf{m}} \Phi(\mathbf{m}) = \frac{1}{2}||\mathbf{A}\mathbf{m} - \mathbf{d}||_2^2. \tag{7}$$

Example

We solve the gradiometry problem, expressed as a minimization problem in equation 7, and obtain a slowness model given a known wavefield. The wavefield is discretized and the temporal and spatial derivative are approximated by finite difference operators of 2nd- and 10th-order of accuracy, respectively. Given a wavfield, we construct the **A** matrix and **d**. The minimization problem is solved with a linear conjugate-gradient algorithm (Aster et al., 2005).

The true slowness squared model is shown on the left in Figure 1. The model is 2.0x2.0 km and has 10 m grid spacing, it is composed of a background slowness squared of 1 s^2/km^2 and contains four Gaussian anomalies. Two shots were propagated through this model in which their positions were outside of the model domain (in order to obey the assumption in equation 2) using a wavelet containing 2 Hz to 15 Hz. The right of Figure 1 illustrates a snapshot of the wavefield at 2.0 s. The wavefield is sampled every 0.01 seconds to avoid temporal aliasing.

For this example, the wavefield, $\mathrm{p}(\mathbf{x},t)$, sampled with 10 m spacing and is known throughout the domain. We start the conjugate-gradient scheme with a constant slowness squared starting model, shown on the left in Figure 2, and performed a total of 500 conjugate gradient iterations. The model after the first conjugate gradient update is shown on the left in Figure 2. The final model is shown on the right of Figure 2 and the normalized objective function values are shown in Figure 3 with log scale.

Gradiometry recovers the true slowness squared model almost perfectly. This example illustrates that when the wavefield is known in the entire domain, we can recover the true Earth properties using wavefield gradiometry.

WAVEFIELD RECONSTRUCTION

Wavefield Reconstruction Inversion (WRI) aims to reconstruct an optimal wavefield given: a set of observations of the unknown true wavefield, a background earth model,

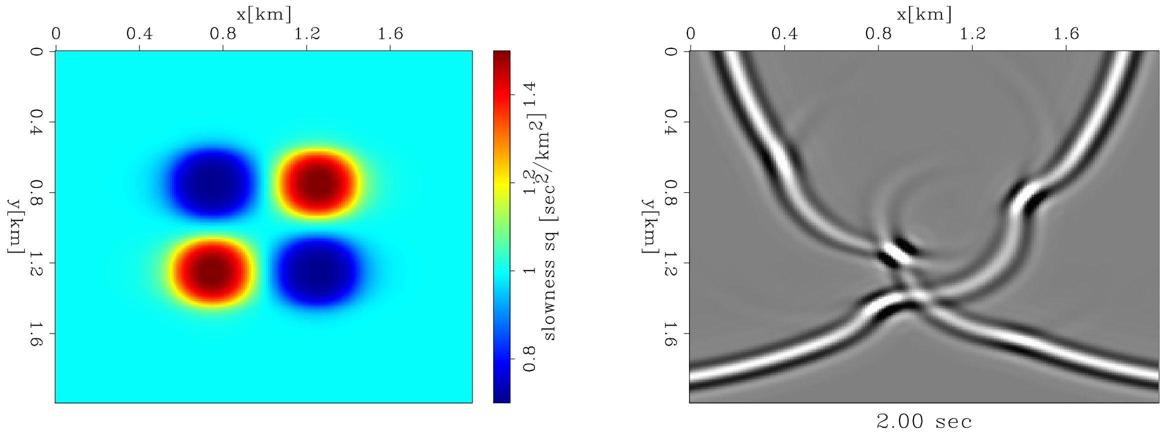

Figure 1: Slowness squared model with Gaussian anomalies (left) and snapshot of two shots propagating through the model at 2.00 seconds (right). [**CR**]

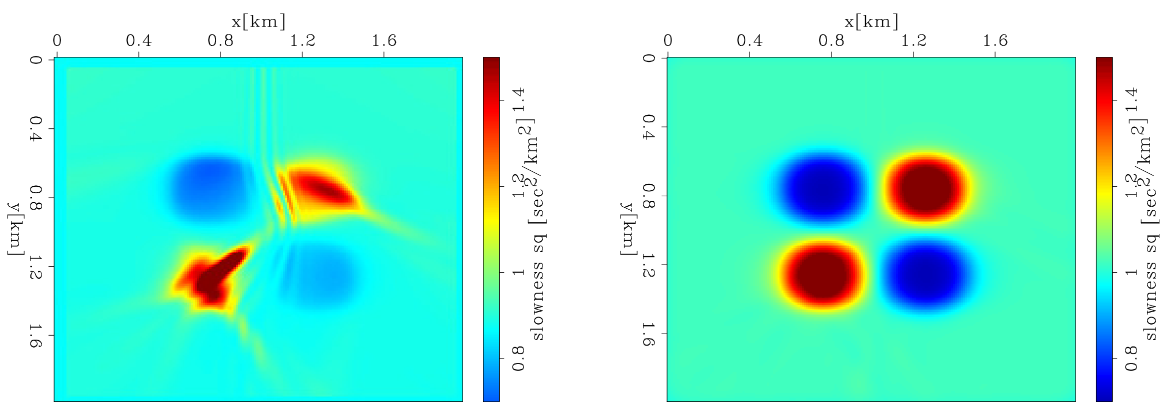

Figure 2: Slowness squared model after the first linear conjugate gradient iteration (left) and after 500 conjugate gradient iterations (right). [**CR**]

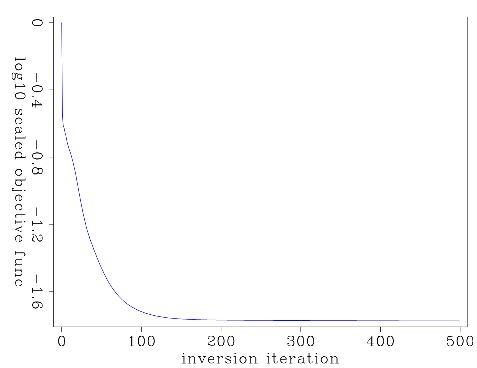

Figure 3: Normalized objective function values over 500 gradiometry conjugate gradient iterations plotted in log scale. [**CR**]

source functions, and boundary conditions. This wavefield is reconstructed inside a domain, fully sampled for all discrete grid points in time and space. In general, the medium parameters of the earth are not known. Given an estimate of the medium parameters, WRI finds an optimal wavefield that matches the observed wavefield while simultaneously obeying the wave equation in a least-squares sense.

Here, we again work with a two-dimensional acoustic isotropic wave equation and show the feasibility of reconstructing the wavefield from a set of observations and background slowness model. The example illustrates that we can recover a wavefield that traveled through a slowness model with four Gaussian anomalies.

Theory

Similar to gradiometry, we begin with the two-dimensional acoustic isotropic wave equation, assume the source function is zero, and rewrite the equation as an operator acting on the wavefield:

$$\mathrm{s}(\mathbf{x})^2 \frac{\partial^2 \mathrm{p}(\mathbf{x},t)}{\partial t^2} - \bigtriangledown^2 \mathrm{p}(\mathbf{x},t) = f(\mathbf{x},t), \tag{8}$$

$$\mathrm{s}(\mathbf{x})^2 \frac{\partial^2 \mathrm{p}(\mathbf{x},t)}{\partial t^2} - \bigtriangledown^2 \mathrm{p}(\mathbf{x},t) = 0, \tag{9}$$

$$\left[\mathrm{s}(\mathbf{x})^2 \frac{\partial^2}{\partial t^2} - \bigtriangledown^2\right] \mathrm{p}(\mathbf{x},t) = 0 \tag{10}$$

$$\mathbf{H}(\mathbf{s})\mathbf{p} = 0, \tag{11}$$

Nore that this formulation of the wave equation differs from that of gradiometry. In fact, the operator is now linear with respect to the wavefield, $\mathbf{p}$, and nonlinear with respect to the earth slowness model, $\mathbf{s}$. If equation 11 is satisfied, the wavefield obeys the wave equation, given an earth model, in the entire domain.

In WRI we want to find a wavefield that obeys the wave equation while simultaneously matching an observed wavefield, $\mathbf{d}$. In realistic applications, $\mathbf{d}$ will have sampling that more sparse than $\mathbf{p}$. For example, $\mathbf{d}$ may be recorded on some sparse grid or from receiver lines. We use the operator $\mathbf{K}$ that transforms $\mathbf{p}$ to the sampling of $\mathbf{d}$.

Given a background slowness model, $\mathbf{s}$, and a set of wavefield observations, $\mathbf{d}$, a wavefield can be found in least-squares sense, by minimizing:

$$\arg\min_{\mathbf{p}} \Phi_\epsilon(\mathbf{p}) = \frac{1}{2}||\mathbf{K}\mathbf{p} - \mathbf{d}||_2^2 + \frac{\epsilon^2}{2}||\mathbf{H}(\mathbf{s})\mathbf{p}||_2^2, \tag{12}$$

Example

We solve the wavefield reconstruction formulation, expressed as a minimization problem in equation 12, to solve for an optimal wavefield from a sparse observed wavefield and a known slowness squared model. This WRI example uses the same setup as the gradiometry example described previously. We also assume to know the true slowness model, **s**, and observed the true wavefield every 50 meters. Figure 4 shows snapshots of the observed wavefield, **d**, at various time steps. The observations at 50 meters were chosen to ensure spatial aliasing with the slowness model and wavelet used in the example.

We start the conjugate gradient algorithm from a zero estimate of the wavefield. Figure 5 shows snapshots of the reconstructed wavefield, **p**, after 2000 conjugate gradient iterations. WRI recovers an optimal wavefield that matches observations of a true wavefield and obeys the wave equation in our domain of interest. The optimal wavefield minimizes both terms in a least-squares sense, and contains minimal artifacts when compared to the true wavefield seen in Figure 6. The normalized objective function values for all iterations are shown in Figure 7 with log scale. This wavefield has a spatial sampling sufficiently dense to be affective as an input of gradiometry.

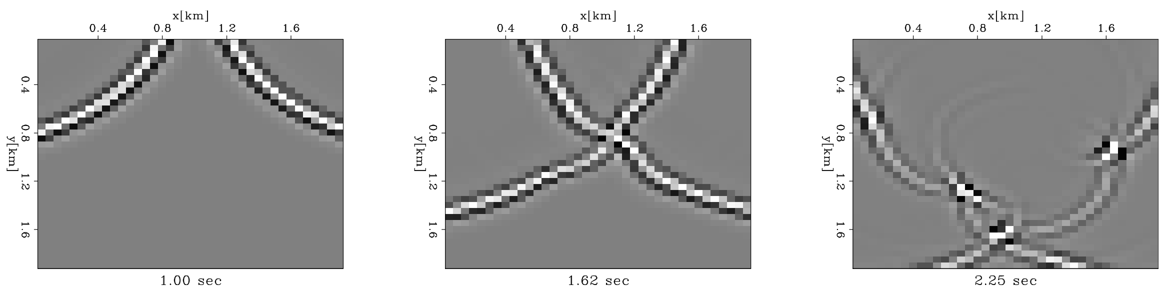

Figure 4: Snapshots of the observed wavefield at 1.00 s, 1.62 s, and 2.25 s. The spatial sampling is 50 m to emulate a realistic geophone spacing from a 2D seismic survey. [**CR**]

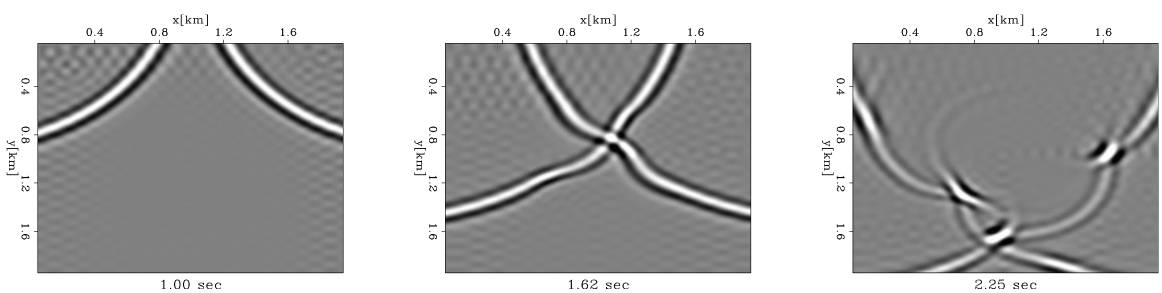

Figure 5: Snapshots of the optimal wavefield after 2000 conjugate-gradient iterations of WRI at 1.00 s, 1.62 s, and 2.25 s. The spatial sampling is 10 m which is high enough to use in a gradiometry inversion. [**CR**]

ALTERNATING INVERSION SCHEME

By alternating between WRI and gradiometry, each solved by conjugate-gradient algorithms, an earth model can be recovered from a set of observations of the wavefield

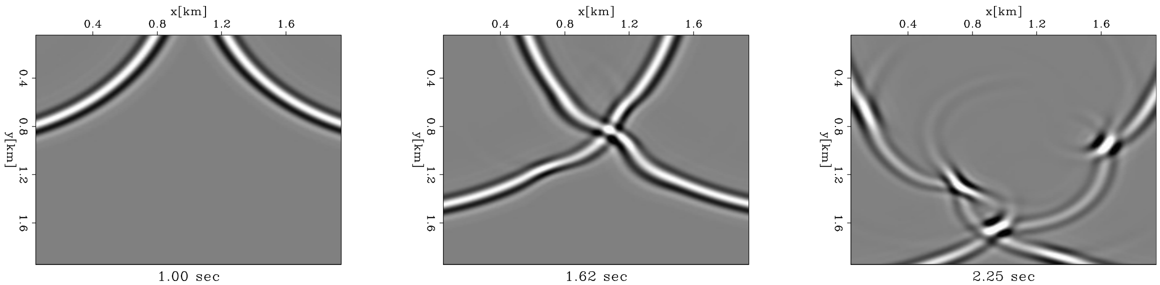

Figure 6: Snapshots of the true wavefield at 1.00 s, 1.62 s, and 2.25 s. [**CR**]

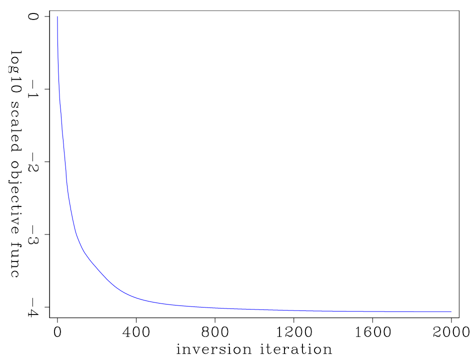

Figure 7: Normalized objective function values over 2000 WRI conjugate gradient iterations plotted in log scale. [**CR**]

using an estimate of the medium parameters as starting model. We start the scheme with WRI to find an optimal wavefield over the entire model domain, from the observations and the starting model. Using this optimal wavefield, we perform wavefield gradiometry to find a new earth model. With the new earth model, we again solve WRI to find a new optimal wavefield, and proceed iteratively. This process is formalized in Algorithm 1 and is repeated for n iterations to find a final estimate of the reconstructed wavefield in the entire domain and of the earth model.

Algorithm 1 Alternating Gradiometry and Wavefield Reconstruction

given some observed wavefield, d
given a starting earth model, s_0
$i = 0$
while $i < n$ **do** ▷ for n iterations
    $p_i \leftarrow WRI \leftarrow d, s_i$ ▷ invert for optimal wavefield
    $s_{i+1} \leftarrow gradiometry \leftarrow p_i$ ▷ invert for new earth model
    $i = i + 1$
end while
return s_n ▷ final earth model

Example

This example shows ten iterations of alternating WRI and gradiometry (Algorithm 1 for n=10). Using the same setup as the gradiometry and WRI examples above, we begin with only the sparsely sampled observed data every 50 meters and an initial slowness squared model guess that is constant. We perform 2000 conjugate gradient iterations of WRI. The optimal wavefield solution is displayed on left side in Figure 8. This wavefield and the constant starting model are then used as the inputs to gradiometry. The right of Figure 8 shows the squared slowness model after 500 conjugate gradient iterations of gradiometry. This model represents the output of the first iteration of the alternating WR and gradiometry inversion, and would be input in the next instance of WRI. The normalized objective function values for every WRI and gradiometry iteration are displayed in Figure 9. This alternatig scheme is repeated for ten iterations resulting in a final optimal wavefield and slowness model illustrated in Figure 10.

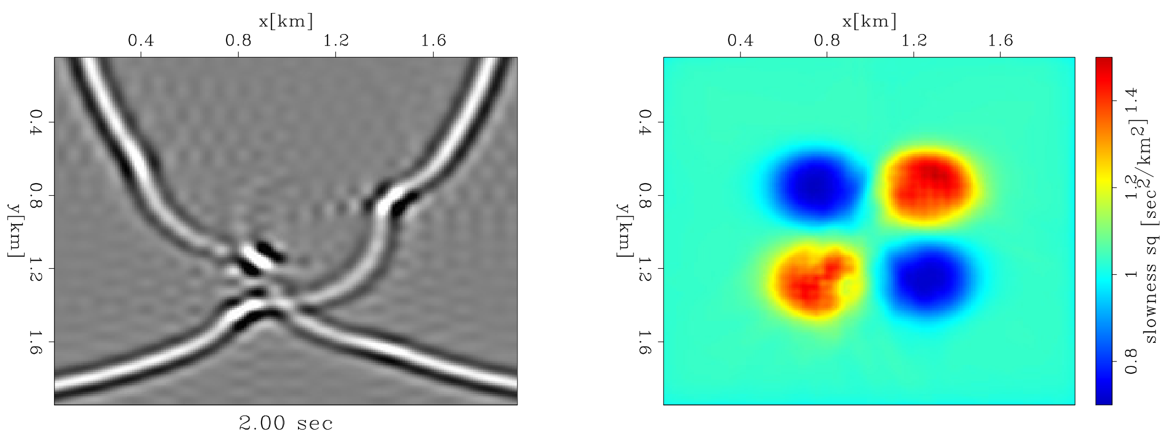

Figure 8: After one iteration of Algorithm 1 the optimal wavefield result of WRI sampled at 10 meters (left) and the final earth model when using this optimal wavefield. [**CR**]

DISCUSSION AND CONCLUSION

Ten iterations of WRI followed by gradiometry results in an earth model that is representative of the true model used to create the observed wavefield data. With sparse data and an incorrect slowness model, WRI recovers a wavefield that obeys that wave equation and the observations in a least-squares sense with residual errors. But, the recovered wavefield contains information about the background slowness that can then be used to update the earth model with gradiometry. The competing objective function terms in WRI explains the acquisition footprint seen in the obtained slowness model estimate. Performing further iterations may resolve this footprint as WRI will be able to more closely rectify the observed data and the wave equation, given a better estimate for the earth model.

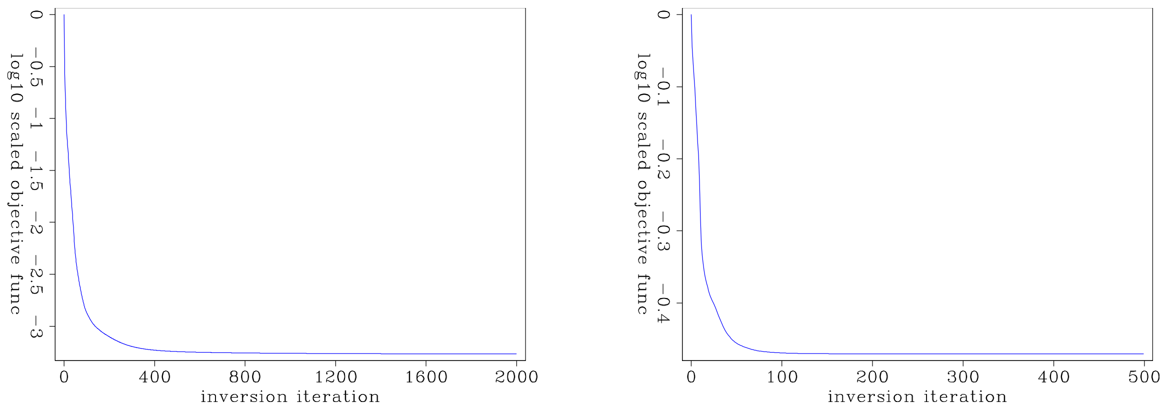

Figure 9: Normalized objective functions from the first iteration of Algorithm 1. The objective function of WRI (left) has good convergence when using an incorrect starting model. The objective function of gradiometry (right) reduces to only 35%. [**CR**]

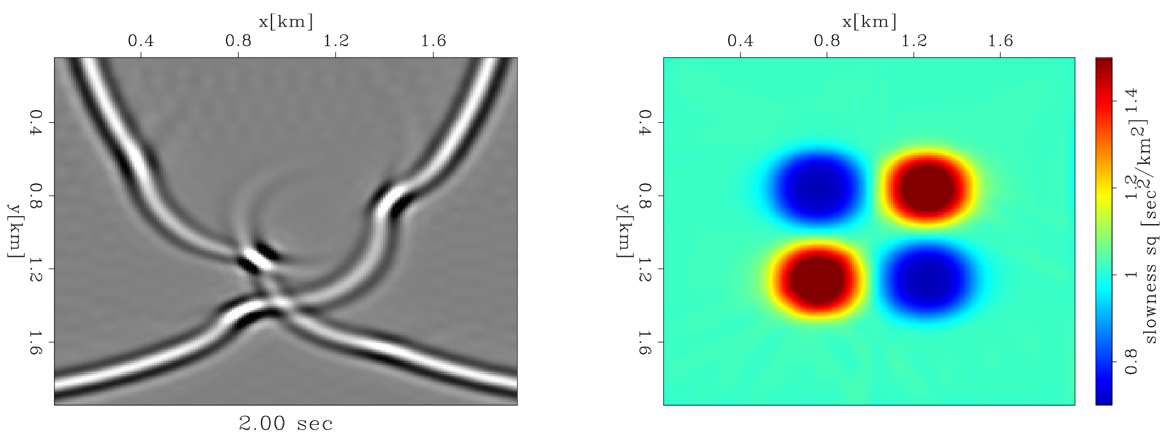

Figure 10: After ten iterations of Algorithm 1 the optimal wavefield result of WRI sampled at 10 meters (left) and the final earth model when using this optimal wavefield. Both the wavefield and slowness model are significantly closer to the truth as compared to the result of the first iteration in Figure 8 [**CR**]

This study illustrates initial results of a wave equation inversion scheme in the that does not require forward or adjoint wave propagation, but can recover background earth model perturbations using only sparse observed seismic data. The implementation in the time domain may allow its application to larger and higher dimensional problems.

ACKNOWLEDGEMENTS

I wish to thank the affiliate members of the Stanford Exploration Project for their financial support and guidance. Furthermore, this topic was brought to my attention by my academic advisor Biondo Biondi. Without his valuable insights and numerous discussions with Bob Clapp, this work would not have gotten off the ground. Finally, I thank my officemates Ettore Biondi and Guillaume Barnier for their good spirit, endless patience, and priceless guidance.

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