Surface waves in vertically heterogeneous media

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ABSTRACT

This work is aimed at understanding the theoretical basics of surface wave propagation and the way they naturally arise in the study of wave equation. The analytical derivations shed light on their key features such as velocity dispersion and existence of several modes of propagation and potentially will lead to an intuitively clear picture of this complex phenomenon. This, in turn, will make the interpretation of surface waves on the real seismic records easier and allow understanding of the methods for their analysis.

INTRODUCTION

Commmonly in the seismic exploration, surface waves are treated as noise which a lot of effort is usually put in suppressing them. However, they are an important constituent of a total recorded wavefield and carry valuable information about the subsurface. Their properties can be used to retrieve elastic parameters of the earth and velocities of near-surface body waves that are crucial for building velocity models and, therefore, for seismic imaging problems. This is why it is important to understand the physics of surface waves and study how media properties influence their propagation.

A very useful tool in studying elastic wave equation is the notion of vector potentials. In short, displacement vector field ($\mathbf{u}$) is separated into curl-free ($\nabla \times \phi = 0$) and divergence-free ($\nabla \cdot \psi = 0$) components which simplifies the wave equation by splitting it into P- and S-components and makes it feasible for further analysis:

$$\mathbf{u} = \nabla\phi + \nabla \times \psi.$$

Thus the P-wave displacement is simply $\mathbf{u}_P = \nabla\phi$ and the displacement for S-wave is $\mathbf{u}_S = \nabla \times \psi$. In the case of the plane wave propagating in the (x, z)-plane $\mathbf{u}_P = (\partial\phi/\partial x, 0, \partial\phi/\partial z), \mathbf{u}_S = (-\partial\psi/\partial z, 0, \partial\psi/\partial x)$ (Aki and Richards, 2002). After noticing this, it is easy to solve problems in terms of the scalar potentials and find corresponding displacements using these formulas.

Another important formula extensively used in studying elastic wave equation is Hooke's law that relates stress (τ_{ij}) and strain tensors ($e_{kl} = 1/2(\partial u_k/\partial x_l + \partial u_l/\partial x_k)$, with u_k being k-th component of displacement vector field) via the elastic coefficients

(c_{ijkl}). In case of isotropic medium these equations take the following form (with λ, μ being Lame parameters and δ_{ij} - Kronecker symbol):

$$\tau_{ij} = c_{ijkl} e_{kl},$$
$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

Combining all the aforementioned methods and laws, a general way of analytically solving the wave equation in 1D media consists of first, writing the solution in terms of potentials (ϕ or ψ) in the form of $A_i \exp{(i\mathbf{k}_i \cdot \mathbf{x} - i\omega t)}$ within each i-th layer, computing corresponding stress and displacements fields and finally matching the boundary conditions at the layers' interfaces (e.g. assuring strain and displacement are continuous through each boundary).

RAYLEIGH WAVES

The most distinctive features of Rayleigh waves (e.g. dispersion, existence of several modes of propagation, slow amplitude decay) are well known. They arise naturally in the solutions of spherical waves propagation (Lamb problem) when analyzing different integration paths in the complex plane (Aki and Richards, 2002). This analysis, however, is complicated and requires knowledge of complex analysis. Analysis of plane waves, nevertheless, is easier and more intuitive, which is the reason why it is the most common starting point in many analytical derivations related to wave propagation in heterogeneous media.

Theory

Imagine a plane P-wave propagating in the half-plane from below and reflecting down from the free-surface boundary (Figure 1). In terms of the angle i the slowness of the incident P-wave is

$$\mathbf{s} = \left(\frac{\sin i}{V_P}, 0, \frac{-\cos i}{V_P}\right).$$

The slownesses of the reflected P- and SV-waves are:

$$\left(\frac{\sin i}{V_P}, 0, \frac{\cos i}{V_P}\right) \text{ and } \left(\frac{\sin j}{V_S}, 0, \frac{\cos j}{V_S}\right).$$

Now we are looking for a solution to the wave equation in the exponential form in terms of potentials with total potential consisting of the following:

$$\begin{aligned} \phi &= \phi^{inc} + \phi^{refl}, \\ \phi^{inc} &= A \exp\left[i\omega\left(\frac{\sin i}{V_P}x - \frac{\cos i}{V_P}z - t\right)\right], \\ \phi^{refl} &= B \exp\left[i\omega\left(\frac{\sin i}{V_P}x + \frac{\cos i}{V_P}z - t\right)\right]. \end{aligned} \tag{1}$$

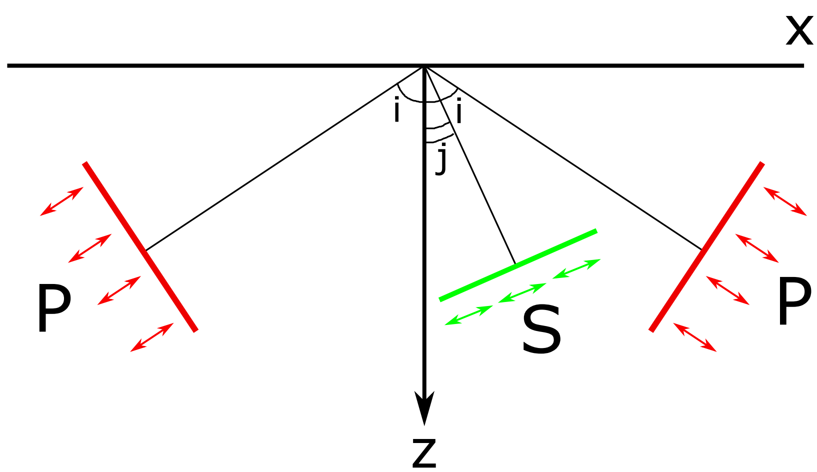


Figure 1: Reflection of a plane P-wave generating reflected P- and reflected S-wave. [**NR**]

where the amplitudes A and B can be thought of as amplitude spectra of 2D Fourier transform and are constant for a given frequency and wavenumber.

The amplitude of the reflected SV-wave is given by the following:

$$\psi = \psi^{refl} = C \exp\left[i\omega\left(\frac{\sin j}{V_S}x + \frac{\cos j}{V_S}z - t\right)\right]. \tag{2}$$

This is the starting point for analyzing plane wave reflection from a boundary and can lead to Zoeppritz equation by matching the stresses and displacements at the boundaries and finding explicit expressions for all the amplitudes A, B, C (Aki and Richards, 2002).

Inhomogeneous waves

So far we have used geometrical interpretation and reasoning to come up with the solutions in the form equation 1. Generally speaking, however, any function of the form $\exp\left[i\omega(s_x x \pm s_z z - t)\right]$ is a solution of the wave equation as long as it satisfies the dispersion relation $s_x^2 + s_z^2 = 1/c^2$, where c is a body wave velocity. This fact can be easily verified simply plugging in the solution of such form into the wave equation.

The horizontal slowness is called also ray parameter p and is well known to be constant in the vertically heterogeneous media. Therefore, we can rewrite our previous expression for the plane P-wave slowness as follows:

$$\mathbf{s} = \left(\frac{\sin i}{V_P}, 0, \pm\frac{\cos i}{V_P}\right) = (p, 0, \pm\sqrt{1/V_P^2 - p^2}).$$

Here, we can ask another question that surprisingly leads to a new type of waves. What if the horizontal component of slowness $s_x = p > 1/V_P$? This observation inevitably leads to a vertical slowness to be imaginary number, since the expression under the square root is negative (in fact, all the reflection and transmission coefficients are becoming complex as well). Hence, the exponents in the equations 1 and 2 become

$$\exp{[i\omega(px \pm \sqrt{1/V_P^2 - p^2} - t)]} = \exp{[i\omega(px - t)]} \exp{[\mp\omega\sqrt{p^2 - 1/V_P^2}z]}.$$

It is obvious now that the wave of this type exponentially decays with depth and propagates with slower horizontal velocity than that of media (because $p > 1/V_P$). These waves are called inhomogeneous (or evanescent).

As it was previously described, that the amplitudes A, B, C of incident P- and reflected P- and S-waves can be thought of as amplitude spectra of a 2D Fourier transform. In 2D, remembering the geometrical interpretation of wavenumbers

$$A(k_x, 0, k_z) = A(\sin i, 0, \cos i),$$

we can see that the steady-state displacement amplitudes (namely its x and z components) at $x = z = t = 0$ are dependent on angle of propagation i. Rewriting this in terms of ray parameter $p > 1/V_P$, we get the expression for inhomogeneous wave displacement amplitudes:

$$A(pV_P, 0, i\sqrt{p^2V_P^2 - 1}) \exp{[i\omega(px - t)]} \exp{[-\omega\sqrt{p^2 - 1/V_P^2}z]}.$$

This equation highlights an interesting property of inhomogeneous waves. If we consider just the real part of the expression above, it is easy to see that x- and z-components of displacement are oscillating with $\pi/2$-phase shift (if one is varying as cosine, the other will be varying as a sine wave). This observation essentially means that the displacement trajectory forms an ellipse rotating clockwise (prograde).

What if now the plane wave is propagating along the ray with $p > 1/V_S > 1/V_P$? This condition will lead to the P- and S-inhomogeneous waves:

$$\begin{aligned} &P(pV_P, 0, i\sqrt{p^2V_P^2 - 1}) \exp{[i\omega(px - t)]} \exp{[-\omega\sqrt{p^2 - 1/V_P^2}z]}, \\ &S(i\sqrt{p^2V_S^2 - 1}, 0, -pV_S) \exp{[i\omega(px - t)]} \exp{[-\omega\sqrt{p^2 - 1/V_S^2}z]}, \end{aligned} \tag{3}$$

where the directions of steady-state displacements of the S-wave are easily distinguishable to be perpendicular to the directions of P-wave. It can also be shown that the z-component for S-wave is larger than x-component (ellipse elongated vertically) (Alsop et al., 1974). The P-wave displacements, on the other hand, are elongated horizontally.

Rayleigh function

We have seen the general expression of the inhomogeneous wave in the half-space. However, we have to make sure that these solutions satisfy boundary conditions at a free-surface ($\tau_{zx} = \tau_{zz} = 0$). Computing the stresses from the displacements and Hooke's law, we derive to the following set of equations (Aki and Richards, 2002):

$$\begin{aligned} 2pV_PV_Si\sqrt{p^2 - 1/V_P^2}P + (1 - 2V_S^2p^2)S = 0 \quad (\tau_{zx} = 0) \\ (1 - 2V_S^2p^2)P - 2(V_S^3p/V_P)i\sqrt{p^2 - 1/V_S^2}S = 0 \quad (\tau_{zz} = 0) \end{aligned} \tag{4}$$

Solving for P or S leads us to the expression called the Rayleigh function $R(p)$ that necessarily needs to be equal to zero:

$$R(p) = \left(\frac{1}{V_S^2} - 2p^2\right)^2 - 4p^2\sqrt{p^2 - \frac{1}{V_P^2}}\sqrt{p^2 - \frac{1}{V_S^2}} = 0.$$

This equation can be rewritten in terms of Rayleigh wave velocity $V_R = V_S/\sin i$ (Brekhovskikh, 1960):

$$q = \frac{1 - 8s + 24s^2 - 16s^3}{(4s)^2(1 - s)}, \tag{5}$$

where

$$q = \left(\frac{V_S}{V_P}\right)^2, s = \left(\frac{V_S}{V_R}\right)^2.$$

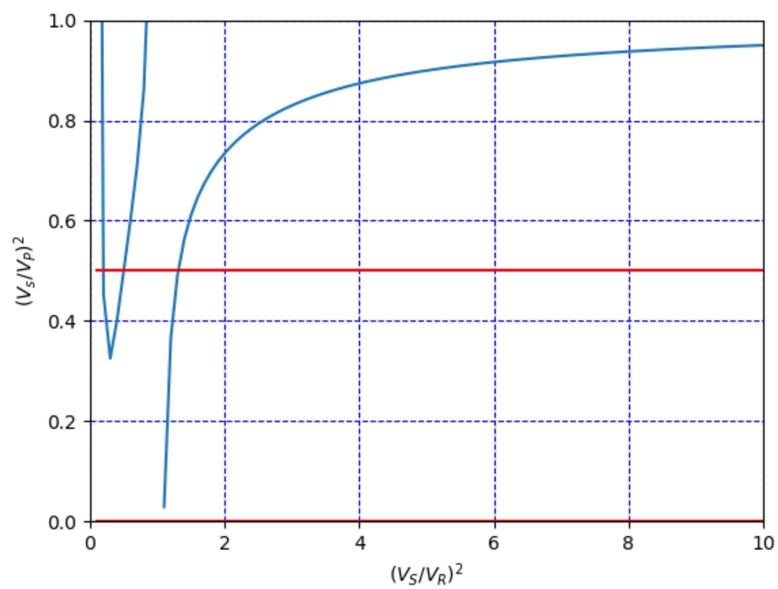


Figure 2: Graphical representation of finding roots of Rayleigh function with $V_P =$ 2000 m/s and $V_S = 1000$ m/s. The blue line shows the right hand side of equation 5. Finding roots is equivalent to finding intersection of the blue line with lines $q = const$. Red lines show the boundaries of realistic V_S/V_P ratio. [**ER**]

Thus for a given V_P/V_S ratio we can find the corresponding Rayleigh velocity as an intersection of straight line $q = const$ with the blue curve corresponding to the right hand side in equation 5 (Figure 2). For the real media we know that this ratio lies in the range $[0, 0.5]$, where the blue curve is almost straight. Thus for any real media the ratio V_S/V_R is almost constant and therefore, velocity of a Rayleigh wave is dependent mostly on the S-wave velocity (and it is slightly smaller than V_S).

Therefore, it seems reasonable to try to invert for S-wave velocity using Rayleigh wave velocities while the P-wave velocity would be hard to recover. Another important observation is that Rayleigh waves are not-dispersive in the half-space (since there is no dependency on the frequency in the Rayleigh function in equation 4).

Numerical examples

First, I replicate the motion of inhomogeneous wave by calculating the analytical expression for their evolution with time (equation 3).

If we track the evolution with time we can see that, indeed, the two elliptical clockwise motions of P- (Figure 3) and S-waves (Figure 4) combined produce an elliptical motion of Rayleigh wave rotating in the counterclockwise direction (Figure 5).

Next, let's observe the particle motion in the actual propagation. Here I have used a 2D elastic finite-difference code to generate seismograms in 3 simple models: constant velocity of V_P =2000 m/s, V_S = 1000 m/s; one layer of V_P =3000 m/s, V_S = 1500 m/s embedded in the background model 1; and two layer model with one additional layer added to the model 2 with $V_P = 2500, V_S = 1250$ m/s.

The first thing to notice here is that, indeed, in case of half-space, the Rayleigh wave is not dispersive (Figure 7a). However, once we start adding layers, the surface wave get dispersive (so called geometrical dispersion). Therefore, it is not necessarily anelastic effects that cause dispersion but simple layering.

The simulation outputs the particle velocity field (v_x, v_z). To get the displacement I integrated them in time. Then I produced a video of displacement evolution with time (on Figure 8 the snapshots of these videos are shown).

The observed motions of the Rayleigh waves do indeed follow the elliptical trajectories rotating in the opposite direction of wave propagation and decaying with depth. In case of several layers, we can see existence of several such ellipses that may correspond to waves coming from different layers and having different velocities (different modes of propagation).

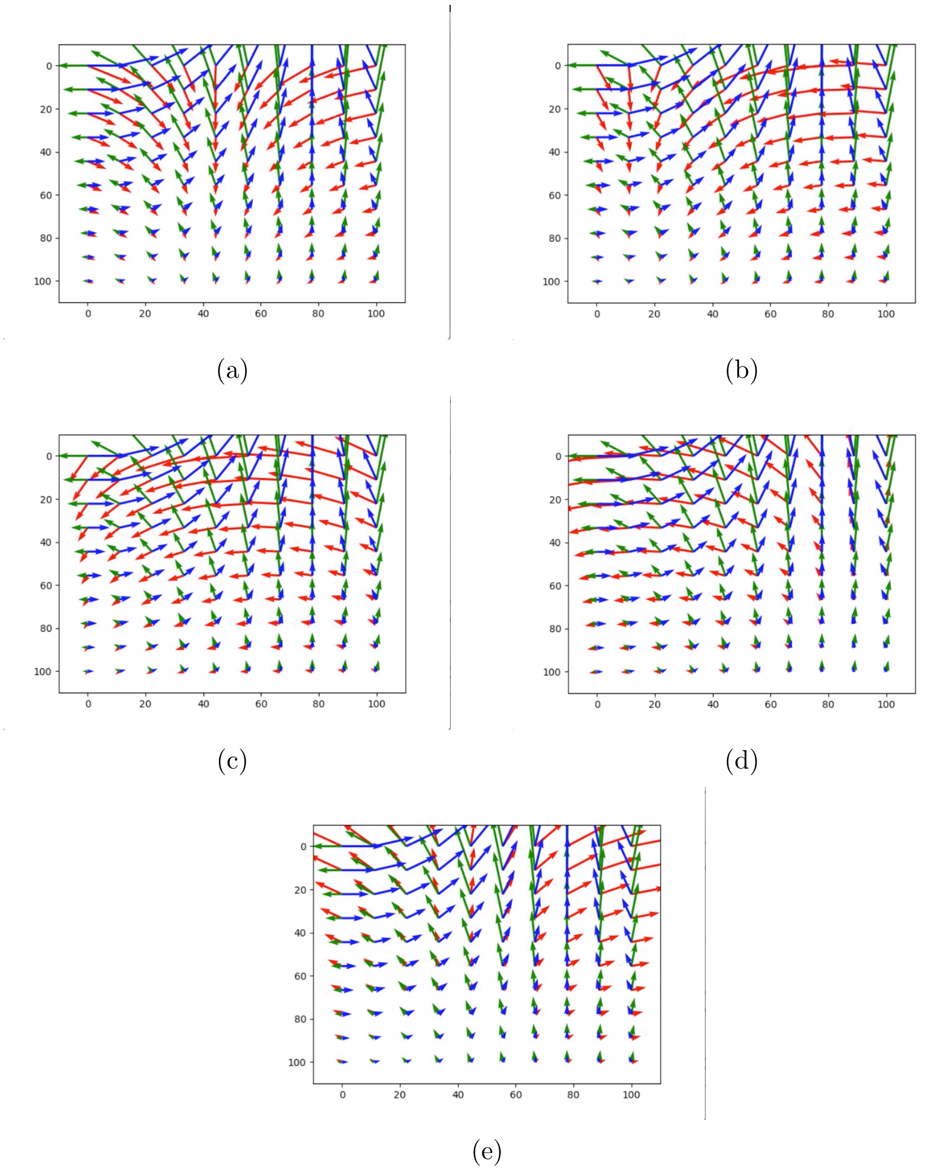
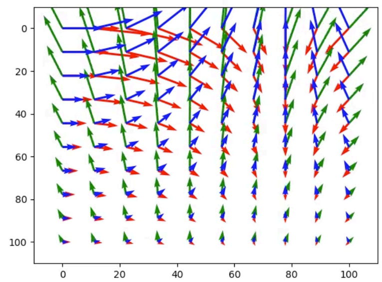
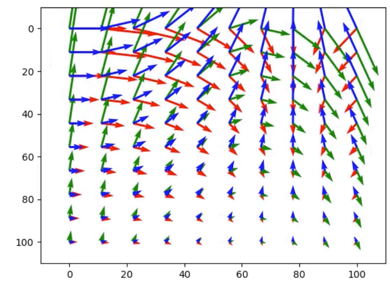


Figure 3: Inhomogeneous P-wave motion (shown by red arrows). Figures a-b-c-d-e correspond to consecutive snapshots in time of particle movement. The green and blue arrows are fixed. The vertical axis corresponds to depth, the horizontal – to spatial distance. [**ER**]



(a)



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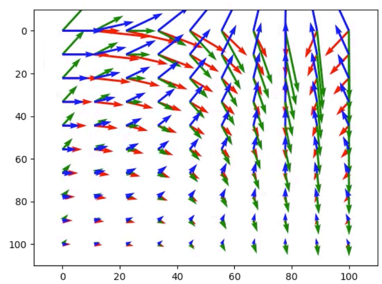

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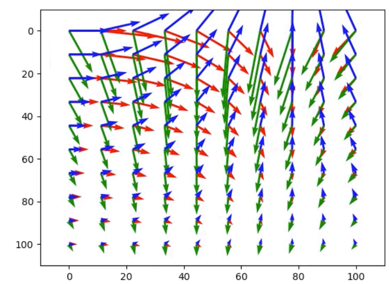

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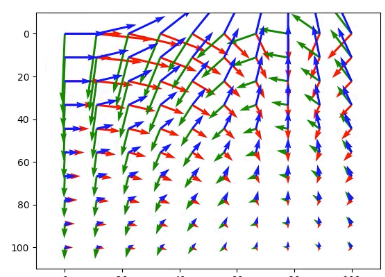
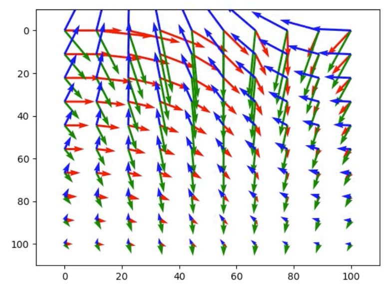
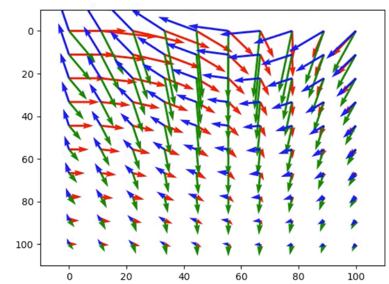


(e)

Figure 4: Inhomogeneous S-wave motion (shown by green arrows). Figures a-b-c-d-e correspond to consecutive snapshots in time of particle movement. The red and blue arrows are fixed. The vertical axis corresponds to depth, the horizontal – to spatial distance. [**ER**]



(a)



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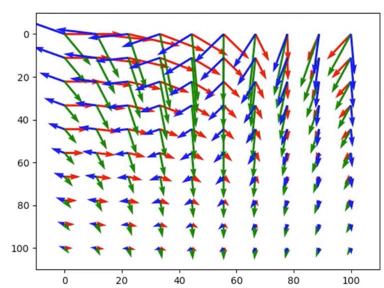

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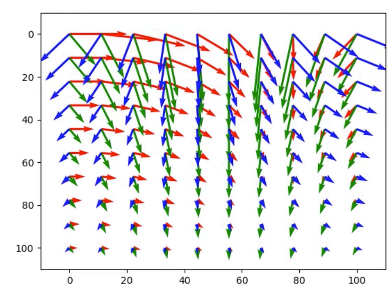

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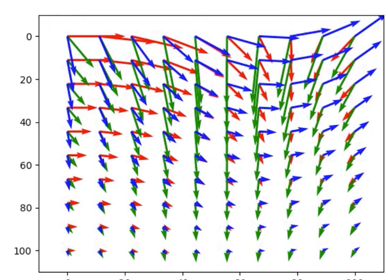
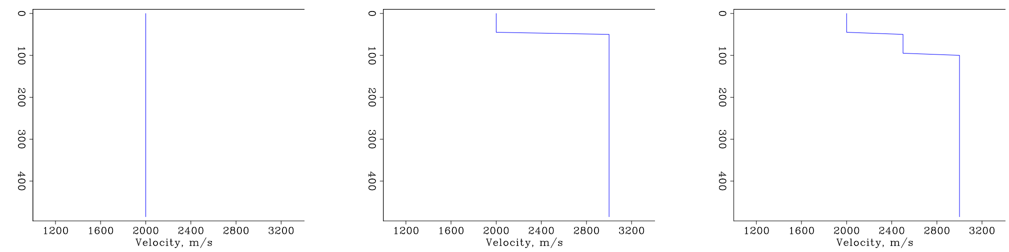


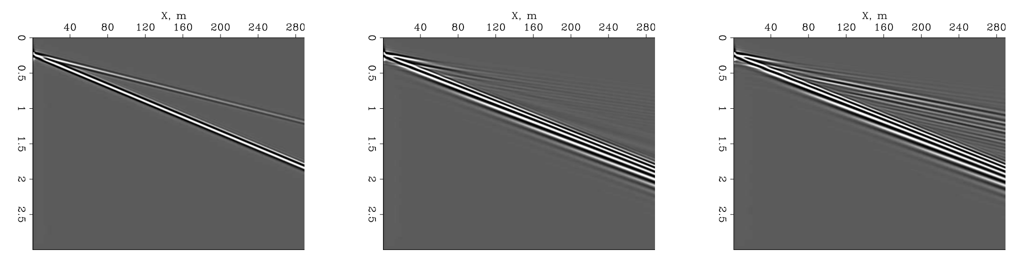
(e)

Figure 5: Rayleigh wave motion (shown by blue arrows). Figures a-b-c-d-e correspond to consecutive snapshots in time of particle movement. The green and red arrows are fixed. The vertical axis corresponds to depth, the horizontal – to spatial distance. [**ER**]



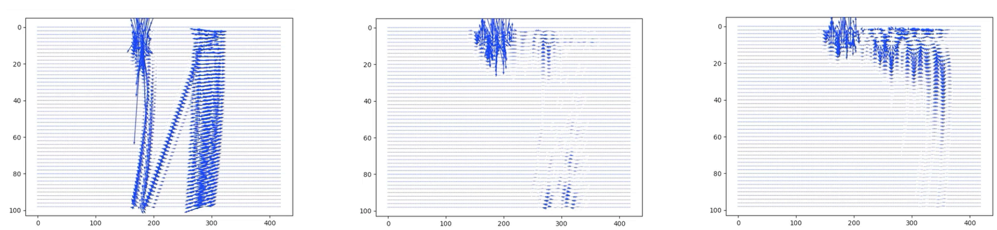
(a) (b) (c)

Figure 6: Models of P-wave velocity used in the simulation: a – constant velocity, b – one-layer model, c – two-layer model. [**ER**]



(a) (b) (c)

Figure 7: Data simulated in different models: a – constant velocity, b – one-layer model, c – two-layer model. [**ER**]



(a) (b) (c)

Figure 8: Snapshots of displacement fields evolution (vertical axis is depth, horizontal–offset from the source): a – constant velocity, b – one-layer model, c – two-layer model. [**CR**]

LOVE WAVES

So far, we have investigated how Rayleigh waves originate from a plane wave in (x, z) plane (P and SV). In this section we turn attention to the SH-wave propagating in the homogeneous layer 10 km thick over a homogeneous halfspace wtih $V_{S1} = 3000$ m/s, $V_{S2} = 5000$ m/s, $\rho_1 = 2.8\ g/cm^3$ and $\rho_2 = 3.2\ g/cm^3$ (Figure 9). This setting gives rise to the new type of surface waves with displacements oriented out-of-plane perpendicular to the direction of propagation.

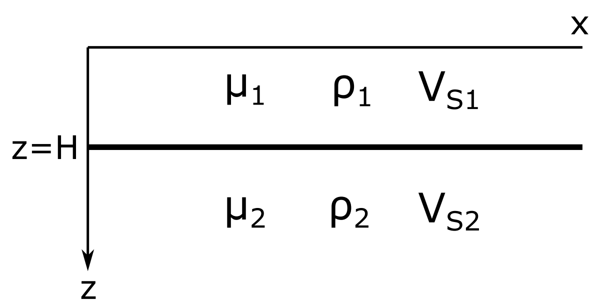


Figure 9: A homogeneous layer over homogeneous halfspace.

Theory

Again the strategy is to write the solution to the wave equation in the first layer and second layer and match the boundary conditions.

Because the SH-wave is perfectly decoupled from P-SV, it obeys a following wave equations:

$$\frac{\partial^2 \mathbf{u}_{SH}}{\partial t^2} = \frac{\mu_1}{\rho_1}\left(\frac{\partial^2 \mathbf{u}_{SH}}{\partial x^2} + \frac{\partial^2 \mathbf{u}_{SH}}{\partial z^2}\right), \quad 0 < z < H$$

$$\frac{\partial^2 \mathbf{u}_{SH}}{\partial t^2} = \frac{\mu_2}{\rho_2}\left(\frac{\partial^2 \mathbf{u}_{SH}}{\partial x^2} + \frac{\partial^2 \mathbf{u}_{SH}}{\partial z^2}\right), \quad z > H$$

We are seeking the solutions of the form:

$$\mathbf{u}(x, y, z, t) = S(z) \exp\left[i(kx - \omega t)\right].$$

It is straightforward to solve for the displacements and obtain the solution in the following form:

$$\mathbf{u}_{SH} = \left[S_1^d e^{-\nu_1 z} + S_1^u e^{\nu_1 z}\right] \exp i(kx - \omega t), \quad 0 \leq z \leq H$$

$$\mathbf{u}_{SH} = \left[S_2^d e^{-\nu_2 z} + S_2^u e^{\nu_2 z}\right] \exp i(kx - \omega t), \quad H \leq z$$

where

$$\nu_i = \sqrt{k^2 - \omega^2 / V_{S_i}^2} \quad \text{with k being a wavenumber}$$

$S_1^d, S_1^u, S_2^d, S_2^u$ are amplitudes of the downgoing and upgoing waves, that need to be found by matching boundary conditions. $S_2^u = 0$ due to the radiation condition and energy decay as $z \to \infty$. Because of the free-surface boundary conditions $S_1^d = S_1^u$. Therefore, there are just two unknowns to be found. Again requiring continuity of the displacement and stresses across the boundary $z = H$:

$$2S_1^d \cos{(i\nu_1 H)} = S_2^d e^{-\nu_2 H},$$
$$2i\mu_1\nu_1 S_1^d \sin i\nu_1 H = \mu_2\nu_2 S_2^d e^{-\nu_2 H},$$

or

$$\frac{S_2^d}{S_2^d} = \frac{2\cos{(i\nu_1 H)}}{e^{-\nu_2 H}} = \frac{2i\mu_1\nu_1 \sin i\nu_1 H}{\mu_2\nu_2 e^{-\nu_2 H}}.$$

This equality can be recast in the form used for further analysis and finding phase velocities $c = \omega/k$:

$$\tan \omega H \sqrt{\frac{1}{V_{S_1}^2} - \frac{1}{c^2}} = \frac{\mu_2}{\mu_1} \frac{\sqrt{\frac{1}{c^2} - \frac{1}{V_{S_2}^2}}}{\sqrt{\frac{1}{V_{S_1}^2} - \frac{1}{c^2}}}. \quad (6)$$

It is easy to plot both curves on the left- and right-hand side of the Equation 6 to find the solutions.

We can see that in this specific case (this particular frequency, thickness and elastic properties) there are 3 solutions (Figure 10), which means essentially that there are three modes existing for this type of waves.

Mimicking Love waves

To understand how dispersion manifests itself in the wave propagation we can try to mimic Love-wave propagation in the one-layer model.

First, using the values of $\rho_1, \rho_2, V_{S_1}, V_{S_2}$ we calculate the phase velocities for a range of frequencies (0 to 0.4 Hz). This can be done by finding the intersection of the curves (as in Figure 10) and using only the lowest values of phase velocity (corresponding to fundamental mode). The resulting phase velocity goes from 5 km/s (velocity in the lower halfspace) at low frequencies and approaches 3 km/s (velocity in the layer) for high frequencies (Figure 11).

From phase velocity it is easy to obtain group velocity by differentiating dw/dk. An interesting observation to make here looking at the group velocity is that there is a range of frequencies (starting from $\approx 0.07Hz$) for which two distinct frequencies have the same group velocity.

We are looking for solution (waveform) at the distance of 500 km from the source by simply summing harmonic waves with corresponding frequencies and wavenumbers:

$$s(x = 500km, t) = \int_{-\infty}^{\infty} \exp{[i(k(\omega)x - \omega t)]} d\omega.$$

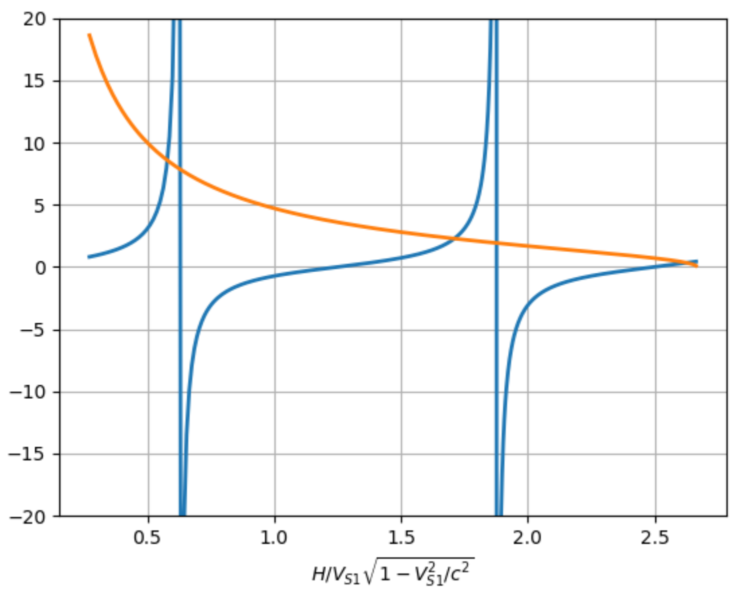


Figure 10: Graphic solution for the dispersion of monochromatic Love wave of 0.4 Hz in the previously described Earth model (left-hand side of equation 6 shown in blue, right-hand side shown in red). The intersection of these curves correspond to the solutions. The discontinuities of the blue curve are the ones of tangent and do not represent actual values. [**ER**]

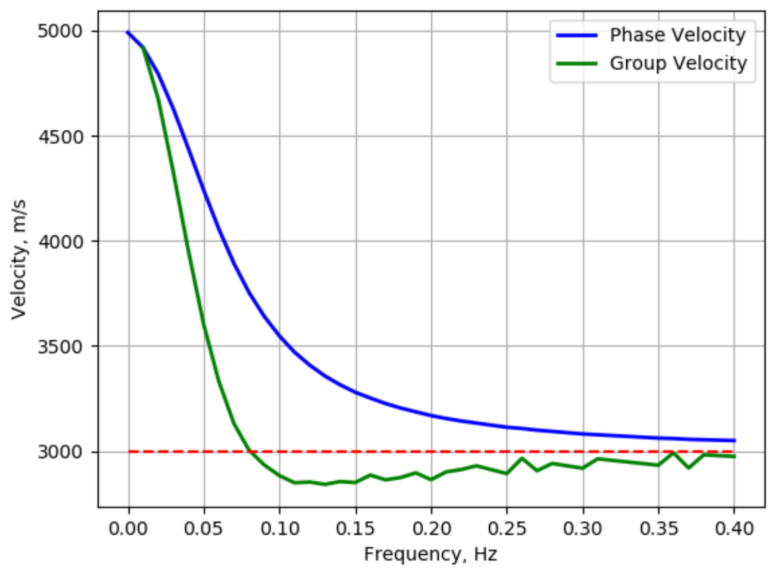


Figure 11: Phase and group velocities of Love wave. [**ER**]

If we think about this problem in terms of mathematics, this is a problem of evaluating oscillatory integral. A common way of analyzing such integrals is finding the points of stationary phase (where derivative of phase function w.r.t. integration variable goes to zero). These very points will contribute to the final result of integration.

That's why it is interesting to see the phase function $(kx - \omega t)$ behavior for a given distance and distinct time moments going from 80 s to 200 s with increment of 20 s. It can be seen in the Figure 12 that the phase at times 80 and 200 s does not have stationary points, which is why we can expect no events at those times (oscillatory integral sum to zero). Starting from 100 s we begin seeing stationary points going to higher frequencies as we increase time.

Now let us zoom into the times starting from 166 s (t=500km/3s corresponding to the direct wave arrival). An oddity here is that starting from this time and going into consecutive times we can see two stationary points at every phase curve: one going from low to high frequencies as we increase time, and the second going the opposite direction – from high to low frequency.

Finally sum all the harmonic waves from 0 Hz to 0.4 Hz frequency to get the final "mimicked" trace (Figure 14). Looking at this result we see the previous predictions in action. Starting from 100 s until 166 s there is only one frequency where phase is stationary, thus we observe the increase of frequency as we advance in time. At time t = 166 s we see high frequencies arriving that correspond to those stationary

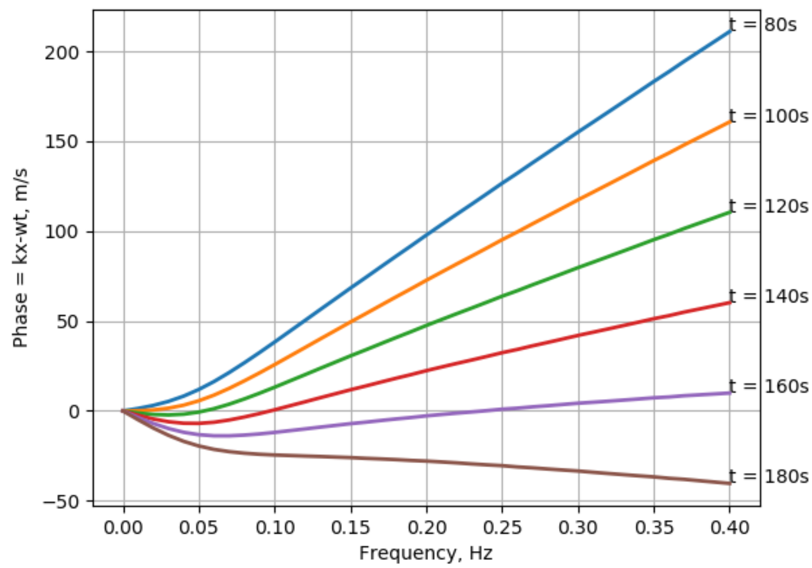


Figure 12: Phase function of Love wave for different times. [**ER**]

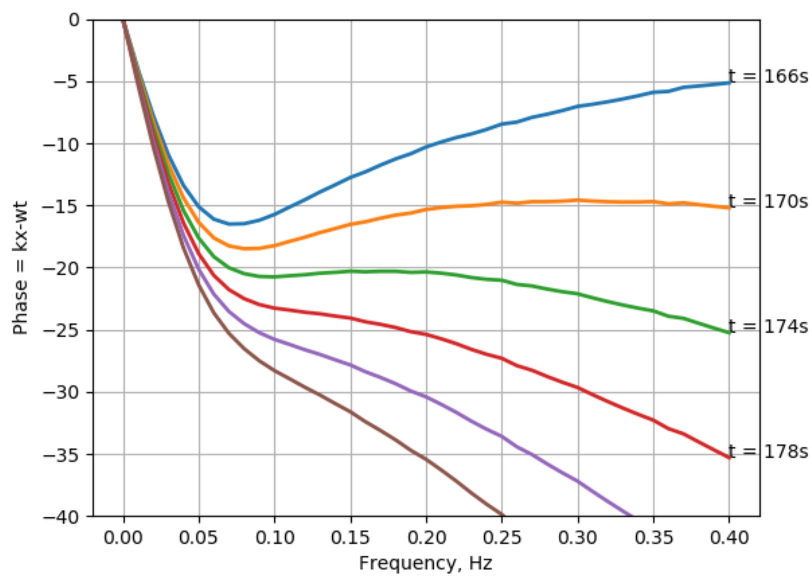


Figure 13: Phase function of Love wave for different times (zoomed). [**ER**]

points coming from right to left as we increase time in Figure 13). This is anomalous dispersion superimposed onto the normal dispersion (stationary points coming from left to right in Figure 13). We see that even if this analysis was simple, it is quite successful in explaining how all the pieces combine to create a final observable dispersed waveform of Love wave.

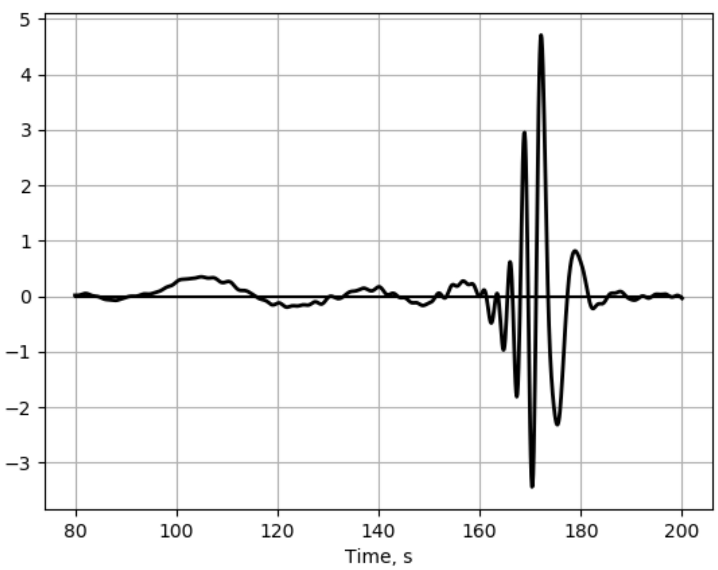


Figure 14: Mimicked trace of Love wave. [**ER**]

CONCLUSION

Surface waves is a complex phenomenon associated with difficulties in the seismic data processing. This paper is an attempt in understanding why they are different and discerning their physical and mathematical properties to demystify their behavior. They carry useful information that can be used in recovering subsurface properties and therefore, if we are successful in modeling them and explaining their physical behavior we might get closer to their suppression from the seismic record and aim at the inverse problem of body wave velocity reconstruction.

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